Research Article

Pointwise Analog of the Stečkin Approximation Theorem

Włodzimierz Łenski

*Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra,
Ulica Szafrana 4a, 65-516 Zielona Góra, Poland*

Correspondence should be addressed to Włodzimierz Łenski; w.lenski@wmie.uz.zgora.pl

Received 11 November 2012; Accepted 20 January 2013

Academic Editor: Roberto Renò

Copyright © 2013 Włodzimierz Łenski. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We show the pointwise version of the Stečkin theorem on approximation by de la Vallée-Poussin means. The result on norm approximation is also derived.

1. Introduction

Let L^p ($1 \leq p < \infty$) $[C]$ be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p th power (continuous) over $Q = [-\pi, \pi]$, and let $X^p = L^p$ when $1 \leq p < \infty$ or $X^p = C$ when $p = \infty$.

Let us define the norms of $f \in X^p$ as

$$\begin{aligned} \|f\| &= \|f\|_{X^p} = \|f(\cdot)\|_{X^p} \\ &:= \begin{cases} \left\{ \int_Q |f(x)|^p dx \right\}^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup_{x \in Q} |f(x)| & \text{when } p = \infty, \end{cases} \\ \|f\|_{x,\delta} &= \|f\|_{X^p,x,\delta} = \|f(\cdot)\|_{X^p,x,\delta} := \sup_{0 < h \leq \delta} \|f(\cdot)\|_{X^p,x,h}^\circ \\ &:= \begin{cases} \sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |f(x+t)| \right\} & \text{when } p = \infty, \end{cases} \end{aligned} \quad (1)$$

where

$$\begin{aligned} \|f\|_{x,\delta}^\circ &= \|f\|_{X^p,x,\delta}^\circ = \|f(\cdot)\|_{X^p,x,\delta}^\circ \end{aligned}$$

$$:= \begin{cases} \left\{ \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\ \sup_{0 < |h| \leq \delta} |f(x+h)| & \text{when } p = \infty \end{cases} \quad (\delta > 0). \quad (2)$$

We note additionally that

$$\|f\|_{C,x,0} = \|f\|_{C,x,0}^\circ = |f(x)|. \quad (3)$$

Consider the trigonometric Fourier series of f

$$Sf(x) = \frac{a_0(f)}{2} + \sum_{k=0}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \quad (4)$$

with the partial sums $S_k f$.

Let

$$\sigma_{n,m} f(x) := \frac{1}{m+1} \sum_{k=n-m}^n S_k f(x) \quad (m \leq n = 0, 1, 2, \dots). \quad (5)$$

As a measure of approximation by the previous quantities, we use the pointwise characteristics

$$\begin{aligned} w_x f(\delta) &= w_x f(\delta)_{X^p} := \|\Delta_x f(\cdot)\|_{X^p,x,\delta} \end{aligned}$$

$$= \begin{cases} \sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{-h}^h |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |\Delta_x f(t)| \right\} & \text{when } p = \infty, \end{cases} \quad (6)$$

compared to [1], and

$$\begin{aligned} w_x^\circ f(\delta) &= w_x^\circ f(\delta)_{X^p} := \|\Delta_x f(\cdot)\|_{X^p, x, \delta}^\circ \\ &= \begin{cases} \left\{ \frac{1}{2\delta} \int_{-\delta}^{\delta} |\Delta_x f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup_{0 < |t| \leq \delta} |\Delta_x f(t)| & \text{when } p = \infty, \end{cases} \end{aligned} \quad (7)$$

and also

$$\begin{aligned} \Omega_x f\left(\frac{\pi}{n+1}\right) &= \Omega_x f\left(\frac{\pi}{n+1}\right)_{X^p} := \frac{1}{n+1} \sum_{k=0}^n w_x f\left(\frac{\pi}{k+1}\right)_{X^p}, \\ \Omega_x^\circ f\left(\frac{\pi}{n+1}\right) &= \Omega_x^\circ f\left(\frac{\pi}{n+1}\right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n w_x^\circ f\left(\frac{\pi}{k+1}\right)_{X^p}, \end{aligned} \quad (8)$$

where

$$\Delta_x f(t) := f(x+t) - f(x), \quad (9)$$

constructed on the base of definition of X^p -points (Lebesgue points (L^p -points) or points of continuity (C-points)). We also use the modulus of continuity of f in the space X^p defined by the formula

$$\omega f(\delta) = \omega f(\delta)_{X^p} := \sup_{0 < |h| \leq \delta} \|\Delta_x f(h)\|_{X^p} \quad (10)$$

and its arithmetic mean

$$\Omega f\left(\frac{\pi}{n+1}\right) = \Omega f\left(\frac{\pi}{n+1}\right)_{X^p} = \frac{1}{n+1} \sum_{k=0}^n \omega f\left(\frac{\pi}{k+1}\right)_{X^p}. \quad (11)$$

