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Research Article

Perturbations of Regularized Determinants of Operators in a Banach Space

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Let X be a separable Banach space with the approximation property. For an integer $p \geq 1$, let Γ_p be a quasinormed ideal of compact operators in X with a quasinorm N_{Γ_p}, such that $\sum_{k=1}^{\infty} |\lambda_k(A)|^p \leq a_p N_{\Gamma_p}^p(A)$ $(A \in \Gamma_p)$, where $\lambda_k(A)$ are the eigenvalues of A and a_p is a constant independent of A. We suggest upper and lower bounds for the regularized determinants of operators from Γ_p as well as bounds for the difference between determinants of two operators. Applications to the p-summing operators, Hille-Tamarkin integral operators, Hille-Tamarkin matrices, Schatten-von Neumann operators, and Lorentz operator ideals are discussed.

1. Statement of the Main Result

Let X be a separable Banach space with the approximation property and the unit operator I. Let $E_p(z)$ be the Weierstrass primary factor:

$$E_1(z) = (1-z), \quad E_p(z) = (1-z)\exp\left[\sum_{m=1}^{p-1}\frac{z^m}{m}\right] \quad (p = 2, 3, \ldots;\ z \in \mathbb{C}). \tag{1}$$

For a Riesz operator A whose eigenvalues counted with their algebraic multiplicities are denoted by $\lambda_k(A)$ $(k = 1, 2, \ldots)$, introduce the p-regularized determinant

$$\det_p(I - A) = \prod_{k=1}^{\infty} E_p(\lambda_k(A)), \tag{2}$$

provided

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^p < \infty. \tag{3}$$

The classical theory of regularized determinants for Schatten-von Neumann operators has a long history, which is presented, in particular, in [1, 2]. König [3] developed the theory of regularized determinants for absolutely p-summing operators $(2 < p < \infty)$ in a Banach space. In [2, 4], following the classical pattern, regularized determinants are defined for operators of the form $I + A$, in a Banach space where not necessarily A itself but at least some power A^m admits a trace. The idea is to replace in all formulas the undefined traces by zero.

Let SN_p $(p = 1, 2, \ldots)$ be the von Neumann-Schatten ideal of compact operators A in a separable Hilbert space H with the finite norm $N_p(A) = [\mathrm{Trace}(AA^*)^{p/2}]^{1/p}$, where A^* is adjoint to A. The following inequalities are well-known:

$$\begin{gathered}\left|\det_p(I - A)\right| \leq \exp\left[d_p N_p^p(A)\right], \\ \left|\det_p(I - A) - \det_p(I - B)\right| \\ \leq N_p(A - B)\exp\left[d_p\left(1 + \left[N_p(A) + N_p(B)\right]\right)^p\right]\end{gathered} \tag{4}$$

with an unknown constant d_k, see the books [1, page 1106] and [2, page 194]. In [5, 6] these inequalities were slightly improved. In [7] it was proved that one can take $d_p = \gamma_p$, where

$$\gamma_p := \frac{p-1}{p} \quad (p \neq 1; p \neq 3), \qquad \gamma_1 = \gamma_3 = 1. \tag{5}$$

In this paper we investigate a quasinormed ideal Γ_p of compact operators in X with a quasinorm $N_{\Gamma_p}(\cdot)$. That is,

$N_{\Gamma_p}(\cdot)$ satisfies all the usual properties of a norm, with the exception of the triangular inequality, which is replaced by

$$N_{\Gamma_p}(A+B) \le b(\Gamma_p)\left(N_{\Gamma_p}(A)+N_{\Gamma_p}(B)\right) \tag{6}$$

with a constant $b(\Gamma_p)$ independent of A, B. Moreover, for an integer $1 \le p < \infty$ and any $A \in \Gamma_p$, the inequality

$$\sum_{k=1}^{\infty}|\lambda_k(A)|^p \le a_p N_{\Gamma_p}^p(A) \tag{7}$$

holds, where a_p is a constant independent of A (but dependent on Γ_p). The aim of this paper is to generalize inequalities (4) to operators from Γ_p. In addition, a lower bound for $\det_p(I-A)$ is established.

Now we are in a position to formulate our main result.

