Research Article

Perturbations of Regularized Determinants of Operators in a Banach Space

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Let $X$ be a separable Banach space with the approximation property. For an integer $p \geq 1$, let $\Gamma_p$ be a quasinormed ideal of compact operators in $X$ with a quasinorm $N_{\Gamma_p}(\cdot)$, such that $\sum_{k=1}^{\infty} |\lambda_k(A)|^p \leq a_p N_{\Gamma_p}(A) \ (A \in \Gamma_p)$, where $\lambda_k(A)$ are the eigenvalues of $A$ and $a_p$ is a constant independent of $A$. We suggest upper and lower bounds for the regularized determinants of operators from $\Gamma_p$ as well as bounds for the difference between determinants of two operators. Applications to the $p$-summing operators, Hille-Tamarkin integral operators, Hille-Tamarkin matrices, Schatten-von Neumann operators, and Lorentz operator ideals are discussed.

1. Statement of the Main Result

Let $X$ be a separable Banach space with the approximation property and the unit operator $I$. Let $E_p(z)$ be the Weierstrass primary factor:

$$E_1(z) = (1 - z), \quad E_p(z) = (1 - z) \exp \left[ \sum_{m=1}^{p-1} \frac{z^m}{m} \right] \quad (p = 2, 3, \ldots; z \in \mathbb{C}).$$

For a Riesz operator $A$ whose eigenvalues counted with their algebraic multiplicities are denoted by $\lambda_k(A)$ ($k = 1, 2, \ldots$), introduce the $p$-regularized determinant

$$\det_p(I - A) = \prod_{k=1}^{\infty} E_p(\lambda_k(A)), \quad (2)$$

provided

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^p < \infty. \quad (3)$$

The classical theory of regularized determinants for Schatten-von Neumann operators has a long history, which is presented, in particular, in [1, 2]. König [3] developed the theory of regularized determinants for absolutely $p$-summing operators ($2 < p < \infty$) in a Banach space. In [2, 4], following the classical pattern, regularized determinants are defined for operators of the form $I + A$, in a Banach space where not necessarily $A$ itself but at least some power $A^m$ admits a trace. The idea is to replace in all formulas the undefined traces by zero.

Let $SN_p (p = 1, 2, \ldots)$ be the von Neumann-Schatten ideal of compact operators $A$ in a separable Hilbert space $H$ with the finite norm $N_p(A) = [\text{Trace}(AA^*)^{p/2}]^{1/p}$, where $A^*$ is adjoint to $A$. The following inequalities are well-known:

$$|\det_p(I - A)| \leq \exp \left[ d_p N_p^p(A) \right],$$

$$|\det_p(I - A) - \det_p(I - B)| \leq N_p(A - B) \exp \left[ d_p (1 + [N_p(A) + N_p(B)])^{p'} \right] \quad (4)$$

with an unknown constant $d_k$, see the books [1, page 1106] and [2, page 194]. In [5, 6] these inequalities were slightly improved. In [7] it was proved that one can take $d_p = \gamma_p$, where

$$\gamma_p := \frac{p - 1}{p} \quad (p \neq 1; p \neq 3), \quad \gamma_1 = \gamma_3 = 1. \quad (5)$$

In this paper we investigate a quasinormed ideal $\Gamma_p$ of compact operators in $X$ with a quasinorm $N_{\Gamma_p}(\cdot)$. That is,
Then \( g_1(\lambda) \) is an entire function and
\[
 f(C) - f(\tilde{C}) = g_1 \left( \frac{1}{2} \right) - g_1 \left( -\frac{1}{2} \right). \tag{14}
\]

Thanks to the Cauchy integral,
\[
g_1 \left( \frac{1}{2} \right) - g_1 \left( -\frac{1}{2} \right) = \frac{1}{2\pi i} \int_{|z|=|1/2+r|} \frac{g_1(z) \, dz}{z^2 - (1/4)} \quad (r > 0). \tag{15}
\]

Hence,
\[
\left| g_1 \left( \frac{1}{2} \right) - g_1 \left( -\frac{1}{2} \right) \right| \leq \frac{1}{2} \sup_{|z|=|1/2+r|} \left| g_1(z) \right|. \tag{16}
\]

In addition, by (11),
\[
|g_1(z)| = \left| f \left( \frac{1}{2} (C + \tilde{C}) + z (C - \tilde{C}) \right) \right| \leq G \left( n \left( \frac{1}{2} (C + \tilde{C}) + z (C - \tilde{C}) \right) \right)
\]
\[
\leq G \left( \frac{1}{2} cN(C + \tilde{C}) + \left( \frac{1}{2} + r \right) cN(C - \tilde{C}) \right) \quad (|z| = \frac{1}{2} + r). \tag{17}
\]

Therefore according to (15),
\[
\left| f(C) - f(\tilde{C}) \right| = \left| g_1 \left( \frac{1}{2} \right) - g_1 \left( -\frac{1}{2} \right) \right| \leq \frac{1}{2} G \left( \frac{1}{2} cN(C + \tilde{C}) + \left( \frac{1}{2} + r \right) cN(C - \tilde{C}) \right). \tag{18}
\]

Taking \( r = 1/N(C - \tilde{C}) \), we get the required result. \( \Box \)

We need also the following result, proved in [7, Lemma 2.3].