We can observe that, for $f \in X^{\tilde{p}}$ and $\tilde{p} \geq p$,

$$\|w_\cdot f(\delta)_{X^p}\|_C \leq \omega f(\delta)_C, \quad (12)$$

whence

$$\begin{aligned} \|\Omega_\cdot f(\delta)_{X^p}\|_C &\leq \Omega f(\delta)_C, \\ \|w_\cdot^\circ f(\delta)_{X^p}\|_{X^p} &\leq \omega f(\delta)_{X^p}, \end{aligned} \quad (13)$$

whence

$$\|\Omega^\circ f(\delta)_{X^p}\|_{X^p} \leq \Omega f(\delta)_{X^p}. \quad (14)$$

Let us introduce one more measure of pointwise approximation analogous to the best approximation of function f

by trigonometric polynomials T of the degree at most n ($T \in H_n$)

$$E_n(f)_{X^p} := \inf_{T \in H_n} \{\|f(\cdot) - T(\cdot)\|_{X^p}\}; \quad (15)$$

Namely,

$$\begin{aligned} E_n(f, x; \delta) &= E_n(f, x; \delta)_{X^p} := \inf_{T \in H_n} \{\|f(\cdot) - T(\cdot)\|_{X^p, x, \delta}\} \\ &= \begin{cases} \inf_{T \in H_n} \left\{ \sup_{0 < h \leq \delta} \left[\frac{1}{2h} \int_{-h}^h |f(x+t) - T(x+t)|^p dt \right]^{1/p} \right\} & \text{when } 1 \leq p < \infty, \\ \inf_{T \in H_n} \left\{ \sup_{0 < |h| \leq \delta} |f(x+h) - T(x+h)| \right\} & \text{when } p = \infty, \end{cases} \\ E_n^\circ(f, x; \delta) &= E_n^\circ(f, x; \delta)_{X^p} := \inf_{T \in H_n} \{\|f(\cdot) - T(\cdot)\|_{X^p, x, \delta}^\circ\}. \end{aligned} \quad (16)$$

We will also use its arithmetic mean

$$\begin{aligned} F_{n,m}(f, x) &= F_{n,m}(f, x)_{X^p} := \frac{1}{m+1} \sum_{k=0}^m E_n\left(f, x; \frac{\pi}{k+1}\right)_{X^p}, \\ F_{n,m}^\circ(f, x) &= F_{n,m}^\circ(f, x)_{X^p} := \frac{1}{m+1} \sum_{k=0}^m E_n^\circ\left(f, x; \frac{\pi}{k+1}\right)_{X^p}. \end{aligned} \quad (17)$$

Using these characteristics, we will show the pointwise version of the Stečkin [2] generalization of the Fejér-Lebesgue theorem. As a corollary, we will obtain the mentioned original result of Stečkin on norm approximation as well as the result of Tanović-Miller [3].

By K we will designate either an absolute constant or a constant depending on some parameters, not necessarily the same of each occurrence.

2. Statement of the Results

At the beginning, we formulate the partial solution of the considered problem.

Theorem 1 (see [4]). *If $f \in X^p$, then, for any positive integer $m \leq n$ and all real x ,*

$$\begin{aligned}
|\sigma_{n,m}f(x) - f(x)| &\leq \pi^2 E_{n-m}^\circ\left(f, x, \frac{\pi}{2n-m+1}\right)_{X^p} \\
&\quad + 6F_{n-m,m}^\circ(f, x)_{X^p} \\
&\quad + \int_{\pi/(2n-m+1)}^{\pi/(m+1)} \frac{E_{n-m}^\circ(f, x, t)_{X^p}}{t} dt \\
&\quad + E_{n-m}^\circ(f, x; 0)_{X^p}, \tag{18} \\
|\sigma_{n,m}f(x) - f(x)| &\leq (6 + \pi^2) F_{n-m,m}(f, x)_{X^p} \\
&\quad \times \left[1 + \ln \frac{n+1}{m+1}\right] \\
&\quad + E_{n-m}(f, x; 0)_{X^p}.
\end{aligned}$$

Now, we can present the main result on pointwise approximation.

Theorem 2. *If $f \in X^p$, then, for any positive integer $m \leq n$ and all real x ,*

$$\begin{aligned}
|\sigma_{n,m}f(x) - f(x)| &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p} + F_{n-m+v,v}(f, x)_{X^p}}{m+v+1} \\
&\quad + E_{2n}(f, x; 0)_{X^p}. \tag{19}
\end{aligned}$$

Remark 3. Theorem 2, in case $f \in C$, immediately yields the result of Stečkin [2]. The result of Stečkin type also holds if instead of C we consider the spaces X^p with $1 < p < \infty$. Then, in the proof, we need the Hardy-Littlewood estimate of the maximal function.

