Research Article

A Mixed Problem with an Integral Two-Space-Variables Condition for Parabolic Equation with The Bessel Operator

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We study a mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator. The existence and uniqueness of the solution in functional weighted Sobolev space are proved. The proof is based on a priori estimate “energy inequality” and the density of the range of the operator generated by the problem considered.

1. Introduction

The importance of boundary value problems with integral boundary condition has been pointed out by Samarskiĭ [1] and problems with integral conditions for parabolic equations were treated by Kamynin [2], Ionkin [3], Yurchuk [4], Benouar and Yurchuk [5], Bouziani [6], Bouziani and Benouar [7, 8], and Mesloub and Bouziani [9]. Other parabolic problems also arise in plasma physics by Samarskiĭ [1], heat conduction by Cannon [10], Ionkin [3], dynamics of ground waters by Nakhushev [11], Vodakhova [12], Kartynnik [13], and Lin [14]. Regular case of this problem is studied in [15]. The problem where the equation contains an operator of the form $(\partial/\partial x)(a(x,t)(\partial u/\partial x))$, instead of Bessel operator, is treated in [16]. Similar problems for second-order parabolic equations are investigated by the potential method in [17]. Later problems with an integral two-space-variables condition for parabolic equations were treated by Marhoune [18], Marhoune and Lakhal [19]. Motivated by this, we study a mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator.

2. Setting of the Problem

In the rectangular domain $\Omega = (0, 1) \times (0, T)$, with $T < \infty$, we consider the following equation:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} \right) = f(x, t), \quad (1)$$

with the initial data

$$\ell u = u(x, 0) = \varphi(x), \quad x \in (0, 1), \quad (2)$$

Neumann boundary condition

$$\frac{\partial u}{\partial x}(1, t) = 0, \quad (3)$$

and the integral condition

$$\int_0^\alpha xu(x, t) dx + \int_\beta^1 xu(x, t) dx = 0, \quad (4)$$

$$\beta > \alpha > 0, \quad \alpha + \beta = 1, \quad t \in (0, T),$$

where φ is a known function.

We shall assume that the functions φ and f satisfy the compatibility condition with (4), that is,

$$\int_0^\alpha x\varphi(x) dx + \int_\beta^1 x\varphi(x) dx = 0, \quad (5)$$

$$\int_0^\alpha xf(x, t) dx + \int_\beta^1 xf(x, t) dx = 0. \quad (6)$$

The presence of integral terms in boundary condition can, in general, greatly complicate the application of standard functional or numerical techniques, specially the integral two-space-variables condition. Then to avoid this difficulty, we introduce a technique for transferring this problem to another classically less complicated one that does not contain integral conditions. For that, we establish the following lemma.

Lemma 1. *Problem (1)–(4) is equivalent to the following problem (PR):*

$$\begin{aligned}\mathcal{L}u &= \frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} \right) = f(x, t), \\ \ell u &= u(x, 0) = \varphi(x), \quad x \in (0, 1), \\ \alpha \frac{\partial u}{\partial x}(\alpha, t) &= \beta \frac{\partial u}{\partial x}(\beta, t), \quad \frac{\partial u}{\partial x}(1, t) = 0.\end{aligned}\quad (\text{PR})$$

Proof. Let $u(x, t)$ be a solution of (1)–(4), we prove that

$$\frac{\partial u}{\partial x}(\alpha, t) = \frac{\partial u}{\partial x}(\beta, t). \quad (7)$$

So, multiplying (1) by x and integrating with respect to x over $(0, \alpha)$ and $(\beta, 1)$ and taking into account (4) and (6), we obtain

$$x \frac{\partial u}{\partial x} \Big|_{x=0}^{x=\alpha} dx + x \frac{\partial u}{\partial x} \Big|_{x=\beta}^{x=1} dx = 0, \quad \text{this implies} \quad (8)$$

$$\alpha \frac{\partial u}{\partial x}(\alpha, t) + \frac{\partial u}{\partial x}(1, t) - \beta \frac{\partial u}{\partial x}(\beta, t) = 0.$$