Theorem 1. *Let $A, B \in \Gamma_p$ for an integer $p \ge 1$. Then*

$$\left|\det_p(I-A)\right| \le e^{a_p\gamma_p N_{\Gamma_p}^p(A)}, \tag{8}$$

$$\begin{aligned}&\left|\det_p(I-A)-\det_p(I-B)\right|\\ &\le N_{\Gamma_p}(A-B)\\ &\quad\times\exp\left[\gamma_p a_p b(\Gamma_p)\left(1+\frac{1}{2}\left[N_{\Gamma_p}(A-B)+N_{\Gamma_p}(A+B)\right]\right)^p\right].\end{aligned} \tag{9}$$

This theorem is proved in the next section.

Note that if N_{Γ_p} is a norm, then $b(\Gamma_p)=1$; if $\Gamma_p = SN_p$, then $a_p = 1$.

2. Proof of Theorem 1

Let Y be an quasinormed space, that is, it is a linear space with a quasinorm $N(\cdot) = N_Y(\cdot)$. Namely,

$$N\left(C+\widetilde{C}\right) \le c\left(N(C)+N\left(\widetilde{C}\right)\right) \quad \left(C, \widetilde{C} \in Y\right) \tag{10}$$

with a constant $c = c_Y$.

Lemma 2. *For all $C, \widetilde{C} \in Y$, let $f(C+\lambda\widetilde{C})$ be a scalar-valued entire function of $\lambda \in \mathbb{C}$ and there be a monotone nondecreasing function $G : [0,\infty) \to [0,\infty)$, such that*

$$|f(C)| \le G(N(C)) \tag{11}$$

for all $C \in Y$. Then

$$\begin{aligned}&\left|f(C)-f\left(\widetilde{C}\right)\right|\\ &\quad\le N\left(C-\widetilde{C}\right)G\left(c\left(1+\frac{1}{2}N\left(C-\widetilde{C}\right)+\frac{1}{2}N\left(C+\widetilde{C}\right)\right)\right).\end{aligned} \tag{12}$$

Proof. Put

$$g_1(\lambda) = f\left(\frac{1}{2}\left(C+\widetilde{C}\right)+\lambda\left(C-\widetilde{C}\right)\right). \tag{13}$$

Then $g_1(\lambda)$ is an entire function and

$$f(C)-f\left(\widetilde{C}\right) = g_1\left(\frac{1}{2}\right)-g_1\left(-\frac{1}{2}\right). \tag{14}$$

Thanks to the Cauchy integral,

$$\begin{aligned}&g_1\left(\frac{1}{2}\right)-g_1\left(-\frac{1}{2}\right)\\ &\quad=\frac{1}{2\pi i}\int_{|z|=1/2+r}\frac{g_1(z)\,dz}{(z-1/2)(z+1/2)} \quad (r>0).\end{aligned} \tag{15}$$

Hence,

$$\begin{aligned}\left|g_1\left(\frac{1}{2}\right)-g_1\left(-\frac{1}{2}\right)\right| &\le \left(\frac{1}{2}+r\right)\sup_{|z|=1/2+r}\frac{|g_1(z)|}{|z^2-(1/4)|}\\ &\le \frac{1}{r}\sup_{|z|=1/2+r}|g_1(z)|.\end{aligned} \tag{16}$$

In addition, by (11),

$$\begin{aligned}|g_1(z)| &= \left|f\left(\frac{1}{2}\left(C+\widetilde{C}\right)+z\left(C-\widetilde{C}\right)\right)\right|\\ &\le G\left(N\left(\frac{1}{2}\left(C+\widetilde{C}\right)+z\left(C-\widetilde{C}\right)\right)\right)\\ &\le G\left(\frac{1}{2}cN\left(C+\widetilde{C}\right)+\left(\frac{1}{2}+r\right)cN\left(C-\widetilde{C}\right)\right)\\ &\qquad\left(|z|=\frac{1}{2}+r\right).\end{aligned} \tag{17}$$

Therefore according to (15),

$$\begin{aligned}&\left|f(C)-f\left(\widetilde{C}\right)\right|\\ &\quad=\left|g_1\left(\frac{1}{2}\right)-g_1\left(-\frac{1}{2}\right)\right|\\ &\quad\le\frac{1}{r}G\left(\frac{1}{2}cN\left(C+\widetilde{C}\right)+\left(\frac{1}{2}+r\right)cN\left(C-\widetilde{C}\right)\right).\end{aligned} \tag{18}$$

Taking $r = 1/N(C-\widetilde{C})$, we get the required result. ∎

We need also the following result, proved in [7, Lemma 2.3].

