Research Article

Fixed Point Theorems for a Class of Weakly C-Contractive Mappings in a Setting of 2-Banach Space

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Received 26 September 2012; Accepted 1 December 2012

Academic Editor: Baoding Liu

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In this paper, the authors have proved some existence theorems of fixed points for a class of weakly C-contractive mappings in a setting of 2-Banach space. The authors have supported the results with the help of suitable examples.

1. Introduction

The theory of 2-Banach spaces was investigated by Gähler in [1] and Iseki in [2] who proved some fixed point theorems in such spaces and these new spaces have subsequently been studied by many authors. For example, we refer to [3, 4] where authors have dealt with mappings that are of contractive nature. Our findings as presented here have added further findings in fixed point theory in a 2-Banach space by working with a class of mappings so general that take care of those as found in [5, 6].

We now state some definitions before presenting our main results.

2. Preliminaries

Definition 1. Let X be a real linear space and $\|\cdot, \cdot\|$ a non-negative real valued function defined on X satisfying the following conditions:

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
- (iii) $\|x, \alpha y\| = |\alpha| \|x, y\|$, α being real for all $x, y \in X$,
- (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

Then $\|\cdot, \cdot\|$ is called a 2-norm and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space. So a 2-norm $\|x, y\|$ always satisfies $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and all scalars α .

Definition 2. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is said to be a Cauchy sequence if $\lim_{n,m \rightarrow \infty} \|x_n - x_m, a\| = 0$ for all a in X .

Definition 3. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is said to be convergent in X if there exists a point x in X such that $\lim_{n \rightarrow \infty} \|x_n - x, a\| = 0$ for all a in X . If $\{x_n\}$ converges to x , we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 4. A linear 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X . A complete linear 2-normed space X is called a 2-Banach space.

The main purpose of our paper is to find out the existence of fixed point for a class of weakly C-contractive results due to Choudhury [7] in a setting of 2-Banach space. We have also established a common fixed point theorem for such type of mappings with a supporting example. Some allied theorems have also been presented here.

3. Main Results

Definition 5. A mapping $T : (X, \|\cdot, \cdot\|) \rightarrow (X, \|\cdot, \cdot\|)$ is said to be weakly C-contractive or a weak C-contraction if for all x, y, a in X ,

$$\begin{aligned} \|T(x) - T(y), a\| &\leq \frac{1}{2} [\|x - T(y), a\| + \|y - T(x), a\|] \\ &\quad - \phi[\|x - T(y), a\|, \|y - T(x), a\|], \end{aligned} \quad (1)$$

where $\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\phi(x, y) = 0$ if and only if $x = y = 0$.

Theorem 6. *Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and let $T : X \rightarrow X$ be a weak C-contraction. Then T has a unique fixed point in X .*

Proof. Let $x_0 \in X$. Let $x_{n+1} = T(x_n)$, $n = 1, 2, \dots$

If $x_n = x_{n+1} = T(x_n)$, then T has a fixed point x_n .

Assume that $x_n \neq x_{n+1}$. Now for any $a \in X$,

$$\begin{aligned} \|x_n - x_{n+1}, a\| &= \|T(x_{n-1}) - T(x_n), a\| \\ &\leq \frac{1}{2} [\|x_{n-1} - T(x_n), a\| + \|x_n - T(x_{n-1}), a\|] \\ &\quad - \phi[\|x_{n-1} - T(x_n), a\|, \|x_n - T(x_{n-1}), a\|] \\ &\leq \frac{1}{2} [\|x_{n-1} - x_n, a\| + \|x_n - x_{n+1}, a\|] \\ &\quad - \phi[\|x_{n-1} - x_{n+1}, a\|, 0]. \end{aligned} \quad (2)$$

Therefore,

$$\|x_n - x_{n+1}, a\| \leq \|x_{n-1} - x_n, a\|. \quad (3)$$

So $\{\|x_n - x_{n+1}, a\|\}$ is a monotone decreasing sequence of real numbers and hence $\{\|x_n - x_{n+1}, a\|\}$ is convergent and let $\|x_n - x_{n+1}, a\| \rightarrow r$ (say) as $n \rightarrow \infty$