Then, from (3), we obtain

$$\alpha \frac{\partial u}{\partial x}(\alpha, t) = \beta \frac{\partial u}{\partial x}(\beta, t). \quad (9)$$

Let now $u(x, t)$ be a solution of (PR), we are bound to prove that

$$\int_0^\alpha xu(x, t) dx + \int_\beta^1 xu(x, t) dx = 0. \quad (10)$$

So, multiplying (1) by x and integrating with respect to x over $(0, \alpha)$ and $(\beta, 1)$ and taking into account

$$\alpha \frac{\partial u}{\partial x}(\alpha, t) = \beta \frac{\partial u}{\partial x}(\beta, t), \quad \frac{\partial u}{\partial x}(1, t) = 0, \quad (11)$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t} \int_0^\alpha xu(x, t) dx &= \int_0^\alpha xf(x, t) dx, \\ \frac{\partial}{\partial t} \int_\beta^1 xu(x, t) dx &= \int_\beta^1 xf(x, t) dx,\end{aligned}\quad (12)$$

combining the two preceding equations, and from (6) we get

$$\int_0^\alpha xu(x, t) dx + \int_\beta^1 xu(x, t) dx = 0. \quad (13)$$

□

3. A Priori Estimate

The method used here is one of the most efficient functional analysis methods in solving partial differential equations with integral conditions, the so-called a priori estimate method or the energy-integral method. This method is essentially based on the construction of multipliers for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the stated problem. But here we use the energy inequality method for the equivalent problem (PR) given in Lemma 1; so to investigate the posed problem, we introduce the needed function spaces.

We introduce function spaces needed in our investigation. We denote by $L_\rho^2(\Omega)$ the weighted Lebesgue space that consists of all measurable functions u equipped with the finite norm

$$\|u\|_{L_\rho^2(\Omega)} = \left(\int_\Omega \rho(x) |u(x, \cdot)|^2 dx \right)^{1/2}, \quad (14)$$

where $\rho(x) = (x^2 + x)$. If $\rho(x) = 1$, $L_\rho^2(\Omega)$ are identified with the standard spaces $L^2(\Omega)$.

In this paper, we prove the existence and the uniqueness for solution of the problem (1)–(4) as a solution of the operator equation

$$Lu = \mathcal{F}, \quad (15)$$

where $L = (\mathcal{L}, \ell)$, with domain of definition B consisting of functions $u \in L_\rho^2(\Omega)$ such that $\partial u / \partial t$, $\partial u / \partial x$, $\partial^2 u / \partial x^2$, $\partial^2 u / \partial t \partial x \in L_\rho^2(\Omega)$, and u satisfy condition (4); the operator L is considered from B to F , where B is the Banach space consisting of all functions $u(x, t)$ having a finite norm

$$\begin{aligned}\|u\|_B^2 &= \int_\Omega (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\ &+ \sup_{0 \leq \tau \leq T} \int_0^1 (x^2 + x) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx,\end{aligned}\quad (16)$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega f^2 dx dt + \int_0^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \quad (17)$$

is finite.

Theorem 2. *For any function $u \in B$, one has the estimate*

$$\|u\|_B \leq c \|Lu\|_F, \quad (18)$$

where c is a positive constant independent of u .

Proof. Multiplying (1) by the following Mu :

$$Mu = \begin{cases} (x^2 - x) \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t}, & 0 \leq x \leq \alpha, \\ -x \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t}, & \alpha \leq x \leq \beta, \\ (x^2 - x) \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t}, & \beta \leq x \leq 1, \end{cases} \quad (19)$$

