Research Article

Bessel Transform of \((k, \gamma)\)-Bessel Lipschitz Functions

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Received 7 January 2013; Revised 12 March 2013; Accepted 13 March 2013

Academic Editor: Nasser Saad

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Using a generalized translation operator, we obtain an analog of Theorem 5.2 in Younis (1986) for the Bessel transform for functions satisfying the \((k, \gamma)\)-Bessel Lipschitz condition in \(L^2(\mathbb{R}^+\)).

1. Introduction and Preliminaries

Younis Theorem 5.2 [1] characterized the set of functions in \(L^2(\mathbb{R})\) satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms; namely, we have the following.

Theorem 1 (see [1]). Let \(f \in L^2(\mathbb{R})\). Then the following are equivalent:

1. \(|f(x + h) - f(x)|_2 = O(h^\alpha/(\log(1/h))^\beta)\) as \(h \to 0, 0 < \alpha < 1, \beta > 0\),
2. \(\int_{|x|\geq r} |\mathcal{F}(f)(x)|^2 dx = O(r^{-2\alpha}(\log r)^{-2\beta})\) as \(r \to \infty\),

where \(\mathcal{F}\) stands for the Fourier transform of \(f\).

In this paper, we obtain a generalization of Theorem 1 for the Bessel transform. For this purpose, we use a generalized translation operator.

Assume that \(L_{2,\alpha}(\mathbb{R}^+)\), \(\alpha > -1/2\) is the Hilbert space of measurable functions \(f(t)\) on \(\mathbb{R}^+_\) with finite norm

\[
\|f\|_{2,\alpha} = \left( \int_0^{\infty} |f(x)|^2 x^{2\alpha+1} \, dx \right)^{1/2}.
\]

Let

\[
B = \frac{d^2}{dt^2} + \frac{(2\alpha + 1)}{t} \frac{d}{dt}
\]

be the Bessel differential operator.

For \(\alpha \geq -1/2\), we introduce the Bessel normalized function of the first kind \(j_\alpha\) defined by

\[
j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left( \frac{z}{2} \right)^{2n},
\]

where \(\Gamma\) is the gamma function (see [2]).

The function \(y = j_\alpha(x)\) satisfies the differential equation

\[
By + y = 0,
\]

with the initial conditions \(y(0) = 1\) and \(y'(0) = 0\). \(j_\alpha(z)\) is function infinitely differentiable, even, and, moreover, entirely analytic.

Lemma 2. For \(x \in \mathbb{R}^+_\), the following inequality is fulfilled:

\[
|1 - j_\alpha(x)| \geq c,
\]

with \(x \geq 1\), where \(c > 0\) is a certain constant which depends only on \(\alpha\).

Proof. Analog of Lemma 2.9 is in [3].

Lemma 3. The following inequalities are valid for Bessel function \(j_\alpha\):

1. \(|j_\alpha(x)| \leq 1, \text{ for all } x \in \mathbb{R}^+_\),
2. \(1 - j_\alpha(x) = O(x^2), 0 \leq x \leq 1\).

Proof. See [4].
The Bessel transform we call the integral transform from 
\[ f(t) \lambda^{2\alpha+1} dt, \quad \alpha \in \mathbb{R}^+. \] (6)
The inverse Bessel transform is given by the formula
\[ f(t) = (2^{\alpha} \Gamma(\alpha + 1))^{-1} \int_0^{\infty} \tilde{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda. \] (7)
We have the Parseval's identity
\[ \| f \|_{2\alpha} = \frac{2^{\alpha} \Gamma(\alpha + 1)}{\Gamma(1/2) \Gamma(\alpha + (1/2))}. \] (8)

In \( L_{2\alpha}(\mathbb{R}_+), \) consider the generalized translation operator \( T_h \) defined by
\[ T_h f(t) = c_\alpha \int_0^t f \left( \sqrt{t^2 + h^2 - 2th \cos \varphi} \right) \sin^{2\alpha} \varphi d\varphi, \] (9)
where
\[ c_\alpha = \left( \int_0^\pi \sin^{2\alpha} \varphi d\varphi \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2) \Gamma(\alpha + (1/2))}. \] (10)

The following relations connect the generalized translation operator and the Bessel transform; in [7] we have
\[ (T_h \tilde{f})(\lambda) = j_\alpha(\lambda h) \tilde{f}(\lambda). \] (11)

2. Main Result

In this section we give the main result of this paper. We need first to define \((k, \gamma)\)-Bessel Lipschitz class.

Definition 4. Let \( 0 < k < 1 \) and \( \gamma \geq 0. \) A function \( f \in L_{2\alpha}(\mathbb{R}_+) \) is said to be in the \((k, \gamma)\)-Bessel Lipschitz class, denoted by Lip\((k, \gamma, 2)\), if
\[ \| T_h f(t) - f(t) \|_{2\alpha} = O \left( \frac{h^k}{(\log(1/h))^{2\gamma}} \right), \quad h \to 0. \] (12)

Our main result is as follows.