We now show that $r = 0$. If not, then

$$\begin{aligned} \|x_n - x_{n+1}, a\| &= \|T(x_{n-1}) - T(x_n), a\| \\ &\leq \frac{1}{2} \|x_{n-1} - x_{n+1}, a\| \\ &\leq \frac{1}{2} [\|x_{n-1} - x_n, a\| + \|x_n - x_{n+1}, a\|]. \end{aligned} \quad (4)$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} r &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \|x_{n-1} - x_{n+1}, a\| \\ &\quad + \frac{1}{2} \lim_{n \rightarrow \infty} \|x_n - x_{n+1}, a\| \leq \frac{1}{2}r + \frac{1}{2}r, \end{aligned} \quad (5)$$

implying that

$$\lim_{n \rightarrow \infty} \|x_{n-1} - x_{n+1}, a\| = 2r. \quad (6)$$

Also from (2) and using continuity of ϕ , we get $r \leq r - \phi[2r, 0]$ showing that $\phi[2r, 0] \leq 0$, a contradiction, unless

$r = 0$. So, $\|x_n - x_{n+1}, a\| \rightarrow 0$ as $n \rightarrow \infty$, that is

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, a\| = 0. \quad (7)$$

We shall now show that $\{x_n\}$ is Cauchy sequence in X .

If not, then there exists a $\varepsilon > 0$ and for each positive integer k , there exist integers $n(k)$ and $m(k)$ with $k < m(k) < n(k)$ such that $\|x_{m(k)} - x_{n(k)}, a\| \geq \varepsilon$ and $\|x_{m(k)} - x_{n(k)-1}, a\| < \varepsilon$. Then

$$\begin{aligned} \varepsilon &\leq \|x_{m(k)} - x_{n(k)}, a\| = \|T(x_{m(k)-1}) - T(x_{n(k)-1}), a\| \\ &\leq \frac{1}{2} [\|x_{m(k)-1} - x_{n(k)}, a\| + \|x_{n(k)-1} - x_{m(k)}, a\|] \\ &\quad - \phi[\|x_{m(k)-1} - x_{n(k)}, a\|, \|x_{n(k)-1} - x_{m(k)}, a\|]. \end{aligned} \quad (8)$$

Also,

$$\begin{aligned} \varepsilon &\leq \|x_{m(k)} - x_{n(k)}, a\| \leq \|x_{m(k)} - x_{n(k)-1}, a\| \\ &\quad + \|x_{n(k)-1} - x_{n(k)}, a\| \\ &< \varepsilon + \|x_{n(k)-1} - x_{n(k)}, a\|. \end{aligned} \quad (9)$$

So by using (7), we obtain

$$\lim_{n \rightarrow \infty} \|x_{m(k)} - x_{n(k)}, a\| = \varepsilon. \quad (10)$$

Now

$$\begin{aligned} \|x_{m(k)} - x_{n(k)-1}, a\| &\leq \|x_{m(k)} - x_{m(k)-1}, a\| \\ &\quad + \|x_{m(k)-1} - x_{n(k)}, a\| \\ &\quad + \|x_{n(k)} - x_{n(k)-1}, a\|. \end{aligned} \quad (11)$$

Also

$$\begin{aligned} \|x_{m(k)-1} - x_{n(k)}, a\| &\leq \|x_{m(k)-1} - x_{m(k)}, a\| \\ &\quad + \|x_{m(k)} - x_{n(k)}, a\|. \end{aligned} \quad (12)$$

Therefore,

$$\lim_{k \rightarrow \infty} \|x_{m(k)-1} - x_{n(k)}, a\| = \varepsilon. \quad (13)$$

Similarly, it can be shown that

$$\lim_{k \rightarrow \infty} \|x_{n(k)-1} - x_{m(k)}, a\| = \varepsilon. \quad (14)$$