and integrating over Ω^τ , where $\Omega^\tau = (0, 1) \times (0, \tau)$,

$$\begin{aligned} & \int_{\Omega^\tau} \mathcal{L}u \cdot Mu dx dt \\ &= \left[\begin{aligned} & \int_{\Omega_\alpha^\tau} (x^2 - x) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\ &+ \int_{\Omega_\alpha^\tau} (1 - x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\ &+ \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\ &- \int_{\Omega_\alpha^\tau} (x - x^2) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &- \int_{\Omega_\alpha^\tau} (x + 1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\ &= \int_{\Omega_\alpha^\tau} f \cdot \left((x^2 - x) \frac{\partial u}{\partial x} \right. \\ &\quad \left. + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt, \quad \Omega_\alpha^\tau, \\ & \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\ &- \int_{\Omega_{\alpha,\beta}^\tau} x \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt + \int_{\Omega_{\alpha,\beta}^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\ &+ \int_{\Omega_{\alpha,\beta}^\tau} x \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &- \int_{\Omega_{\alpha,\beta}^\tau} (x + 1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\ &= \int_{\Omega_{\alpha,\beta}^\tau} f \cdot \left(-x \frac{\partial u}{\partial x} \right. \\ &\quad \left. + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt, \quad \Omega_{\alpha,\beta}^\tau, \\ & \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\ &+ \int_{\Omega_\beta^\tau} (x^2 - x) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\ &+ \int_{\Omega_\beta^\tau} (1 - x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\ &+ \int_{\Omega_\beta^\tau} (x - x^2) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &- \int_{\Omega_\beta^\tau} (x + 1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\ &= \int_{\Omega_\beta^\tau} f \cdot \left((x^2 - x) \frac{\partial u}{\partial x} \right. \\ &\quad \left. + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt, \quad \Omega_\beta^\tau. \end{aligned} \right] \quad (20) \end{aligned}$$

Integration by parts, we obtain

$$\begin{aligned} & - \int_{\Omega_\alpha^\tau} (x - x^2) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &= \int_0^\tau \left(\frac{\alpha}{2} - \frac{\alpha^2}{2} \right) \left(\frac{\partial u}{\partial x}(\alpha, t) \right)^2 dt \\ &\quad - \int_{\Omega_\alpha^\tau} \left(\frac{1}{2} - x \right) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \\ & - \int_{\Omega_\alpha^\tau} (x + 1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\ &= - \int_0^\tau (\alpha^2 + \alpha) \frac{\partial u}{\partial x}(\alpha, t) \frac{\partial u}{\partial t}(\alpha, t) dt \\ &\quad + \int_{\Omega_\alpha^\tau} x \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt + \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\ &\quad - \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx, \\ & \int_{\Omega_{\alpha,\beta}^\tau} x \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &= \frac{\beta}{2} \int_0^\tau \left(\frac{\partial u}{\partial x}(\beta, t) \right)^2 dt - \frac{\alpha}{2} \int_0^\tau \left(\frac{\partial u}{\partial x}(\alpha, t) \right)^2 dt \\ &\quad - \frac{1}{2} \int_{\Omega_{\alpha,\beta}^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \\ & - \int_{\Omega_{\alpha,\beta}^\tau} (x + 1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\ &= - \int_0^\tau (\beta^2 + \beta) \frac{\partial u}{\partial x}(\beta, t) \frac{\partial u}{\partial t}(\beta, t) dt \\ &\quad + \int_0^\tau (\alpha^2 + \alpha) \frac{\partial u}{\partial x}(\alpha, t) \frac{\partial u}{\partial t}(\alpha, t) dt \\ &\quad + \int_{\Omega_{\alpha,\beta}^\tau} x \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\ &\quad + \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\ &\quad - \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx, \\ & \int_{\Omega_\beta^\tau} (x - x^2) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} dx dt \\ &= \int_0^\tau \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \left(\frac{\partial u}{\partial x}(\beta, t) \right)^2 dt \\ &\quad - \int_{\Omega_\beta^\tau} \left(\frac{1}{2} - x \right) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega_\beta^\tau} (x+1) \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} dx dt \\
& = \int_0^\tau (\beta^2 + \beta) \frac{\partial u}{\partial x}(\beta, t) \frac{\partial u}{\partial t}(\beta, t) dt \\
& \quad + \int_{\Omega_\beta^\tau} x \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& \quad + \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
& \quad - \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx,
\end{aligned} \tag{21}$$