Theorem 5. Let \( f \in L_{2\alpha}(\mathbb{R}_+) \). Then the following are equivalents

1. \( f \in \text{Lip}(k, \gamma, 2) \).
2. \( \int_0^\infty |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2k} / (\log r)^{2\gamma}), \quad r \to +\infty. \)

Proof. (1) \( \Rightarrow \) (2) Assume that \( f \in \text{Lip}(k, \gamma, 2) \). Then we have
\[ \| T_h f(t) - f(t) \|_{2\alpha}^2 = \frac{1}{(2^{\alpha} \Gamma(\alpha + 1))^{2\gamma}} \int_0^\infty |1 - j_\alpha(\lambda h)|^2 |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda. \] (13)

If \( \lambda \in [1/h, 2/h] \) then \( \lambda h \geq 1 \) and Lemma 2 implies that
\[ 1 \leq \frac{1}{c_\alpha} |1 - j_\alpha(\lambda h)|. \] (14)

Then
\[ \int_0^{2/h} \left| \tilde{f}(\lambda) \right|^2 \lambda^{2\alpha+1} d\lambda \]
\[ = \frac{1}{c_\alpha} \int_0^{2/h} |1 - j_\alpha(\lambda h)|^2 |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \]
\[ \leq \frac{1}{c_\alpha} \int_0^{\infty} |1 - j_\alpha(\lambda h)|^2 |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \]
\[ = O \left( \frac{h^k}{(\log(1/h))^{2\gamma}} \right). \] (15)

We obtain
\[ \int_r^{2r} \left| \tilde{f}(\lambda) \right|^2 \lambda^{2\alpha+1} d\lambda \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}}, \] (16)
where \( C \) is a positive constant.

So that
\[ \int_0^{\infty} \left| \tilde{f}(\lambda) \right|^2 \lambda^{2\alpha+1} d\lambda = \left[ \int_r^{2r} + \int_{2r}^{4r} + \cdots \right] |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \]
\[ \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} + C \frac{(2r)^{-2k}}{(\log 2r)^{2\gamma}} + C \frac{(4r)^{-2k}}{(\log 4r)^{2\gamma}} + \cdots \]
\[ \leq C \frac{r^{-2k}}{(\log r)^{2\gamma}} \left( 1 + 2^{-2k} + (2^{-2k})^2 + (2^{-2k})^3 + \cdots \right) \]
\[ \leq CK \frac{r^{-2k}}{(\log r)^{2\gamma}}, \] (17)
where \( K = (1 - 2^{-2k})^{-1} \) since \( 2^{-2k} < 1. \)

This proves that
\[ \int_0^{\infty} \left| \tilde{f}(\lambda) \right|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{r^{-2k}}{(\log r)^{2\gamma}} \right) \quad \text{as } r \to +\infty. \] (18)

(2) \( \Rightarrow \) (1) Suppose now that
\[ \int_0^{\infty} \left| \tilde{f}(\lambda) \right|^2 \lambda^{2\alpha+1} d\lambda = O \left( \frac{r^{-2k}}{(\log r)^{2\gamma}} \right) \quad \text{as } r \to +\infty. \] (19)

We write
\[ \int_0^{\infty} |1 - j_\alpha(\lambda h)|^2 |\tilde{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2, \] (20)
where
\begin{align*}
I_1 &= \int_0^{1/h} \left| 1 - j_a (\lambda h) \right|^2 \left| \tilde{f} (\lambda) \right|^2 \lambda^{2a+1} d\lambda, \\
I_2 &= \int_{1/h}^{\infty} \left| 1 - j_a (\lambda h) \right|^2 \left| \tilde{f} (\lambda) \right|^2 \lambda^{2a+1} d\lambda.
\end{align*}
(21)

Estimate the summands \(I_1\) and \(I_2\) from above. It follows from the inequality \(|j_a (\lambda h)| \leq 1\) that
\begin{align*}
I_2 &= \int_{1/h}^{\infty} \left| 1 - j_a (\lambda h) \right|^2 \left| \tilde{f} (\lambda) \right|^2 \lambda^{2a+1} d\lambda \\
&\leq 4 \int_{1/h}^{\infty} \left| \tilde{f} (\lambda) \right|^2 \lambda^{2a+1} d\lambda = O \left( \frac{h^{2k}}{(\log (1/h))^{2\gamma}} \right).
\end{align*}
(22)

To estimate \(I_1\), we use the inequality (2) of Lemma 3. Set
\begin{equation}
\phi (x) = \int_x^{\infty} \left| \tilde{f} (\lambda) \right|^2 \lambda^{2a+1} d\lambda.
\end{equation}
(23)

Using integration by parts, we obtain
\begin{align*}
I_1 &\leq -C_1 h^2 \int_0^{1/h} s^2 \phi' (s) ds \\
&\leq -C_1 \phi \left( \frac{1}{h} \right) + 2C_1 h^2 \int_0^{1/h} s \phi (s) ds \\
&\leq C_2 h^2 \int_0^{1/h} s \phi (s) ds \\
&\leq C_2 h^2 \int_0^{1/h} s s^{-2k} (\log s)^{-2\gamma} ds \\
&\leq C_2 h^{2k} (\log (1/h))^{-2\gamma},
\end{align*}
(24)

where \(C_1, C_2, \) and \(C_2\) are positive constants and this ends the proof. \(\Box\)

References