So from (8), we get

$$\varepsilon \leq \frac{1}{2} (\varepsilon + \varepsilon) - \phi[\varepsilon, \varepsilon] \quad (15)$$

implying that $\phi[\varepsilon, \varepsilon] \leq 0$, a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence and therefore $\{x_n\}$ is a convergent in X and let $x_n \rightarrow z \in X$ (say) as $n \rightarrow \infty$. Now for any $a \in X$,

$$\begin{aligned} \|z - T(z), a\| &\leq \|z - x_{n+1}, a\| + \|x_{n+1} - T(z), a\| \\ &\leq \|z - x_{n+1}, a\| + \|T(x_n) - T(z), a\| \\ &\leq \|z - x_{n+1}, a\| \\ &\quad + \frac{1}{2} [\|z - T(x_n), a\| + \|x_n - T(z), a\|] \\ &\quad - \phi[\|z - T(x_n), a\|, \|x_n - T(z), a\|]. \end{aligned} \quad (16)$$

So by the continuity of ϕ and taking $n \rightarrow \infty$, we get

$$\begin{aligned} \|z - T(z), a\| &\leq \frac{1}{2} [\|z - T(z), a\| - \phi[0, \|z - T(z), a\|]] \\ &\leq \frac{1}{2} \|z - T(z), a\| \end{aligned} \quad (17)$$

implying that $z = T(z)$.

We shall now show that the fixed point z is unique. If possible let $z_1 \in X$ be another fixed point of T , then for any $a \in X$

$$\begin{aligned} \|z - z_1, a\| &= \|T(z) - T(z_1), a\| \\ &\leq \frac{1}{2} [\|z - T(z_1), a\| + \|z_1 - T(z), a\|] \\ &\quad - \phi[\|z - T(z_1), a\|, \|z_1 - T(z), a\|] \\ &\leq \|z - z_1, a\| - \phi[\|z - z_1, a\|, \|z - z_1, a\|], \end{aligned} \quad (18)$$

which by property of ϕ is a contradiction unless $\|z - z_1, a\| = 0$ for all $a \in X$ that is $z = z_1$.

This completes the proof. $\square$

Example 7. Let X be $\mathbb{R}^3$ and consider the following 2-norm on X as

$$\|x, y\| = \left| \det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|, \quad (19)$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$. Then $(X, \|\cdot, \cdot\|)$ is a 2-Banach space (see [8, 9]).

$T : (X, \|\cdot, \cdot\|) \rightarrow (X, \|\cdot, \cdot\|)$ is defined by $T(x, y, z) = (0, 0, 1)$ for all $(x, y, z) \in X$. Also let $T(0, 0, 1) = (0, 0, 1)$.

Now setting

$$\phi\left(\left(t, t'\right)\right) = \frac{1}{2} \left[t + t'\right]; \quad (t, t') \in [0, \infty) \times [0, \infty). \quad (20)$$

$\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\phi((t, t')) = 0$ if and only if $t = t' = 0$. Now $T(x) = T(x_1, x_2, x_3) = (0, 0, 1)$ and $T(y) = T(y_1, y_2, y_3) = (0, 0, 1)$. Take $a = (a_1, a_2, a_3) \in X$.

Now

$$\|T(x) - T(y), a\| = \left| \det \begin{pmatrix} i & j & k \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} \right| = 0. \quad (21)$$

Also,

$$\begin{aligned} \|T(x) - T(y), a\| &\leq \frac{1}{2} [\|x - T(y), a\| + \|y - T(x), a\|] \\ &\quad - \phi[\|x - T(y), a\|, \|y - T(x), a\|]. \end{aligned} \quad (22)$$

All the conditions of Theorem 6 are satisfied.

Hence T has a unique fixed point $(0, 0, 1)$ in $\mathbb{R}^3$.

Theorem 8. Let F and G be two self-mappings of a 2-Banach space X . Suppose that for any $a \in X$,

$$\begin{aligned} \|F(x) - G(y), a\| &\leq \frac{1}{2} [\|x - G(y), a\| + \|y - F(x), a\|] \\ &\quad - \phi[\|x - G(y), a\|, \|y - F(x), a\|], \end{aligned} \quad (23)$$

where $\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\phi(x, y) = 0$ if and only if $x = y = 0$. Then F and G have a unique common fixed point in X .