and by using the Cauchy's ε -inequality, we get

$$\begin{aligned}
& \int_{\Omega_\alpha^\tau} x^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& \leq \frac{\varepsilon}{2} \int_{\Omega_\alpha^\tau} x^2 \left(\frac{\partial u}{\partial x} \right)^2 dx dt + \frac{1}{2\varepsilon} \int_{\Omega_\alpha^\tau} x^2 \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
& \leq \frac{1}{4} \int_{\Omega_\alpha^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{4(\alpha^2 + \alpha)} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{(\alpha^2 + \alpha)}{2} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt, \\
& \int_{\Omega_\alpha^\tau} f \cdot \left((x^2 - x) \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt \\
& \leq (\alpha^2 + \alpha)^2 \int_{\Omega_\alpha^\tau} f^2 dx dt + \frac{1}{4} \int_{\Omega_\alpha^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{4(\alpha^2 + \alpha)} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{2(\alpha^2 + \alpha)} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt, \\
& \int_{\Omega_{\alpha,\beta}^\tau} f \cdot \left(-x \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt \\
& \leq \left(\frac{(\beta - \alpha)((\beta^2 - \alpha^2) + (\beta - \alpha))}{2} + \frac{1}{2} \right) \\
& \quad \times \int_{\Omega_{\alpha,\beta}^\tau} f^2 dx dt + \frac{1}{4} \int_{\Omega_{\alpha,\beta}^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{4((\beta^2 - \alpha^2) + (\beta - \alpha))}
\end{aligned}$$

$$\begin{aligned}
& \times \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{2} \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2, \\
& - \int_{\Omega_\beta^\tau} x^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt \\
& \leq \frac{\varepsilon}{2} \int_{\Omega_\beta^\tau} x \left(\frac{\partial u}{\partial x} \right)^2 dx dt + \frac{1}{2\varepsilon} \int_{\Omega_\beta^\tau} x \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
& \leq \frac{\varepsilon(1 - \beta)}{2} \int_{\Omega_\beta^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + \frac{1}{2\varepsilon} \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
& \leq \frac{1}{4} \int_{\Omega_\beta^\tau} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
& \quad + (1 - \beta) \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt, \\
& \int_{\Omega_\beta^\tau} f \cdot \left((x^2 - x) \frac{\partial u}{\partial x} + (x^2 + x) \frac{\partial u}{\partial t} \right) dx dt \\
& \leq ((1 - \beta)^2 + (1 - \beta)) \int_{\Omega_\beta^\tau} f^2 + \frac{1}{4} \int_{\Omega_\beta^\tau} \left(\frac{\partial u}{\partial x} \right)^2 \\
& \quad + \frac{1}{4((1 - \beta)^2 + (1 - \beta))} \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 \\
& \quad + \frac{1}{2((1 - \beta)^2 + (1 - \beta))} \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2.
\end{aligned} \tag{22}$$

Substituting (21)–(22) into $\int_{\Omega^\tau} \mathcal{L}u \cdot \text{Mud}x dt$, we obtain

$$\begin{aligned}
& \left(1 - \frac{\alpha^2 + \alpha}{2} - \frac{1}{2(\alpha^2 + \alpha)} \right) \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
& \quad + \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
& \quad + \int_0^\tau \left(\frac{\alpha}{2} - \frac{\alpha^2}{2} \right) \left(\frac{\partial u}{\partial x}(\alpha, t) \right)^2 dt \\
& \quad - \int_0^\tau (\alpha^2 + \alpha) \frac{\partial u}{\partial x}(\alpha, t) \frac{\partial u}{\partial t}(\alpha, t) dt
\end{aligned}$$