Lemma 3. *For any integer $p \ge 1$ and all $z \in \mathbb{C}$, one has $|E_p(z)| \le \exp[\gamma_p|z|^p]$.*

Proof of Theorem 1. By the previous lemma

$$\begin{aligned}\left|\det_p(I-A)\right| &= \prod_{k=1}^{\infty}\left|E_p(\lambda_k(A))\right|\\ &\le \prod_{k=1}^{\infty}e^{\gamma_p|\lambda_k(A)|^p}\\ &= \exp\left[\gamma_p\sum_{k=1}^{\infty}|\lambda_k(A)|^p\right].\end{aligned} \tag{19}$$

Now (8) follows from the latter inequality and (7).

Moreover, (8) and Lemma 2 imply (9). ∎

3. Lower Bounds

Let $1 \notin \sigma(A)$ and L be *a Jordan curve connecting* 0 *and* 1, *lying in the disc* $\{z \in \mathbb{C} : |z| \le 1\}$ *and such that*

$$\phi_A := \inf_{s\in L; k=1,2,\ldots} |1 - s\lambda_k(A)| > 0. \tag{20}$$

Let $l = |L|$ be the length of L. For example, if A does not have the eigenvalues on $[1, \infty)$, then one can take $L = [0, 1]$. In this case $l = 1$ and

$$\phi_A = \inf_{s\in[0,1]; k=1,2,\ldots} |1 - s\lambda_k(A)|. \tag{21}$$

If the spectral radius $r_s(A)$ of A is less than one, then $l = 1$, $\phi_A \ge 1 - r_s(A)$.

Theorem 4. *Let* $A \in \Gamma_p$ $(p = 1, 2, \ldots)$, $1 \notin \sigma(A)$, *and condition* (20) *hold. Then*

$$\left|\det_p(I - A)\right| \ge e^{-(la_p N^p_{\Gamma_p}(A)/\phi_A)}. \tag{22}$$

Proof. We have

$$\det_p(I - zA) = \prod_{j=1}^{\infty} E_p(z\lambda_j) \quad (\lambda_j = \lambda_j(A)). \tag{23}$$

Clearly,

$$\frac{d}{dz}\det_p(I - zA) = \sum_{k=1}^{\infty} \frac{dE_p(z\lambda_k)}{dz} \prod_{j=1, j\ne k}^{\infty} E_p(z\lambda_j),$$

$$\frac{dE_p(z\lambda_j)}{dz} = \left[-\lambda_j + (1 - z\lambda_j)\sum_{m=0}^{p-2} z^m \lambda_j^{m+1}\right] \exp\left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m}\right]. \tag{24}$$

But

$$-\lambda_j + (1 - z\lambda_j)\sum_{m=0}^{p-2} z^m \lambda_j^{m+1} = -z^{p-1}\lambda_j^p, \tag{25}$$

since

$$\sum_{m=0}^{p-2} z^m \lambda_j^m = \frac{1 - (z\lambda_j)^{p-1}}{1 - z\lambda_j}. \tag{26}$$

So

$$\begin{aligned}\frac{dE_p(z\lambda_j)}{dz} &= -z^{p-1}\lambda_j^p \exp\left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m}\right] \\ &= -\frac{z^{p-1}\lambda_j^p}{1 - z\lambda_j} E_p(z\lambda_j).\end{aligned} \tag{27}$$

Hence,

$$\frac{d}{dz}\det_p(I - zA) = h(z)\det_p(I - zA), \tag{28}$$

where

$$h(z) := -z^{p-1}\sum_{k=1}^{\infty} \frac{\lambda_k^p(A)}{1 - z\lambda_k(A)}. \tag{29}$$

Consequently,

$$\det_p(I - A) = \exp\left[\int_L h(s)\,ds\right]. \tag{30}$$

But $|s| \le 1$ for any $s \in L$, and thus by (7)

$$\begin{aligned}\left|\int_L h(s)\,ds\right| &\le \sum_{k=1}^{\infty} |\lambda_k(A)|^p \int_L \frac{|s|^{p-1}|ds|}{|1 - s\lambda_k(A)|} \\ &\le \frac{l}{\phi_A}\sum_{k=1}^{\infty} |\lambda_k(A)|^p \le \frac{la_p}{\phi_A} N^p_{\Gamma_p}(A).\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}\left|\det_p(I - A)\right| &= \left|\exp\left[\int_L h(s)\,ds\right]\right| \\ &\ge \exp\left[-\left|\int_L h(s)\,ds\right|\right] \\ &\ge \exp\left[-N^p_{\Gamma_p} la_p \phi_A^{-1}(A)\right],\end{aligned} \tag{32}$$

as claimed. ∎

Since

$$\begin{aligned}&\left|\det_p(I - B)\right| \\ &\quad \ge \left|\det_p(I - A)\right| - \left|\det_p(I - A) - \det_p(I - B)\right|,\end{aligned} \tag{33}$$

Theorems 1 and 4 imply the following result.