Proof. For $x_0 \in X$, define $\{x_n\}_{n \geq 1}$ by $x_{2n+1} = F(x_{2n})$, $x_{2n+2} = G(x_{2n+1})$. So for any $a \in X$,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, a\| &= \|F(x_{2n}) - G(x_{2n+1}), a\| \\ &\leq \frac{1}{2} [\|x_{2n} - G(x_{2n+1}), a\| \\ &\quad + \|x_{2n+1} - F(x_{2n}), a\|] \\ &\quad - \phi[\|x_{2n} - G(x_{2n+1}), a\|, \\ &\quad \|x_{2n+1} - F(x_{2n}), a\|] \\ &\leq \frac{1}{2} [\|x_{2n} - x_{2n+2}, a\|] \\ &\leq \frac{1}{2} [\|x_{2n} - x_{2n+1}, a\| \\ &\quad + \|x_{2n+1} - x_{2n+2}, a\|]. \end{aligned} \quad (24)$$

Hence

$$\|x_{2n+1} - x_{2n+2}, a\| \leq \|x_{2n} - x_{2n+1}, a\|. \quad (25)$$

Similarly

$$\|x_{2n} - x_{2n+1}, a\| \leq \|x_{2n-1} - x_{2n}, a\|. \quad (26)$$

Thus $\{\|x_n - x_{n+1}, a\|\}$ is a monotone decreasing sequence of real numbers and hence is convergent and

Let

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, a\| = r \geq 0 \quad (\text{say}) \quad (27)$$

We now claim that $r = 0$.

If not,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, a\| &\leq \frac{1}{2} \|x_{2n} - x_{2n+2}, a\| \\ &\leq \frac{1}{2} [\|x_{2n} - x_{2n+1}, a\| \\ &\quad + \|x_{2n+1} - x_{2n+2}, a\|]. \end{aligned} \quad (28)$$

So,

$$\lim_{n \rightarrow \infty} \|x_{2n} - x_{2n+2}, a\| = 2r \quad (29)$$

from (24) taking $n \rightarrow \infty$

$$r \leq r - \phi[2r, 0] \quad (30)$$

implying that $\phi[2r, 0] \leq 0$, a contradiction.

Therefore, $r = 0$.

Now for any integer $p > 0$

$$\begin{aligned} \|x_n - x_{n+p}, a\| &\leq \|x_n - x_{n+1}, a\| + \|x_{n+1} - x_{n+2}, a\| \\ &\quad + \cdots + \|x_{n+p-1} - x_{n+p}, a\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (31)$$

So $\{x_n\}$ is Cauchy in X and hence by completeness of X , let

$$\lim_{n \rightarrow \infty} x_n = z \in X. \quad (32)$$

Now for all $a \in X$,

$$\begin{aligned} \|z - F(z), a\| &\leq \|z - x_{n+2}, a\| + \|x_{n+2} - F(z), a\| \\ &\leq \|z - x_{2n+2}, a\| + \|F(z) - G(x_{2n+1}), a\|. \end{aligned} \quad (33)$$

By routine check up, we can see that for sufficiently large values of n , Hence, $z = Fz$.

Similarly we can show that $z = G(z)$.

We shall now show that z is unique.

If not let $z_1 \in X$ be another fixed point of F and G . Then for any $a \in X$

$$\begin{aligned} \|z - z_1, a\| &= \|F(z) - G(z_1), a\| \\ &\leq \frac{1}{2} [\|z - z_1, a\| + \|z_1 - z, a\|] \\ &\quad - \phi[\|z - z_1, a\|, \|z_1 - z, a\|]. \end{aligned} \quad (34)$$

So $\|z - z_1, a\| \leq \|z - z_1, a\| - \phi[\|z - z_1, a\|, \|z_1 - z, a\|]$, showing that $\phi[\|z - z_1, a\|, \|z_1 - z, a\|] \leq 0$. So $z - z_1 = 0$ implies $z = z_1$. $\square$

Example 9. Let X be $\mathbb{R}^3$ and consider the following 2-norm on X as

$$\|x, y\| = \left| \det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|, \quad (35)$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$. Then $(X, \|\cdot, \cdot\|)$ is a 2-Banach space (see [8, 9]).