**Lemma 3.** For any integer \( p \geq 1 \) and all \( z \in \mathbb{C} \), one has
\[ |E_p(z)| \leq \exp |y_p| |z|^p. \]

**Proof of Theorem 1.** By the previous lemma
\[
\left| \det_p(I - A) \right| = \prod_{k=1}^{\infty} \left| E_p(\lambda_k(A)) \right| \leq \prod_{k=1}^{\infty} \exp \left[ y_p |\lambda_k(A)|^p \right] \leq \exp \left[ y_p \sum_{k=1}^{\infty} |\lambda_k(A)|^p \right]. \tag{19}
\]

Now (8) follows from the latter inequality and (7). Moreover, (8) and Lemma 2 imply (9). \( \Box \)
3. Lower Bounds

Let \( 1 \notin \sigma(A) \) and \( L \) be a Jordan curve connecting 0 and 1, lying in the disc \( \{ z \in \mathbb{C} : |z| \leq 1 \} \) and such that

\[
\phi_A := \inf_{x \in L, k=1,2,...} |1 - s\lambda_k(A)| > 0. \tag{20}
\]

Let \( l = |L| \) be the length of \( L \). For example, if \( A \) does not have the eigenvalues on \( [1, \infty) \), then one can take \( L = [0,1] \). In this case \( l = 1 \) and

\[
\phi_A = \inf_{x \in [0,1]} |1 - s\lambda_k(A)|. \tag{21}
\]

If the spectral radius \( r_s(\sigma) \) of \( \sigma \) is less than one, then \( l = 1 \), \( \phi_A \geq 1 - r_s(\sigma) \).

**Theorem 4.** Let \( A \in \Gamma_p (p = 1, 2, \ldots), 1 \notin \sigma(A), \) and condition (20) hold. Then

\[
|\det_p (I - A)| \geq e^{-(\lambda_1 N_p^p(\sigma)/\phi_A)}. \tag{22}
\]

**Proof.** We have

\[
\det_p (I - zA) = \prod_{j=1}^{\infty} E_p (z\lambda_j) \quad (\lambda_j = \lambda_j (A)). \tag{23}
\]

Clearly,

\[
\frac{d}{dz} \det_p (I - zA) = \sum_{k=1}^{\infty} \frac{d E_p (z\lambda_k)}{dz} \prod_{j=1, j \neq k}^{\infty} E_p (z\lambda_j),
\]

\[
\frac{d E_p (z\lambda_j)}{dz} = \left[ -\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1} \right] \exp \left[ \sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m} \right]. \tag{24}
\]

But

\[
-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1} = -z^{p-1} \lambda_j^p, \tag{25}
\]

since

\[
\sum_{m=0}^{p-2} z^m \lambda_j^m = \frac{1 - (z\lambda_j)^{p-1}}{1 - z\lambda_j}. \tag{26}
\]

So

\[
\frac{d E_p (z\lambda_j)}{dz} = -z^{p-1} \lambda_j^p \exp \left[ \sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m} \right] = -z^{p-1} \lambda_j^p E_p (z\lambda_j). \tag{27}
\]

Hence,

\[
\frac{d}{dz} \det_p (I - zA) = h(z) \det_p (I - zA), \tag{28}
\]

where

\[
h(z) := -z^{p-1} \sum_{k=1}^{\infty} \frac{\lambda_k^p (A)}{1 - z\lambda_k (A)}. \tag{29}
\]

Consequently,

\[
\det_p (I - A) = \exp \left[ \int_L h(s) \, ds \right]. \tag{30}
\]

But \( |s| \leq 1 \) for any \( s \in L \), and thus by (7)

\[
\left| \int_L h(s) \, ds \right| \leq \sum_{k=1}^{\infty} |\lambda_k (A)| \int_L \frac{|s|^{p-1} |ds|}{|1 - s\lambda_k (A)|} \leq \sum_{k=1}^{\infty} |\lambda_k (A)|^p \leq \frac{l}{\phi_A} N_p^p (\sigma). \tag{31}
\]

Therefore,

\[
|\det_p (I - A)| \geq e^{-\frac{l}{\phi_A} N_p^p (\sigma)}. \tag{32}
\]

as claimed. \( \Box \)

Since

\[
|\det_p (I - B)| \geq |\det_p (I - A) - \det_p (I - A) - \det_p (I - B)|, \tag{33}
\]

Theorems 1 and 4 imply the following result.