At every X^p -point x of f

$$\Omega_x f(\gamma)_{X^p} = o_x(1) \quad \text{as } \gamma \rightarrow 0+ \tag{20}$$

and thus from Theorem 1, we obtain the corollary which states the result of the Tanović-Miller type [3].

Corollary 4. *If $f \in X^p$, then, for any positive integer $m \leq n$ at every X^p -point x of f ,*

$$|\sigma_{n,m}f(x) - f(x)| = o_x(1) \left[1 + \ln \frac{n+1}{m+1}\right] \quad \text{as } n \rightarrow \infty. \tag{21}$$

3. Auxiliary Results

In order to prove our theorems, we require some lemmas.

Lemma 5. *If T_n is the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p}$, then it is also the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p, x, \delta}$ for any $\delta \in [0, \pi]$.*

Proof. From the inequalities

$$\begin{aligned}
\|E_n(f, \cdot, \delta)_{X^p}\|_{X^p} &\geq \|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} \\
&= \|\|f - T_{n,\delta}\|_{X^p, \cdot, \delta}\|_{X^p} = \|f - T_{n,\delta}\|_{X^p} \\
&\geq \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \\
\|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} &\leq \|\|f - T_n\|_{X^p, \cdot, \delta}\|_{X^p} \\
&= \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \tag{22}
\end{aligned}$$

where $T_{n,\delta}$ and T_n are the trigonometric polynomials of the degree at most n of the best approximation of $f \in X^p$ with respect to the norms $\|\cdot\|_{X^p, x, \delta}$ and $\|\cdot\|_{X^p}$, respectively, we obtain relation

$$\|f - T_{n,\delta}\|_{X^p} = \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \tag{23}$$

whence $T_{n,\delta} = T_n$ for any $\delta \in [0, \pi]$ by uniqueness of the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p}$ (e.g., see [5] p. 96). We can also observe that for such T_n and any $h \in [0, \delta]$,

$$\begin{aligned}
\|f - T_n\|_{X^p, x, h}^\circ &= E_n^\circ(f, x, h)_{X^p} \leq E_n(f, x, \delta)_{X^p} \\
&\leq \|f - T_n\|_{X^p, x, \delta}. \tag{24}
\end{aligned}$$

Hence,

$$E_n(f, x, \delta)_{X^p} = \|f - T_n\|_{X^p, x, \delta} \tag{25}$$

and our proof is complete. $\square$

Lemma 6. *If $n \in \mathbb{N}_0$ and $\delta > 0$, then $E_n(f, x; \delta)_{X^p}$ is nonincreasing function of n and nondecreasing function of δ . These imply that for $m, n \in \mathbb{N}$ the function $F_{n,m}(f, x)_{X^p}$ is nonincreasing function of n and m simultaneously.*

Proof. The first part of our statement follows from the property of the norm $\|\cdot\|_{x, \delta}$ and supremum. The second part is a consequence of the calculation

$$\begin{aligned}
\frac{F_{n,m+1}(f, x)_{X^p}}{F_{n,m}(f, x)_{X^p}} &= \frac{m+1}{m+2} \left(1 + \frac{E_n(f, x; \pi/(m+2))_{X^p}}{\sum_{k=0}^m E_n(f, x; \pi/(k+1))_{X^p}}\right) \\
&\leq \frac{m+1}{m+2} \left(1 + \frac{E_n(f, x; \pi/(m+1))_{X^p}}{\sum_{k=0}^m E_n(f, x; \pi/(m+1))_{X^p}}\right) \\
&= \frac{m+1}{m+2} \left(1 + \frac{1}{m+1}\right) = 1. \tag{26}
\end{aligned}$$

$\square$

Lemma 7. *Let $m, n, q \in \mathbb{N}_0$ such that $m \leq n$ and $q \geq m+1$. If $f \in X^p$, then*

$$|\sigma_{n+q,m}f(x) - \sigma_{n,m}f(x)| \leq K F_{n-m,m}(f, x)_{X^p} \sum_{v=0}^{q-1} \frac{1}{m+v+1}. \tag{27}$$

Proof. It is clear that

$$\begin{aligned}\sigma_{n,m}f(x) &= \frac{1}{m+1} \sum_{k=n-m}^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_{n,m}(t) dt,\end{aligned}\quad (28)$$

where

$$\begin{aligned}V_{n,m}(t) &= \frac{1}{m+1} \sum_{k=n-m}^n D_k(t), \\ D_k(t) &= \frac{\sin((2k+1)t/2)}{2 \sin(t/2)}.\end{aligned}\quad (29)$$