Define $F, G : X \rightarrow X$ by $F(x, y, z) = ((1/4)x, (1/4)y, (1/4)z)$ for all $(x, y, z) \in X$ and $G(x, y, z) = ((1/3)x, (1/3)y, (1/3)z)$ for all $(x, y, z) \in X$.

Now setting

$$\phi((t, t')) = \frac{1}{2} \max\{t, t'\}; \quad (t, t') \in [0, \infty) \times [0, \infty). \quad (36)$$

$\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\phi((t, t')) = 0$ if and only if $t = t' = 0$. Take $a = (a_1, a_2, a_3) \in X$.

Now

$$\begin{aligned} \|F(x) - G(y), a\| &= \left| \det \begin{pmatrix} i & j & k \\ \frac{1}{4}x_1 - \frac{1}{3}y_1 & \frac{1}{4}x_2 - \frac{1}{3}y_2 & \frac{1}{4}x_3 - \frac{1}{3}y_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \right| \\ \|x - G(y), a\| &= \left| \det \begin{pmatrix} i & j & k \\ x_1 - \frac{1}{3}y_1 & x_2 - \frac{1}{3}y_2 & x_3 - \frac{1}{3}y_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \right|, \\ \|y - F(x), a\| &= \left| \det \begin{pmatrix} i & j & k \\ y_1 - \frac{1}{3}x_1 & y_2 - \frac{1}{3}x_2 & y_3 - \frac{1}{3}x_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \right|. \end{aligned} \quad (37)$$

By routine calculation, it can be seen that the condition in Theorem 8

$$\begin{aligned} \|F(x) - G(y), a\| &\leq \frac{1}{2} [\|x - G(y), a\| + \|y - F(x), a\|] \\ &\quad - \phi[\|x - G(y), a\|, \|y - F(x), a\|] \end{aligned} \quad (38)$$

is satisfied. Also F and G have a unique common fixed point $(0, 0, 0)$ in X .

Theorem 10. Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space with $T : X \rightarrow X$ and let $\{T_n : X \rightarrow X\}$ be a sequence of mappings such that

(i) for all $u \in X$ and $n = 1, 2, \dots$

$$\begin{aligned} \|T_n(x) - T_n(y), u\| &\leq \frac{1}{2} [\|x - T_n(y), u\| \\ &\quad + \|y - T_n(x), u\|] \\ &\quad - \phi[\|x - T_n(y), u\|, \|y - T_n(x), u\|], \end{aligned} \quad (39)$$

where $\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mapping such that $\phi(x, y) = 0$ if and only if $x = y = 0$,

- (ii) $\lim_{n \rightarrow \infty} T_n(x) = T(x)$ for each $x \in X$. Then T has a unique fixed point z in X such that $z = \lim_{n \rightarrow \infty} z_n$ where z_n is the fixed point of T_n ($n = 1, 2, \dots$).

Proof.

$$\begin{aligned} & \|T(x) - T(y), u\| \\ & \leq \|T(x) - T_n(x), u\| + \frac{1}{2} [\|x - T_n(y), u\| \\ & \quad + \|y - T_n(x), u\|] \\ & \quad - \phi[\|x - T_n(y), u\|, \|y - T_n(x), u\|] \\ & \quad + \|T(y) - T_n(y), u\|. \end{aligned} \quad (40)$$

Now

$$\begin{aligned} & \|x - T_n(y), u\| \leq \|x - T(y), u\| \\ & \quad + \|T_n(y) - T(y), u\| \\ & \longrightarrow \|x - T(y), u\| \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (41)$$

Similarly

$$\|y - T_n(x), u\| \longrightarrow \|y - T(x), u\| \quad \text{as } n \longrightarrow \infty. \quad (42)$$

As ϕ is continuous,

$$\begin{aligned} & \phi[\|x - T_n(y), u\|, \|y - T_n(x), u\|] \\ & \longrightarrow \phi[\|x - T(y), u\|, \|y - T(x), u\|]. \end{aligned} \quad (43)$$

So from (40) taking limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} & \|T(x) - T(y), u\| \leq \frac{1}{2} [\|x - T(y), u\| + \|y - T(x), u\|] \\ & \quad - \phi[\|x - T(y), u\|, \|y - T(x), u\|]. \end{aligned} \quad (44)$$

Hence by Theorem 6, T has a unique fixed point z in X .