Corollary 5. *Let* $A, B \in \Gamma_p$ *for an integer* $p \ge 1$, $1 \notin \sigma(A)$, *and condition* (20) *hold. If, in addition,*

$$\begin{aligned}&\exp\left[-\frac{a_p l N^p_{\Gamma_p}(A)}{\phi_A}\right] \\ &> N_{\Gamma_p}(A - B) \\ &\quad \times \exp\left[a_p \gamma_p b(\Gamma_p)\left(1 + \frac{1}{2}\left(N_{\Gamma_p}(A + B) + N_{\Gamma_p}(A - B)\right)\right)^p\right],\end{aligned} \tag{34}$$

then $I - B$ *is invertible.*

4. Applications

Suppose $1 \le p < \infty$ and that A a linear operator in X. A is said to be p-summing, if there is a constant ν such that

regardless of a natural number m and regardless of the choice $x_1, \dots, x_m \in X$ we have

$$\left[\sum_{k=1}^{m}\|Ax_k\|^p\right]^{1/p} \le \nu \sup\left\{\left[\sum_{k=1}^{m}|(x^*, x_k)|^p\right]^{1/p} : x^* \in X^*,\ \|x^*\| = 1\right\}, \tag{35}$$

cf. [8]. The least ν for which this inequality holds is denoted by $\pi_p(A)$. The set of p-summing operators in X with the finite norm π_p is an ideal, cf. [9], which is denoted by Π_p. By the well-known Theorem 3.7.2 in [9, page 159],

$$\sum_{k=1}^{\infty}|\lambda_k(A)|^p \le \pi_p^p(A) \quad \left(A \in \Pi_p;\ 2 \le p < \infty\right) \tag{36}$$

(see also Theorem 17.4.3 in [? , page 298]). Since $\pi_p(A)$ is a norm, Theorems 1 and 4 imply the following.

Corollary 6. *Let $A, B \in \Pi_p$ for some integer $p \ge 2$. Then $|\det_p(I - A)| \le \exp[\gamma_p \pi_p^p(A)]$ and*

$$\left|\det_p(I - A) - \det_p(I - B)\right| \le \pi_p(A - B)\exp\left[\gamma_p\left(1 + \frac{1}{2}\left[\pi_p(A - B) + \pi_p(A + B)\right]\right)^p\right]. \tag{37}$$

If, in addition, (20) *holds, then*

$$\left|\det_p(I - A)\right| \ge e^{-(l\pi_p^p(A)/\phi_A)}. \tag{38}$$

Furthermore, let $L_\mu^p(\Omega)(\Omega \subset \mathbb{R}^n; 1 < p < \infty)$ be the space of scalar functions f defined on Ω with a finite positive measure μ and the norm

$$\|f\| = \left[\int_\Omega |f(x)|^p d\mu\right]^{1/p}. \tag{39}$$

Let $K : L_\mu^p(\Omega) \to L_\mu^p(\Omega)$ be the integral operator

$$(Kf)(t) = \int_\Omega k(t, s) f(s)\, d\mu \tag{40}$$

whose kernel k defined on $\Omega \times \Omega$ satisfies the condition

$$\widehat{k}_p(K) := \left[\int_\Omega\left(\int_\Omega |k(t, s)|^{p'} d\mu(s)\right)^{p/p'} d\mu(t)\right]^{1/p} < \infty, \tag{41}$$

where $1/p + 1/p' = 1$. Then K is called a (p, p')-Hille-Tamarkin operator. As it is well known [8, page 43], any (p, p')-Hille-Tamarkin operator K is a p-summing operator and

$$\pi_p(K) \le \widehat{k}_p(K). \tag{42}$$

Since $\widehat{k}_p(\cdot)$ is a norm, by Theorems 1 and 4 we get.