Hence, by orthogonality of the trigonometric system,

$$\begin{aligned}\sigma_{n+q,m}f(x) - \sigma_{n,m}f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \\ &\quad \times (V_{n+q,m}(t) - V_{n,m}(t)) dt \\ &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \\ &\quad \times (D_{k+q}(t) - D_k(t)) dt \\ &= \frac{1}{\pi(m+1)} \\ &\quad \times \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \\ &\quad \times \frac{\sin((2k+2q+1)t/2) - \sin((2k+1)t/2)}{2 \sin(t/2)} dt \\ &= \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{-\pi}^{\pi} [f(x+t) - T_{n-m}(x+t)] \\ &\quad \times \frac{\sin(qt/2) \cos((2k+q+1)t/2)}{\sin(t/2)} dt,\end{aligned}\quad (30)$$

with trigonometric polynomial T_{n-m} of the degree at most $n-m$ of the best approximation of f .

Using the notations

$$\begin{aligned}I_1 &= \left[-\frac{\pi}{q}, \frac{\pi}{q}\right], & I_2 &= \left[-\frac{\pi}{m+1}, -\frac{\pi}{q}\right] \cup \left[\frac{\pi}{q}, \frac{\pi}{m+1}\right], \\ I_3 &= \left[-\pi, -\frac{\pi}{m+1}\right] \cup \left[\frac{\pi}{m+1}, \pi\right],\end{aligned}\quad (31)$$

we get

$$\begin{aligned}\Sigma &= \frac{1}{\pi(m+1)} \\ &\quad \times \sum_{k=n-m}^n \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) \\ &\quad \times [f(x+t) - T_{n-m}(x+t)] \\ &\quad \times \frac{\sin(qt/2) \cos((2k+q+1)t/2)}{\sin(t/2)} dt \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \\ \Sigma_1 &\leq \frac{1}{\pi(m+1)} \sum_{k=n-m}^n \int_{I_1} |f(x+t) - T_{n-m}(x+t)| q dt \\ &= \frac{q}{\pi} \int_{I_1} |f(x+t) - T_{n-m}(x+t)| dt \\ &\leq 2E_{n-m}\left(f, x; \frac{\pi}{q}\right)_{X^p}.\end{aligned}\quad (32)$$

We next evaluate the sums Σ_2 and Σ_3 using the partial integrating and Lemma 5. Thus,

$$\begin{aligned}\Sigma_2 &\leq \int_{I_2} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} dt \\ &= 2 \left[\frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right]_{t=\pi/q}^{t=\pi/(m+1)} \\ &\quad + 2 \int_{\pi/q}^{\pi/(m+1)} \frac{1}{t} \left[\frac{1}{2t} \int_{-t}^t |f(x+u) - T_{n-m}(x+u)| du \right] dt \\ &\leq 2E_{n-m}\left(f, x; \frac{\pi}{m+1}\right)_{X^p} + 2 \int_{-\pi}^{\pi} \frac{1}{t} E_{n-m}(f, x; t)_{X^p} dt \\ &\leq 4E_{n-m}\left(f, x; \frac{\pi}{m+1}\right)_{X^p} \left[1 + \ln \frac{q}{m+1} \right] \\ &\leq 4E_{n-m}\left(f, x; \frac{\pi}{m+1}\right)_{X^p} \left[1 + \sum_{v=0}^{q-1} \frac{1}{m+v+1} \right], \\ \Sigma_3 &\leq \frac{1}{m+1} \\ &\quad \times \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} dt \\ &\quad \times \left| \sum_{k=n-m}^n \cos\left(kt + \frac{q+1}{2}t\right) \right| dt\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{m+1} \\
&\quad \times \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t} \\
&\quad \times \left| \frac{2 \sin((n+1)t/2) \cos((2n-m+q+1)t/2)}{2 \sin(t/2)} \right| dt \\
&\leq \frac{\pi}{m+1} \int_{I_3} \frac{|f(x+t) - T_{n-m}(x+t)|}{t^2} dt \\
&= \frac{\pi}{m+1} \left\{ 2 \left[\frac{1}{2t} \int_{-t}^t |f(x+u) \right. \right. \\
&\quad \left. \left. - T_{n-m}(x+u) \right| du \right]_{t=\pi/(m+1)}^{t=\pi} \\
&\quad + 4 \int_{\pi/(m+1)}^{\pi} \frac{1}{t^2} \left[\frac{1}{2t} \int_{-t}^t |f(x+u) \right. \\
&\quad \left. - T_{n-m}(x+u) \right| du \right] dt \Big\} \\
&\leq \frac{\pi}{m+1} \left\{ 2E_{n-m}(f, x; \pi)_{X^p} \right. \\
&\quad \left. + 4 \int_{\pi/(m+1)}^{\pi} \frac{1}{t^2} E_{n-m}(f, x; t)_{X^p} dt \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} \right. \\
&\quad \left. + 2 \int_1^{m+1} \frac{E_{n-m}(f, x; \pi/u)_{X^p}}{\pi^2/u^2} \frac{\pi du}{u^2} \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} \right. \\
&\quad \left. + \frac{2}{\pi} \sum_{k=0}^{m-1} \int_{k+1}^{k+2} E_{n-m}\left(f, x; \frac{\pi}{u}\right)_{X^p} du \right\} \\
&= \frac{2\pi}{m+1} \left\{ E_{n-m}(f, x; \pi)_{X^p} \right. \\
&\quad \left. + \frac{2}{\pi} \sum_{k=0}^{m-1} E_{n-m}\left(f, x; \frac{\pi}{k+1}\right)_{X^p} \right\} \\
&\leq \frac{2\pi + 2/\pi}{m+1} \sum_{k=0}^{m-1} E_{n-m}\left(f, x; \frac{\pi}{k+1}\right)_{X^p},
\end{aligned} \tag{33}$$