If possible, let

$$\lim_{n \rightarrow \infty} z_n \neq z \in X. \quad (45)$$

Now for $u \in X$ let

$$\lim_{n \rightarrow \infty} \|z_n - z, u\| = \alpha > 0 \quad (46)$$

$$\begin{aligned} & \|z - z_n, u\| = \|T(z) - T_n(z_n), u\| \\ & \leq \|T(z) - T_n(z), u\| + \|T_n(z) - T_n(z_n), u\|. \end{aligned} \quad (47)$$

Now

$$\begin{aligned} & \|T_n(z) - T_n(z_n), u\| \leq \frac{1}{2} [\|z - T_n(z_n), u\| \\ & \quad + \|z_n - T_n(z), u\|] \\ & \quad - \phi[\|z - T_n(z_n), u\|, \|z_n - T_n(z), u\|]. \end{aligned} \quad (48)$$

Now

$$\begin{aligned} & \|z_n - T_n(z), u\| \leq \|z_n - z, u\| + \|z - T(z), u\| \\ & \quad + \|T(z) - T_n(z), u\| \\ & = \|z_n - z, u\| + \|T(z) - T_n(z), u\|. \end{aligned} \quad (49)$$

So from (46), (47), (48), and (49), we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|z - z_n, u\| \leq \lim_{n \rightarrow \infty} \|T_n(z) - T_n(z_n), u\| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{2} [\|z - z_n, u\| + \|z_n - z, u\|] \\ & \quad - \lim_{n \rightarrow \infty} \phi[\|z - z_n, u\|, \|z_n - z, u\|] \end{aligned} \quad (50)$$

Hence, $\alpha \leq (1/2)(2\alpha) - \phi[\alpha, \alpha]$ showing that $\phi[\alpha, \alpha] \leq 0$ which is possible only when $\alpha = 0$

Hence $\lim_{n \rightarrow \infty} \|z - z_n, u\| = 0$, implying that $\lim_{n \rightarrow \infty} z_n = z$. $\square$

Theorem 11. Let X be a 2-Banach space, $\{T_n\}$ a sequence of mappings of X onto itself with fixed points z_n . Let T be a mapping of X into itself satisfying the condition of Theorem 6 with fixed point z , such that $T_n \rightarrow T$ uniformly on $\{z_n : n = 1, 2, \dots\}$. Then $\lim_{n \rightarrow \infty} z_n = z \in X$.

Proof. Fix $\varepsilon > 0$. From the uniform convergence of $\{T_n\}$ on $\{z_n : n = 1, 2, \dots\}$, there exists an integer N such that for all $n \geq N$, and for all $u \in X$, $\|T(z_n) - T_n(z_n), u\| < \varepsilon$ for all z_n .

Now

$$\begin{aligned} & \|z - z_n, u\| = \|T(z) - T_n(z_n), u\| \\ & \leq \|T(z) - T(z_n), u\| + \|T(z_n) - T_n(z_n), u\|. \end{aligned} \quad (51)$$

If possible let $\lim_{n \rightarrow \infty} z_n \neq z \in X$ that is there exists $u \in X$,

$$\lim_{n \rightarrow \infty} \|z_n - z, u\| = \alpha > 0. \quad (52)$$

Now for all $u \in X$,

$$\begin{aligned} & \|T(z) - T(z_n), u\| \leq \frac{1}{2} [\|z - T(z_n), u\| + \|z_n - T(z), u\|] \\ & \quad - \phi[\|z - T(z_n), u\|, \|z_n - T(z), u\|]. \end{aligned} \quad (53)$$