$$\begin{aligned}
&\leq (\alpha^2 + \alpha)^2 \int_{\Omega_\alpha^\tau} f^2 dx dt \\
&\quad + \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{2(\alpha^2 + \alpha)} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_\alpha^\tau,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 \\
&\quad + \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
&\quad - \int_0^\tau (\beta^2 + \beta) \frac{\partial u}{\partial x}(\beta, t) \frac{\partial u}{\partial t}(\beta, t) dt \\
&\quad + \int_0^\tau \left(\frac{\alpha}{2} - \frac{\alpha^2}{2} \right) \left(\frac{\partial u}{\partial x}(\alpha, t) \right)^2 dt \\
&\quad \frac{\beta}{2} \int_0^\tau \left(\frac{\partial u}{\partial x}(\beta, t) \right)^2 dt - \frac{\alpha}{2} \int_0^\tau \left(\frac{\partial u}{\partial x}(\alpha, t) \right)^2 dt
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\leq \left(\frac{(\beta - \alpha)^2 (\beta + \alpha + 1)}{2} + \frac{1}{2} \right) \int_{\Omega_{\alpha,\beta}^\tau} f^2 dx dt \\
&\quad + \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{4((\beta^2 - \alpha^2) + (\beta - \alpha))} \\
&\quad \times \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_{\alpha,\beta}^\tau, \\
&\quad \left(\beta - \frac{1}{2((1 - \beta)^2 + (1 - \beta))} \right) \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
&\quad + \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
&\quad \int_0^\tau \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \left(\frac{\partial u}{\partial x}(\beta, t) \right)^2 dt \\
&\quad + \int_0^\tau (\beta^2 + \beta) \frac{\partial u}{\partial x}(\beta, t) \frac{\partial u}{\partial t}(\beta, t) dt \\
&\leq ((1 - \beta)^2 + (1 - \beta))^2 \int_{\Omega_\beta^\tau} f^2 dx dt \\
&\quad + \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{4((1 - \beta)^2 + (1 - \beta))} \\
&\quad \times \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_\beta^\tau.
\end{aligned} \tag{25}$$

Now, by using the conditions (PR) in (23), (24), and (25), respectively, we obtain

$$\begin{aligned}
&\left(1 - \frac{\alpha^2 + \alpha}{2} - \frac{1}{2(\alpha^2 + \alpha)} \right) \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
&\quad + \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
&\leq (\alpha^2 + \alpha)^2 \int_{\Omega_\alpha^\tau} f^2 dx dt + \frac{1}{2} \int_0^\alpha (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{2(\alpha^2 + \alpha)} \int_{\Omega_\alpha^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_\alpha^\tau,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 \\
&\quad + \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
&\leq \left(\frac{(\beta - \alpha)^2 (\beta + \alpha + 1)}{2} + \frac{1}{2} \right) \int_{\Omega_{\alpha,\beta}^\tau} f^2 dx dt \\
&\quad + \frac{1}{2} \int_\alpha^\beta (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{4((\beta^2 - \alpha^2) + (\beta - \alpha))} \\
&\quad \times \int_{\Omega_{\alpha,\beta}^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_{\alpha,\beta}^\tau,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\left(\beta - \frac{1}{2((1 - \beta)^2 + (1 - \beta))} \right) \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
&\quad + \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
&\leq ((1 - \beta)^2 + (1 - \beta))^2 \int_{\Omega_\beta^\tau} f^2 dx dt \\
&\quad + \frac{1}{2} \int_\beta^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
&\quad + \frac{1}{4((1 - \beta)^2 + (1 - \beta))} \\
&\quad \times \int_{\Omega_\beta^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt, \quad \Omega_\beta^\tau.
\end{aligned} \tag{28}$$

Or in Ω^τ , by combining (26), (27), and (28), we get

$$\begin{aligned}
& \min \left(1 - \frac{\alpha^2 + \alpha}{2} - \frac{1}{2(\alpha^2 + \alpha)}, \frac{1}{2}, \right. \\
& \quad \left. \beta - \frac{1}{2((1-\beta)^2 + (1-\beta))} \right) \\
& \quad \times \int_{\Omega^\tau} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt \\
& \quad + \frac{1}{2} \int_0^1 (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \\
& \leq \max \left((\alpha^2 + \alpha)^2, \left(\frac{(\beta - \alpha)^2 (\beta + \alpha + 1)}{2} + \frac{1}{2} \right), \right. \\
& \quad \left. ((1-\beta)^2 + (1-\beta))^2 \right) \\
& \quad \times \int_{\Omega^\tau} f^2 dx dt + \frac{1}{2} \int_0^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
& \quad + \max \left(\frac{1}{2(\alpha^2 + \alpha)}, \frac{1}{4((\beta^2 - \alpha^2) + (\beta - \alpha))}, \right. \\
& \quad \left. \frac{1}{4((1-\beta)^2 + (1-\beta))} \right) \\
& \quad \times \int_{\Omega^\tau} (x^2 + x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt.
\end{aligned} \tag{29}$$