which proves Lemma 7. $\square$

Before formulating the next lemmas, we define a new difference. Let $m, n \in \mathbb{N}_0$ and $m \leq n$. Denote that

$$\tau_{n,m}f(x) := (m+1) \{ \sigma_{n+m+1,m}f(x) - \sigma_{n,m}f(x) \}. \tag{34}$$

Lemma 8. Let $m, n, \mu \in \mathbb{N}_0$ such that $2\mu \leq m \leq n$. If $f \in X^p$, then

$$|\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \ln \frac{m}{\mu}. \tag{35}$$

Proof. The proof follows by the method of Leindler [6]. Namely,

$$\begin{aligned}
&\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x) \\
&= \left(\sum_{k=n+m-2\mu+2}^{n+m+1} - 2 \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)], \\
&|\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \\
&\leq \left| \left(\sum_{k=n+m-2\mu+2}^{n+m+1} - \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)] \right| \\
&\quad + \left| \left(\sum_{k=n+m-\mu+2}^{n+m+1} - \sum_{k=n-\mu+1}^n \right) [S_k f(x) - f(x)] \right| \\
&= \mu |\sigma_{n+m-\mu+1,\mu-1}f(x) - \sigma_{n,\mu-1}f(x)| \\
&\quad + \mu |\sigma_{n+m+1,\mu-1}f(x) - \sigma_{n,\mu-1}f(x)|.
\end{aligned} \tag{36}$$

By Lemma 6, for $2\mu \leq m$,

$$\begin{aligned}
&|\tau_{n,m}f(x) - \tau_{n-\mu,m-\mu}f(x)| \\
&\leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[1 + \ln \frac{(n-\mu+1) + \mu - 1}{\mu} \right] \\
&\quad + K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[1 + \ln \frac{m + \mu - 1}{\mu} \right] \\
&\leq K\mu F_{n-\mu+1,\mu-1}(f, x)_{X^p} \left[1 + \ln \frac{m}{\mu} \right],
\end{aligned} \tag{37}$$

and our proof is complete. $\square$

Lemma 9. Let $m, n \in \mathbb{N}_0$ and $m \leq n$. If $f \in X^p$, then

$$|\tau_{n,m}f(x)| \leq K \sum_{k=n-m}^n F_{k,k-n+m}(f, x)_{X^p}. \tag{38}$$

Proof. Our proof runs parallel with the proof of Theorem 1 in [2].

If $m = 0$, then

$$|\tau_{n,0}f(x)| = |\sigma_{n+1,0}f(x) - \sigma_{n,0}f(x)| \leq KF_{n,0}(f, x)_{X^p}, \tag{39}$$

and if $m = 1$, then

$$\begin{aligned}
|\tau_{n,1}f(x)| &\leq 2 |\sigma_{n+2,1}f(x) - \sigma_{n,1}f(x)| \\
&\leq KF_{n-1,1}(f, x)_{X^p}
\end{aligned}$$

$$\begin{aligned}
&\leq K [F_{n-1,1}(f, x)_{X^p} + F_{n-1,1}(f, x)_{X^p}] \\
&\leq K [F_{n-1,1}(f, x)_{X^p} + F_{n,1}(f, x)_{X^p}]
\end{aligned} \tag{40}$$

by Lemmas 6 and 7.