**Corollary 5.** Let \( A, B \in \Gamma_p, p \geq 1, 1 \notin \sigma(A), \) and condition (20) hold. If, in addition,

\[
\exp \left[ -\frac{a_p L_{N_p^p (A)} (A)}{\phi_A} \right] > N_{\Gamma_p} (A - B)
\]

\[
\times \exp \left[ a_p y_p (\Gamma_p) \left( 1 + \frac{1}{2} \left( N_{\Gamma_p} (A + B) + N_{\Gamma_p} (A - B) \right) \right) \right], \tag{34}
\]

then \( I - B \) is invertible.

4. Applications

Suppose \( 1 \leq p < \infty \) and that \( A \) a linear operator in \( X \). \( A \) is said to be \( p \)-summing, if there is a constant \( \nu \) such that
regardless of a natural number $m$ and regardless of the choice $x_1, \ldots, x_m \in X$ we have
\[
\left[ \sum_{k=1}^{m} \| Ax_k \| \right]^{1/p} \leq v \sup \left\{ \left[ \sum_{k=1}^{m} (x^*, x_k) \right]^{1/p} : x^* \in X^*, \| x^* \| = 1 \right\},
\]
(35)
cf. [8]. The least $v$ for which this inequality holds is denoted by $\pi_p(A)$. The set of $p$-summing operators in $X$ with the finite norm $\pi_p$ is an ideal, cf. [9], which is denoted by $\Pi_p$. By the well-known Theorem 3.7.2 in [9, page 159],
\[
\sum_{k=1}^{\infty} | \lambda_k (A) |^p \leq \pi_p^p (A) \quad (A \in \Pi_p; 2 \leq p < \infty)
\]
(36)
(see also Theorem 17.4.3 in [? , page 298]). Since $\pi_p(A)$ is a norm, Theorems 1 and 4 imply the following.

**Corollary 6.** Let $A, B \in \Pi_p$ for some integer $p \geq 2$. Then
\[
| \det_p (I - A) - \det_p (I - B) | \leq \pi_p (A - B) \exp \left[ \gamma_p \left( 1 + \frac{1}{2} \left[ \pi_p (A - B) + \pi_p (A + B) \right] \right)^p \right].
\]
(37)
If, in addition, (20) holds, then
\[
| \det_p (I - A) | \geq e^{-\left( \ln \pi_p (A)/\lambda_p \right)}.
\]
(38)
Furthermore, let $L^p_{\infty}(\Omega)(\Omega \subset \mathbb{R}^n; 1 < p < \infty)$ be the space of scalar functions $f$ defined on $\Omega$ with a finite positive measure $\mu$ and the norm
\[
\| f \| = \left[ \int_{\Omega} | f (x) |^p d\mu \right]^{1/p}.
\]
(39)
Let $K : L^p_{\infty}(\Omega) \rightarrow L^p_{\infty}(\Omega)$ be the integral operator
\[
(Kf)(t) = \int_{\Omega} k(t, s) f(s) d\mu \quad (Kf)(t) = \int_{\Omega} k(t, s) f(s) d\mu
\]
(40)
whose kernel $k$ defined on $\Omega \times \Omega$ satisfies the condition
\[
\tilde{k}_p (K) := \left[ \int_{\Omega} \left( \int_{\Omega} | k(t, s) |^p d\mu (s) \right)^{1/p} d\mu (t) \right]^{1/p} < \infty,
\]
(41)
where $1/p + 1/p' = 1$. Then $K$ is called a $(p, p')$-Hille-Tamarkin operator. As it is well known [8, page 43], any $(p, p')$-Hille-Tamarkin operator $K$ is a $p$-summing operator and
\[
\pi_p (K) \leq \tilde{k}_p (K).
\]
(42)
Since $\tilde{k}_p (\cdot)$ is a norm, by Theorems 1 and 4 we get.