Using Lemma 1 in [20], one has

$$\begin{aligned}
& \left(\int_{\Omega} (x^2 + x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt + \int_0^1 (x^2 + x) \left(\frac{\partial u(\cdot, \tau)}{\partial x} \right)^2 dx \right) \\
& \leq k \left(\int_{\Omega} f^2 dx dt + \int_0^1 (x^2 + x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \right),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
k = & \max \left((\alpha^2 + \alpha)^2, \left(\frac{(\beta - \alpha)^2 (\beta + \alpha + 1)}{2} + \frac{1}{2} \right), \right. \\
& \left. ((1-\beta)^2 + (1-\beta))^2 \right) \\
& \times \left(\min \left(1 - \frac{\alpha^2 + \alpha}{2} - \frac{1}{2(\alpha^2 + \alpha)}, \frac{1}{2}, \right. \right. \\
& \quad \left. \left. \beta - \frac{1}{2((1-\beta)^2 + (1-\beta))} \right) \right)^{-1} \\
& \times e^{(\max(1/2(\alpha^2 + \alpha), 1/4((\beta^2 - \alpha^2) + (\beta - \alpha)), 1/4((1-\beta)^2 + (1-\beta)))T)}.
\end{aligned} \tag{31}$$

The right-hand side of (30) is independent of τ , hence replacing the left-hand side by its upper bound with respect to τ from 0 to T , we obtain the desired inequality, where $c = (k)^{1/2}$. $\square$

Corollary 3. A solution of the problem (1)–(4) is unique if it exists and depends continuously on $\mathcal{F} \in F$.

4. Existence of Solution

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathcal{F} = (f, \varphi) \in F$.

Theorem 4. Suppose the conditions of Theorem 2 are satisfied. Then the problem (1)–(4) admits a unique solution $u = L^{-1}\mathcal{F}$.

Proof. First we prove that $R(L)$ is dense in F for the special case where $D(L) \equiv B$ is reduced to $D_0(L)$, where $D_0(L) = \{u, u \in D(L) : \ell u = 0\}$. $\square$

Proposition 5. Let the conditions of Theorem 4 be satisfied. if for $\omega \in L^2(\Omega)$ and for all $u \in D_0(L)$, one has

$$\int_{\Omega} \mathcal{L}u \cdot \omega dx dt = 0, \tag{32}$$

then ω vanishes almost everywhere in Ω .

Proof. The scalar product of F is defined by

$$\begin{aligned}
(Lu, \omega)_F &= \int_{\Omega} \mathcal{L}u \cdot \omega dx dt \\
&+ \int_0^1 (x^2 + x) \left(\frac{\partial \ell u}{\partial x} \right) \left(\frac{\partial \omega_0}{\partial x} \right) dx.
\end{aligned} \tag{33}$$

the equality (32) can be written as follows:

$$\int_{\Omega} \frac{\partial u}{\partial t} \cdot \omega dx dt = \int_{\Omega} \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) \cdot \omega dx dt, \tag{34}$$

if we put

$$u = \mathfrak{F}_t(e^{kt}z) = \int_0^t e^{k\tau} z(x, \tau) d\tau, \tag{35}$$

where k is a strictly positive constant and $z, \partial z / \partial x, (\partial / \partial x)(\partial \mathfrak{F}_t(e^{kt}z) / \partial x) \in L^2(\Omega)$, then, u satisfies the boundary conditions in (PR). As a result of (34), we obtain the equality

$$\int_{\Omega} e^{kt} z \omega dx dt = \int_{\Omega} \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \mathfrak{F}_t(e^{kt}z)}{\partial x} \right) \cdot \omega dx dt. \tag{36}$$