Next, we construct the same decreasing sequence (m_s) of integers that was given by Stečkin. Let

$$m_0 = m, \quad m_s = m_{s-1} - \left\lfloor \frac{m_{s-1}}{2} \right\rfloor \quad (s = 1, 2, \dots), \tag{41}$$

where $[y]$ denotes the integral part of y . It is clear that there exists smallest index $t \geq 1$ such that $m_t = 1$ and

$$m = m_0 > m_1 > \dots > m_t = 1. \tag{42}$$

By the definition of the numbers m_s , we have

$$\begin{aligned}
m_s &\geq \frac{m_{s-1}}{2}, \\
m_{s-1} - m_s &= \left\lfloor \frac{m_{s-1}}{2} \right\rfloor \geq \left\lfloor \frac{m_{s-1}}{3} \right\rfloor \quad (s = 1, 2, \dots, t)
\end{aligned} \tag{43}$$

whence

$$\begin{aligned}
m_{t-1} &= 2, \quad m_{t-1} - m_t = 1, \\
m_{s-1} - m_s &\leq m_s \leq 3(m_s - m_{s+1}) \quad (s = 1, 2, \dots, t-1)
\end{aligned} \tag{44}$$

follow.

Under these notations, we get the following equality:

$$\begin{aligned}
\tau_{n,m} f(x) &= \sum_{s=1}^t (\tau_{n-m+m_{s-1}, m_{s-1}} f(x) - \tau_{n-m+m_s, m_s} f(x)) \\
&\quad + \tau_{n-m+m_t, m_t} f(x),
\end{aligned} \tag{45}$$

whence, by $m_t = 1$,

$$\begin{aligned}
|\tau_{n,m} f(x)| &\leq \sum_{s=1}^t |\tau_{n-m+m_{s-1}, m_{s-1}} f(x) - \tau_{n-m+m_s, m_s} f(x)| \\
&\quad + |\tau_{n-m+m_t, m_t} f(x)|
\end{aligned} \tag{46}$$

follows.

It is easy to see that the terms in the sum $\sum_{s=1}^t$, by Lemma 8, with $\mu = m_{s-1} - m_s$ and $m = m_{s-1}$ do not exceed

$$\begin{aligned}
K(m_{s-1} - m_s) F_{n-m+m_s+1, m_{s-1}-m_s}(f, x)_{X^p} \ln \frac{m_{s-1}}{m_{s-1} - m_s}, \\
\text{where } (s = 1, 2, \dots, t-1),
\end{aligned} \tag{47}$$

and by Lemma 7, we get

$$\begin{aligned}
|\tau_{n-m+1,1} f(x)| &\leq 2 |\sigma_{n-m+2,1} f(x) - \sigma_{n-m+1,1} f(x)| \\
&\leq K F_{n-m,1}(f, x)_{X^p}.
\end{aligned} \tag{48}$$

Thus,

$$\begin{aligned}
|\tau_{n,m} f(x)| &\leq K \sum_{s=1}^{t-1} 3(m_s - m_{s+1}) F_{n-m+m_s+1, m_s}(f, x)_{X^p} \ln 3 \\
&\quad + K F_{n-m+2, m-2}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p},
\end{aligned} \tag{49}$$

whence, by the monotonicity of $F_{v,\mu}(f, x)_{X^p}$,

$$\begin{aligned}
|\tau_{n,m} f(x)| &\leq K \left(\sum_{s=1}^{t-1} \sum_{v=m_{s+1}+1}^{m_s} F_{n-m+v+1, v}(f, x)_{X^p} \right. \\
&\quad \left. + \sum_{v=0}^2 F_{n-m+v, m-v-1}(f, x)_{X^p} \right) \\
&\quad + K F_{n-m,1}(f, x)_{X^p} \\
&\leq K \sum_{v=0}^{m_1+1} F_{n-m+v, v}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} \\
&\leq K \sum_{v=0}^m F_{n-m+v, v}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p} \\
&\leq K \sum_{k=n-m}^n F_{k, k-n+m}(f, x)_{X^p} + K F_{n-m,1}(f, x)_{X^p}.
\end{aligned} \tag{50}$$

□

4. Proofs of the Results

Proof of Theorem 2. The proof follows the lines of the proofs of Theorem 4 in [2] and Theorem in [6]. Therefore, let $n > 0$ and $m \leq n$ be fixed. Let us define an increasing sequence $(n_s : s = 0, 1, \dots, t)$ of indices introduced by Stečkin in the following way. Set $n_0 = n$. Assuming that the numbers $n_0, \dots, n_s$ are already defined and $n_s < 2n$, we define n_{s+1} as follows. Let v_s denote the smallest natural number such that

$$F_{n_s-m+v_s, v_s}(f, x)_{X^p} \leq \frac{1}{2} F_{n_s-m, v_s}(f, x)_{X^p} \quad (v = 0, 1, \dots, n). \tag{51}$$

According to the magnitude of v_s , we define

$$n_{s+1} = \begin{cases} n_s - m + 1 & \text{for } v_s \leq m, \\ n_s + v_s & \text{for } m + 1 \leq v_s < 2n + m - n_s, \\ 2n + m & \text{for } v_s \geq 2n + m - n_s. \end{cases} \tag{52}$$

If $n_{s+1} < 2n$, we continue the procedure, and if once $n_{s+1} \geq 2n$, then we stop the construction and define $t := s + 1$.