**Corollary 7.** Let $K$ and $\tilde{K}$ be $(p, p')$-Hille-Tamarkin operators in $L^p_{\infty}(\Omega)$ for an integer $p \geq 2$. Then $| \det_p (I - K) | \leq \exp | \gamma_p \tilde{k}_p (K) |$ and
\[
| \det_p (I - K) - \det_p (I - \tilde{K}) | \leq \tilde{k}_p \left( \tilde{K} - \tilde{K} \right) \exp \left[ \gamma_p \left( 1 + \frac{1}{2} \left[ \tilde{k}_p \left( \tilde{K} - \tilde{K} \right) + \tilde{k}_p \left( \tilde{K} + \tilde{K} \right) \right] \right]^{p} \right].
\]
(43)
If, in addition, condition (20) holds for $A = K$, then
\[
| \det_p (I - K) | \geq e^{-\left( \ln \pi_p (K)/\lambda_p \right)}.
\]
(44)

Now let us consider a linear operator $T$ in $L^p (1 < p < \infty)$ generated by an infinite matrix $(\gamma_{jk})_{j=1}^{\infty}$, satisfying
\[
\tilde{\gamma}_p (T) := \left[ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \gamma_{jk} \right)^{p/p'} \right]^{1/p} < \infty.
\]
(45)
Then $T$ is called a $(p, p')$-Hille-Tamarkin matrix. As it is well known [8, page 43], any $(p, p')$-Hille-Tamarkin matrix $T$ is a $p$-summing operator with
\[
\pi_p (T) \leq \tilde{\gamma}_p (T),
\]
(46)
cf. [9, Sections 5.3.2 and 5.3.3, page 230].
Since $\tilde{\gamma}_p (\cdot)$ is a norm, Theorems 1 and 4 imply the following.

**Corollary 8.** Let $T$ and $\tilde{T}$ be $(p, p')$-Hille-Tamarkin matrices for an integer $p \geq 2$. Then $| \det_p (I - T) | \leq \exp | \gamma_p \tilde{\gamma}_p (T) |$ and
\[
| \det_p (I - T) - \det_p (I - \tilde{T}) | \leq \tilde{\gamma}_p \left( \tilde{T} - \tilde{T} \right) \exp \left[ \gamma_p \left( 1 + \frac{1}{2} \left[ \tilde{\gamma}_p \left( \tilde{T} - \tilde{T} \right) + \tilde{\gamma}_p \left( \tilde{T} + \tilde{T} \right) \right] \right]^{p} \right].
\]
(47)
If, in addition, condition (20) holds for $A = T$, then
\[
| \det_p (I - T) | \geq e^{-\left( \ln \pi_p (T)/\lambda_p \right)}.
\]
(48)

Now let $X = H$ be a separable Hilbert space and $L^{q, r}_{\infty} (q > 1, 0 < r < q)$ the Lorentz ideal of compact operators $T$ with the finite quasinorm
\[
N_{q,r} (T) = \left[ \sum_{k=1}^{\infty} k^{(q/r) - 1} s_k^q (T) \right]^{1/q},
\]
(49)
where $s_k (T)$ are the singular numbers of $T$ taken with their multiplicities. So
\[
N_{q,r} (T + \tilde{T}) \leq c_{q,r} \left( N_{q,r} (T) + N_{q,r} (\tilde{T}) \right)
\]
(50)
\[
(c_{q,r} = \text{const}; T, \tilde{T} \in L_{q,r}).
\]
For the details, see [? , Section 1.1]. By [? , Lemma 1.4],
\[
\sum_{k=1}^{\infty} k^{(q/r) - 1} |\lambda_k(T)|^q \leq c_{q,r} N_{q,r}^p(T).
\] (51)

For an integer \( p \geq 1 \), let \( q > p \) and \( r = qp/(p + q) \). Then simple calculations show that \((q/r) - 1 = q/p \). By the Hölder inequality, for \( d = q/p \), we obtain
\[
\sum_{k=1}^{\infty} |\lambda_k(T)|^p \leq \tau (d) \left( \sum_{k=1}^{\infty} k^d |\lambda_k(T)|^{pd} \right)^{1/d}
\] (52)
with
\[
\tau (d) = \left( \sum_{k=1}^{\infty} k^{-d} \right)^{1/d} \left( \frac{1}{d} + \frac{1}{d} = 1 \right).
\] (53)

So we have
\[
\sum_{k=1}^{\infty} |\lambda_k(T)|^p \leq \tau (q/p) \left( \sum_{k=1}^{\infty} k^{(q/r) - 1} |\lambda_k(T)|^q \right)^{p/q}.
\] (54)

Thus (51) implies the following result.

**Lemma 9.** For an integer \( p \geq 1 \) and a \( q > p \), let \( T \in L_{q,r} \) with \( r = qp/(p + q) \). Then
\[
\sum_{k=1}^{\infty} |\lambda_k(T)|^p \leq c_{q,r} \tau \left( \frac{q}{p} \right) N_{q,r}^p(T).
\] (55)

Now we can directly apply Theorems 1 and 4.

**References**