The left-hand side of (36) shows that the mapping

$$L^2(\Omega) \ni z \longrightarrow \int_{\Omega} \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \mathfrak{F}_t(e^{kt}z)}{\partial x} \right) \cdot \omega dx dt \tag{37}$$

is a continuous linear functional of z . From the right-hand side of (36) there follows that is true if the function ω has the following properties:

$$\begin{aligned} \frac{1}{x} \frac{\partial \omega}{\partial x}, \frac{\partial \mathfrak{F}_t^* \omega}{\partial x}, \frac{1}{x} \frac{\partial \mathfrak{F}_t^* \omega}{\partial x}, \frac{\partial^2 \mathfrak{F}_t^* \omega}{\partial x^2}, \mathfrak{F}_t^* \left(\frac{\omega}{x^2} \right) &\in L^2(\Omega), \\ \frac{\partial \omega}{\partial x}(\alpha, 0) = \frac{\partial \omega}{\partial x}(\beta, 0) = \frac{\partial \omega}{\partial x}(1, 0) = 0, \quad \frac{\partial \omega}{\partial x}(1, t) = 0. \end{aligned} \quad (38)$$

In terms of the given function ω and from the equality (36), we give the function ω in terms of z as follows:

$$\omega = \begin{cases} -\left(x - \frac{x^2}{2}\right) \frac{\partial z}{\partial x}, & 0 \leq x \leq \alpha, \\ -x \frac{\partial z}{\partial x}, & \alpha \leq x \leq \beta, \\ -\left(x - \frac{x^2}{2}\right) \frac{\partial z}{\partial x}, & \beta \leq x \leq 1, \end{cases} \quad (39)$$

and z satisfies the same conditions of the function u in (PR):

$$\alpha \frac{\partial z}{\partial x}(\alpha, t) = \beta \frac{\partial z}{\partial x}(\beta, t), \quad (40)$$

$$\frac{\partial z}{\partial x}(1, t) = 0, \quad z(\alpha, t) = z(\beta, t) = 0. \quad (41)$$

Replacing ω in (36) by its representation (39) and integrating by parts each term of (36) and by taking the conditions (40) and (41), we obtain

$$\begin{aligned} &\int_{\Omega^\tau} e^{kt} z \omega dx dt \\ &= \begin{cases} -\frac{1}{2} \int_0^\tau \left(\alpha - \frac{\alpha^2}{2} \right) \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\alpha, t)}{\partial x} \right)^2 dx dt \\ -\frac{1}{2} \int_{\Omega_\alpha^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_\alpha^\tau = (0, \alpha) \times (0, \tau), \\ -\frac{\beta}{2} \int_0^\tau \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\beta, t)}{\partial x} \right)^2 dx dt \\ +\frac{\alpha}{2} \int_0^\tau \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\alpha, t)}{\partial x} \right)^2 dx dt \\ -\frac{1}{2} \int_{\Omega_{\alpha, \beta}^\tau} \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_{\alpha, \beta}^\tau = (\alpha, \beta) \times (0, \tau), \\ \frac{1}{2} \int_0^\tau \left(\beta - \frac{\beta^2}{2} \right) \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\beta, t)}{\partial x} \right)^2 dx dt \\ -\frac{1}{2} \int_{\Omega_\beta^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_\beta^\tau = (\beta, 1) \times (0, \tau). \end{cases} \end{aligned} \quad (42)$$

So, we obtain

$$\begin{aligned} &\int_{\Omega^\tau} e^{kt} z \omega dx dt \\ &= \begin{cases} -\frac{1}{2} \int_{\Omega_\alpha^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_\alpha^\tau = (0, \alpha) \times (0, \tau), \\ -\frac{1}{2} \int_{\Omega_{\alpha, \beta}^\tau} \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_{\alpha, \beta}^\tau = (\alpha, \beta) \times (0, \tau), \\ -\frac{1}{2} \int_{\Omega_\beta^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \\ \quad \Omega_\beta^\tau = (\beta, 1) \times (0, \tau). \end{cases} \end{aligned} \quad (43)$$