By the previous definition of (n_s) , we have the following obvious properties:

$$\begin{aligned}
t &\geq 1, \quad n = n_0 < n_1 < \dots < n_t, \quad 2n \leq n_t \leq 2n + m, \\
n_{s+1} - n_s &\geq m + 1 \quad (s = 0, 1, \dots, t-1),
\end{aligned} \tag{53}$$

and relations

$$\begin{aligned}
 F_{n_{s+1}-m,v}(f, x)_{X^p} &\leq \frac{1}{2} F_{n_s-m,v}(f, x)_{X^p} \\
 &\text{for } s = 0, 1, \dots, t-2, \\
 \frac{1}{2} F_{n_s-m,v}(f, x)_{X^p} &\leq F_{n_{s+1}-m-1,v}(f, x)_{X^p} \\
 &\text{for } s = 0, 1, \dots, t-1,
 \end{aligned} \tag{54}$$

whenever $n_{s+1} - n_s > m + 1$.

Let us start with

$$\begin{aligned}
 &|\sigma_{n,m} f(x) - f(x)| \\
 &= \sum_{s=0}^{t-1} \left[|\sigma_{n_s,m} f(x) - f(x)| - |\sigma_{n_{s+1},m} f(x) - f(x)| \right] \\
 &\quad + |\sigma_{n_t,m} f(x) - f(x)| \\
 &\leq \sum_{s=0}^{t-1} |\sigma_{n_{s+1},m} f(x) - \sigma_{n_s,m} f(x)| + |\sigma_{n_t,m} f(x) - f(x)| \\
 &= \sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| + |\sigma_{n_t,m} f(x) - f(x)|.
 \end{aligned} \tag{55}$$

Using Theorem 1 and that $2n \leq n_t \leq 2n + m$, we get

$$\begin{aligned}
 &|\sigma_{n_t,m} f(x) - f(x)| \\
 &\leq K F_{n_t-m,m}(f, x)_{X^p} \left[1 + \ln \frac{n_t + 1}{m + 1} \right] \\
 &\quad + |f(x) - T_{n_t-m}(x)| \\
 &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p}}{m + v + 1} + |f(x) - T_{n_t-m}(x)| \\
 &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p} + F_{n-m+v,v}(f, x)_{X^p}}{m + v + 1} \\
 &\quad + |f(x) - T_{n_t-m}(x)| \\
 &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p} + F_{n-m+v,v}(f, x)_{X^p}}{m + v + 1} \\
 &\quad + E_{n_t-m}(f, x; 0)_{X^p} \\
 &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p} + F_{n-m+v,v}(f, x)_{X^p}}{m + v + 1} \\
 &\quad + E_{2n}(f, x; 0)_{X^p}.
 \end{aligned} \tag{56}$$

The estimate of the sum in the right hand side of (55) we derive from the following one

$$\begin{aligned}
 &\left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \\
 &\leq K \sum_{v=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+v,m}(f, x)_{X^p} + F_{n_s-m+v,v}(f, x)_{X^p}}{m + v + 1}.
 \end{aligned} \tag{57}$$

We split the proof of this inequality in two parts. If $n_{s+1} - n_s = m + 1$, then by Lemma 9,

$$\begin{aligned}
 &\left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \leq K \frac{1}{m+1} \sum_{k=n_s-m}^{n_s} F_{k,k-n_s+m}(f, x)_{X^p} \\
 &\leq K \sum_{v=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+v,v}(f, x)_{X^p}}{m + v + 1}.
 \end{aligned} \tag{58}$$

If $n_{s+1} - n_s > m + 1$, then, by Lemma 7,

$$\begin{aligned}
 &\left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \\
 &\leq K F_{n_s-m,m}(f, x)_{X^p} \sum_{v=0}^{n_{s+1}-n_s-1} \frac{1}{m + v + 1},
 \end{aligned} \tag{59}$$

and since $(1/2)F_{n_s-m,m}(f, x)_{X^p} \leq F_{n_{s+1}-m-1,m}(f, x)_{X^p}$, we have

$$\begin{aligned}
 &\left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \\
 &\leq 2K F_{n_{s+1}-m-1,m}(f, x)_{X^p} \sum_{v=0}^{n_{s+1}-n_s-1} \frac{1}{m + v + 1} \\
 &\leq 2K \sum_{v=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+v,m}(f, x)_{X^p}}{m + v + 1}.
 \end{aligned} \tag{60}$$