Corollary 7. *Let K and $\widetilde{K}$ be (p, p')-Hille-Tamarkin operators in $L_\mu^p(\Omega)$ for an integer $p \ge 2$. Then $|\det_p(I - K)| \le \exp[\gamma_p \widehat{k}_p^p(K)]$ and*

$$\left|\det_p(I - K) - \det_p\left(I - \widetilde{K}\right)\right| \le \widehat{k}_p\left(\widetilde{K} - K\right)\exp\left[\gamma_p\left(1 + \frac{1}{2}\left[\widehat{k}_p\left(\widetilde{K} - K\right) + \widehat{k}_p\left(\widetilde{K} + K\right)\right]\right)^p\right]. \tag{43}$$

If, in addition, condition (20) *holds for $A = K$, then*

$$\left|\det_p(I - K)\right| \ge e^{-(l\widehat{k}_p^p(K)/\phi_K)}. \tag{44}$$

Now let us consider a linear operator T in l^p $(1 < p < \infty)$ generated by an infinite matrix $(t_{jk})_{j,k=1}^{\infty}$, satisfying

$$\widehat{\tau}_p(T) := \left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|t_{jk}\right|^{p'}\right)^{p/p'}\right]^{1/p} < \infty. \tag{45}$$

Then T is called a (p, p')-Hille-Tamarkin matrix. As it is well known [8, page 43], any (p, p')-Hille-Tamarkin matrix T is a p-summing operator with

$$\pi_p(T) \le \widehat{\tau}_p(T), \tag{46}$$

cf. [9, Sections 5.3.2 and 5.3.3, page 230].

Since $\widehat{\tau}_p(\cdot)$ is a norm, Theorems 1 and 4 imply the following.

Corollary 8. *Let T and $\widetilde{T}$ be (p, p')-Hille-Tamarkin matrices for an integer $p \ge 2$. Then $|\det_p(I - T)| \le \exp[\gamma_p \widehat{\tau}_p^p(T)]$ and*

$$\begin{aligned}&\left|\det_p(I - T) - \det_p\left(I - \widetilde{T}\right)\right| \\ &\le \widehat{\tau}_p\left(\widetilde{T} - T\right) \\ &\quad \times \exp\left[\gamma_p\left(1 + \frac{1}{2}\left[\widehat{\tau}_p\left(\widetilde{T} - T\right) + \widehat{\tau}_p\left(\widetilde{T} + T\right)\right]\right)^p\right].\end{aligned} \tag{47}$$

If, in addition, condition (20) *holds for $A = T$, then*

$$\left|\det_p(I - T)\right| \ge e^{-(l\widehat{\tau}_p^p(T)/\phi_T)}. \tag{48}$$

Now let $X = H$ be a separable Hilbert space and $L_{q,r}$ $(q > 1,\ 0 < r < q)$ the Lorentz ideal of compact operators T with the finite quasinorm

$$N_{q,r}(T) = \left[\sum_{k=1}^{\infty} k^{(q/r)-1} s_k^q(T)\right]^{1/q}, \tag{49}$$

where $s_k(T)$ are the singular numbers of T taken with their multiplicities. So

$$N_{q,r}\left(T + \widetilde{T}\right) \le c_{q,r}\left(N_{q,r}(T) + N_{q,r}\left(\widetilde{T}\right)\right) \quad \left(c_{q,r} = \text{const};\ T, \widetilde{T} \in L_{q,r}\right). \tag{50}$$

For the details, see [? , Section 1.1]. By [? , Lemma 1.4],

$$\sum_{k=1}^{\infty} k^{(q/r)-1} |\lambda_k(T)|^q \le c_{q,r} N_{q,r}^q(T). \tag{51}$$

For an integer $p \ge 1$, let $q > p$ and $r = qp/(p+q)$. Then simple calculations show that $(q/r) - 1 = q/p$. By the Hölder inequality, for $d = q/p$, we obtain

$$\sum_{k=1}^{\infty} |\lambda_k(T)|^p \le \tau(d) \left(\sum_{k=1}^{\infty} k^d |\lambda_k(T)|^{pd} \right)^{1/d} \tag{52}$$

with

$$\tau(d) = \left(\sum_{k=1}^{\infty} k^{-d'} \right)^{1/d'} \quad \left(\frac{1}{d'} + \frac{1}{d} = 1 \right). \tag{53}$$

So we have

$$\sum_{k=1}^{\infty} |\lambda_k(T)|^p \le \tau(q/p) \left(\sum_{k=1}^{\infty} k^{(q/r)-1} |\lambda_k(T)|^q \right)^{p/q}. \tag{54}$$

Thus (51) implies the following result.

Lemma 9. *For an integer $p \ge 1$ and a $q > p$, let $T \in L_{q,r}$ with $r = qp/(p+q)$. Then*

$$\sum_{k=1}^{\infty} |\lambda_k(T)|^p \le c_{q,r} \tau\left(\frac{q}{p}\right) N_{q,r}^p(T). \tag{55}$$

Now we can directly apply Theorems 1 and 4.

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