Now, by combining over Ω^τ , we get

$$\begin{aligned} &\frac{1}{2} \int_{\Omega_\alpha^\tau} (1-x) e^{kt} z^2 dx dt + \frac{1}{2} \int_{\Omega_{\alpha, \beta}^\tau} e^{kt} z^2 dx dt \\ &+ \frac{1}{2} \int_{\Omega_\beta^\tau} (1-x) e^{kt} z^2 dx dt \\ &= \frac{\alpha^2}{4} \int_0^\tau \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\alpha, t)}{\partial x} \right)^2 dx dt \\ &- \frac{\beta^2}{4} \int_0^\tau \mathfrak{F}_t e^{kt} \left(\frac{\partial z(\beta, t)}{\partial x} \right)^2 dx dt \\ &- \frac{1}{2} \int_{\Omega_\alpha^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt \\ &- \frac{1}{2} \int_{\Omega_{\alpha, \beta}^\tau} \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt \\ &- \frac{1}{2} \int_{\Omega_\beta^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt, \end{aligned} \quad (44)$$

by taking into account the boundary condition (40) of the function z yields

$$\begin{aligned} &\frac{1}{2} \left(\int_{\Omega_\alpha^\tau} (1-x) e^{kt} z^2 dx dt + \int_{\Omega_{\alpha, \beta}^\tau} e^{kt} z^2 dx dt \right. \\ &\quad \left. + \int_{\Omega_\beta^\tau} (1-x) e^{kt} z^2 dx dt \right) \\ &= -\frac{1}{2} \left(\int_{\Omega_\alpha^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt + \int_{\Omega_{\alpha, \beta}^\tau} \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt \right. \\ &\quad \left. + \int_{\Omega_\beta^\tau} (1-x) \mathfrak{F}_t e^{kt} \left(\frac{\partial z}{\partial x} \right)^2 dx dt \right) \\ &\leq 0. \end{aligned} \quad (45)$$

Then, we obtain

$$\begin{aligned} & \frac{1}{2} \left(\int_{\Omega_{\alpha}^{\tau}} (1-x) e^{kt} z^2 dx dt + \int_{\Omega_{\alpha,\beta}^{\tau}} e^{kt} z^2 dx dt \right. \\ & \quad \left. + \int_{\Omega_{\beta}^{\tau}} (1-x) e^{kt} z^2 dx dt \right) \\ & \leq 0. \end{aligned} \quad (46)$$

And thus $z = 0$ in Ω , then $\omega = 0$ in Ω . This proves Proposition 5. $\square$

We return to the proof of Theorem 4. We have already noted that it is sufficient to prove that the set $R(L)$ dense in F .

Suppose that for some $W = (\omega, \omega_0) \in R(L)^{\perp}$ and for all $u \in D(L) \equiv B$, it holds

$$\begin{aligned} & (Lu, \omega)_F \\ & = \int_{\Omega} \mathcal{L}u \cdot \omega dx dt + \int_0^1 (x+1) \left(\frac{\partial \ell u}{\partial x} \right) \left(\frac{\partial \omega_0}{\partial x} \right) dx. \end{aligned} \quad (47)$$

Then we must prove that $W = 0$. Putting $u \in D_0(L)$ in (47), we have

$$\int_{\Omega} \mathcal{L}u \cdot \omega dx dt = 0, \quad u \in D_0(L). \quad (48)$$

Hence Proposition 5 implies that $\omega = 0$. Thus (47) takes the form

$$\int_0^1 (x+1) \left(\frac{\partial \ell u}{\partial x} \right) \left(\frac{\partial \omega_0}{\partial x} \right) dx = 0, \quad u \in D(L). \quad (49)$$

Since the range of the trace operator ℓ is everywhere dense in the Hilbert space F with the norm

$$\left(\int_0^1 \left[(x+1) \left(\frac{\partial \ell u}{\partial x} \right)^2 \right] \right)^{1/2}, \quad (50)$$

the equality (49) implies that $\omega_0 = 0$. Hence $W = 0$ implies $(R(L) = F)$. Therefore, the proof of Theorem 4 is complete.

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