Consequently,

$$\begin{aligned}
 &\sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \\
 &\leq 2K \sum_{s=0}^{t-1} \sum_{v=0}^{n_{s+1}-n_s-1} \frac{F_{n_s-m+v,m}(f, x)_{X^p} + F_{n_s-m+v,v}(f, x)_{X^p}}{m + v + 1}.
 \end{aligned} \tag{61}$$

Since $n_{s+1} - n_s \leq 2n + m - n - 1 = n + m - 1$ for all $s \leq t - 1$, changing the order of summation, we get

$$\begin{aligned}
 &\sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s,m} f(x) \right| \\
 &\leq 2K \sum_{v=0}^{n+m-1} \frac{1}{m + v + 1} \\
 &\quad \times \sum_{s: n_{s+1}-n_s > v} \left[F_{n_s-m+v,m}(f, x)_{X^p} + F_{n_s-m+v,v}(f, x)_{X^p} \right].
 \end{aligned} \tag{62}$$

Using the inequality

$$F_{n_{s+1}-m, \nu}(f, x)_{X^p} \leq \frac{1}{2} F_{n_s-m, \nu}(f, x)_{X^p} \quad (63)$$

$$\text{for } \begin{cases} \nu = 0, 1, 2, \dots, n_{s+1} - n_s - 1, \\ s = 0, 1, 2, \dots, t-2, \end{cases}$$

we obtain

$$\begin{aligned} & \sum_{s: n_{s+1} - n_s > \nu} [F_{n_s-m+\nu, m}(f, x)_{X^p} + F_{n_s-m+\nu, \nu}(f, x)_{X^p}] \\ &= F_{n_p-m+\nu, m}(f, x)_{X^p} + F_{n_p-m+\nu, \nu}(f, x)_{X^p} \\ &+ \sum_{s \geq p+1: n_{s+1} - n_s > \nu} [F_{n_{s+1}-m+\nu, m}(f, x)_{X^p} \\ &\quad + F_{n_{s+1}-m+\nu, \nu}(f, x)_{X^p}] \\ &\leq F_{n_p-m+\nu, m}(f, x)_{X^p} + F_{n_p-m+\nu, \nu}(f, x)_{X^p} \quad (64) \\ &+ \sum_{s: s \geq p+1} F_{n_s-m, m}(f, x)_{X^p} [F_{n_s-m, m}(f, x)_{X^p} \\ &\quad + F_{n_s-m, \nu}(f, x)_{X^p}] \\ &\leq F_{n_p-m+\nu, m}(f, x)_{X^p} + F_{n_p-m+\nu, \nu}(f, x)_{X^p} \\ &\quad + 2 [F_{n_{p+1}-m, m}(f, x)_{X^p} + F_{n_{p+1}-m, \nu}(f, x)_{X^p}] \\ &\leq 3 [F_{n_p-m+\nu, m}(f, x)_{X^p} + F_{n_p-m+\nu, \nu}(f, x)_{X^p}], \end{aligned}$$

where p denotes the smallest index s having the property $n_{s+1} - n_s > \nu$. Hence,

$$\begin{aligned} & \sum_{s=0}^{t-1} \left| \frac{1}{m+1} \tau_{n_s, m} f(x) \right| \\ &\leq K \sum_{\nu=0}^{n+m-1} \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1} \quad (65) \\ &\leq K \sum_{\nu=0}^n \frac{F_{n-m+\nu, m}(f, x)_{X^p} + F_{n-m+\nu, \nu}(f, x)_{X^p}}{m+\nu+1}, \end{aligned}$$

and our proof follows. $\square$

References

- [1] S. Aljančić, R. Bojanić, and M. Tomić, "On the degree of convergence of Fejér-Lebesgue sums," *L'Enseignement Mathématique. Revue Internationale. IIe Série*, vol. 15, pp. 21–28, 1969.
- [2] S. B. Stečkin, "On the approximation of periodic functions by de la Vallée Poussin sums," *Analysis Mathematica*, vol. 4, no. 1, pp. 61–74, 1978.
- [3] N. Tanović-Miller, "On some generalizations of the Fejér-Lebesgue theorem," *Unione Matematica Italiana. Bollettino. B. Serie 6*, vol. 1, no. 3, pp. 1217–1233, 1982.
- [4] W. Łenski, "Pointwise approximation by de la Vallée-Poussin means," *East Journal on Approximations*, vol. 14, no. 2, pp. 131–136, 2008.
- [5] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Academic Press, New York, NY, USA, 1971.
- [6] L. Leindler, "Sharpening of Stečkin's theorem to strong approximation," *Analysis Mathematica*, vol. 16, no. 1, pp. 27–38, 1990.