So for $u \in X$ satisfying (52)

$$\|z - T(z_n), u\| \leq \|z - z_n, u\| + \|T_n(z_n) - T(z_n), u\|. \quad (54)$$

Therefore

$$\lim_{n \rightarrow \infty} \|z - T(z_n), u\| \leq \varepsilon + \lim_{n \rightarrow \infty} \|z - z_n, u\| = \varepsilon + \alpha. \quad (55)$$

As $\varepsilon > 0$ is arbitrary,

$$\lim_{n \rightarrow \infty} \|z - T(z_n), u\| \leq \alpha. \quad (56)$$

Again

$$\|z - z_n, u\| \leq \|z - T(z_n), u\| + \|T(z_n) - T_n(z_n), u\|. \quad (57)$$

Hence

$$\lim_{n \rightarrow \infty} \|z - z_n, u\| \leq \lim_{n \rightarrow \infty} \|z - T(z_n), u\| + \varepsilon. \quad (58)$$

As $\varepsilon > 0$ is arbitrary,

$$\alpha \leq \lim_{n \rightarrow \infty} \|z - T(z_n), u\|. \quad (59)$$

Therefore

$$\lim_{n \rightarrow \infty} \|z - T(z_n), u\| = \alpha. \quad (60)$$

Hence

$$\lim_{n \rightarrow \infty} \phi[\|z - T(z_n), u\|, \|z_n - T(z), u\|] \longrightarrow \phi[\alpha, \alpha]. \quad (61)$$

Taking $n \rightarrow \infty$ in (53), we get $\alpha \leq (1/2)(2\alpha) - \phi[\alpha, \alpha]$, showing that $\phi[\alpha, \alpha] \leq 0$ which possible only when $\alpha = 0$, a contradiction to (52).

Hence $\lim_{n \rightarrow \infty} \|z_n - z, u\| = 0$, implying that $\lim_{n \rightarrow \infty} z_n = z$. $\square$

Theorem 12. Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and for any $x \in X$, $\{T_n(x)\}_{n=1,2,\dots}$ a sequence of mappings of X onto itself. Suppose there exists a sequence of nonnegative integers $\{m_n\}$ such that for all $x, y, u \in X$, and every pair i, j with $i \neq j$ satisfying

$$\begin{aligned} \|T_i^{m_i}(x) - T_j^{m_j}(y), u\| &\leq \frac{1}{2} [\|x - T_j^{m_j}(y), u\| \\ &\quad + \|y - T_i^{m_i}(x), u\|] \\ &\quad - \phi[\|x - T_j^{m_j}(y), u\|, \\ &\quad \|y - T_i^{m_i}(x), u\|], \end{aligned} \quad (62)$$

where $\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous mappings such that $\phi(x, y) = 0$ if and only if $x = y = 0$. Then $\{T_n\}$ have a unique common fixed point in X .

Proof. Define $S_i = T_i^{m_i}$, $i = 1, 2, \dots$

Then

$$\begin{aligned} \|S_i(x) - S_j(y), u\| &\leq \frac{1}{2} [\|x - S_j(y), u\| \\ &\quad + \|y - S_i(x), u\|] \\ &\quad - \phi[\|x - S_j(y), u\|, \\ &\quad \|y - S_i(x), u\|]. \end{aligned} \quad (63)$$

Pick $x_0 \in X$ and define $x_n = S_n(x_{n-1})$, $n = 1, 2, \dots$

Now

$$\begin{aligned} \|x_n - x_{n+1}, u\| &= \|S_n(x_{n-1}) - S_{n+1}(x_n), u\| \\ &\leq \frac{1}{2} [\|x_{n-1} - S_{n+1}(x_n), u\| \\ &\quad + \|x_n - S_n(x_{n-1}), u\|] \\ &\quad - \phi[\|x_{n-1} - S_{n+1}(x_n), u\|, \\ &\quad \|x_n - S_n(x_{n-1}), u\|]. \end{aligned} \quad (64)$$

By routine calculation, we can see that

$$\|x_n - x_{n+1}, u\| \leq \|x_{n-1} - x_n, u\|. \quad (65)$$

Implying that $\{\|x_n - x_{n+1}, u\|\}$ is a monotone decreasing sequence of real numbers and hence is convergent and let

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, u\| = r > 0. \quad (66)$$

We shall now show that $r = 0$.

If not,

$$\begin{aligned} \|x_n - x_{n+1}, u\| &\leq \frac{1}{2} [\|x_{n-1} - x_{n+1}, u\|] \\ &\leq \frac{1}{2} [\|x_{n-1} - x_n, u\| + \|x_n - x_{n-1}, u\|]. \end{aligned} \quad (67)$$

Taking $n \rightarrow \infty$,

$$\begin{aligned} r &\leq \frac{1}{2} \lim_{n \rightarrow \infty} [\|x_{n-1} - x_{n+1}, u\|] \\ &\leq \frac{1}{2} [\lim_{n \rightarrow \infty} \|x_{n-1} - x_n, u\| + \lim_{n \rightarrow \infty} \|x_n - x_{n-1}, u\|] = r. \end{aligned} \quad (68)$$

Hence

$$\lim_{n \rightarrow \infty} [\|x_{n-1} - x_{n+1}, u\|] = 2r. \quad (69)$$

Taking $n \rightarrow \infty$ in (64), we get $r \leq r - \phi[2r, 0]$, showing that $\phi[2r, 0] \leq 0$ which is possible only when $r = 0$.

Now for any integer $p > 0$

$$\begin{aligned} \|x_n - x_{n+p}, u\| &\leq \|x_n - x_{n+1}, u\| + \|x_{n+1} - x_{n+2}, u\| \\ &\quad + \dots + \|x_{n+p-1} - x_{n+p}, u\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (70)$$

Therefore $\{x_n\}$ is Cauchy in X and hence by completeness of X , let

$$\lim_{n \rightarrow \infty} x_n = z \in X. \quad (71)$$

Now for any integer m ,

$$\begin{aligned}
 \|z - S_m(z), u\| &\leq \|z - x_n, u\| + \|x_n - S_m(z), u\| \\
 &= \|z - x_n, u\| + \|S_n(x_{n-1}) - S_m(z), u\| \\
 &\leq \|z - x_n, u\| + \frac{1}{2} [\|x_{n-1} - S_m(z), u\| \\
 &\quad + \|z - S_n(x_{n-1}), u\|] \quad (72) \\
 &\leq \|z - x_n, u\| + \frac{1}{2} [\|x_{n-1} - z, u\| \\
 &\quad + \frac{1}{2} [\|z - S_m(z), u\| + \|z - x_n, u\|].
 \end{aligned}$$

Hence

$$\begin{aligned}
 \frac{1}{2} \|z - S_m(z), u\| &\leq 2 [\|z - x_n, u\| \\
 &\quad + \frac{1}{2} [\|x_{n-1} - z, u\|] \quad (73) \\
 &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty
 \end{aligned}$$

Therefore $\|z - S_m(z), u\| = 0$ implying that $S_m(z) = z$ for all m .

Hence $\{S_m\}$ has a common fixed point z .

If $z_1 \in X$ is another fixed point of $\{S_n\}$, then

$$\begin{aligned}
 \|z - z_1, u\| &= \|S_n(z) - S_n(z_1), u\| \\
 &\leq \frac{1}{2} [\|z - z_1, u\| + \|z - z_1, u\|] \quad (74) \\
 &\quad - \phi [\|z - z_1, u\|, \|z - z_1, u\|].
 \end{aligned}$$

Implying that $\phi[\|z - z_1, u\|, \|z - z_1, u\|] \leq 0$. Hence $z - z_1 = 0$ implying that $z = z_1$.

Now for each m ,

$$T_n(z) = T_n(S_n(z)) = T_n(T_n^{m_n}(z)) = T_n^{m_n}(T_n(z)), \quad (75)$$

showing that $T_n(z)$ is the fixed point of S_n .

Also $S_n(z) = z$ and $S_n(T_n(z)) = T_n(z)$. Hence $z = T_n(z)$ for all n . $\square$

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