Research Article

On the Finite Volume Element Method for Self-Adjoint Parabolic Integrodifferential Equations

Mohamed Bahaj¹ and Anas Rachid²

¹ *Department of Mathematics and Computing Science, Faculty of Sciences and Technology, Hassan 1st University, BP 577 Settat, Morocco*

² *École Nationale Supérieure d'Arts et Métiers-Casablanca, Université Hassan II Mohammedia-Casablanca, BP 150 Mohammedia, Morocco*

Correspondence should be addressed to Anas Rachid; rachid.anas@gmail.com

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Finite volume element schemes for non-self-adjoint parabolic integrodifferential equations are derived and stated. For the spatially discrete scheme, optimal-order error estimates in L^2 , H^1 , and L^p , $W^{1,p}$ norms for $2 \leq p < \infty$, are obtained. In this paper, we also study the lumped mass modification. Based on the Crank-Nicolson method, a time discretization scheme is discussed and related error estimates are derived.

1. Introduction

The main purpose of this paper is to study semidiscrete and full discrete finite volume element method (FVE) for parabolic integrodifferential equation of the form

$$\begin{aligned} u_t - \nabla \cdot (A(x, t) \nabla u) - \int_0^t \nabla \cdot (B(x, t, s) \nabla u(s)) ds \\ = f(x, t), \quad \text{in } \Omega \times (0, T), \\ u = 0, \quad \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, \quad \text{in } \Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\Omega$, and $T < \infty$. Here $A(t)$, a non-self-adjoint second-order strongly elliptic, and $B(t, s)$, an arbitrary second-order linear partial differential operator, both with coefficients depending smoothly on x and t , $f = f(x, t)$ and $u_0(x)$ are known functions, which are assumed to be smooth and satisfy certain compatibility conditions for $x \in \Omega$ and $t = 0$, so that (1) has a unique solution in certain Sobolev space. Problem (1) occurs in nonlocal reactive flows

in porous media, viscoelasticity, and heat conduction through materials with memory.

Finite volume method is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer, and petroleum engineering. The method can be formulated in the finite difference framework or in the Petrov-Galerkin framework. Usually, the former one is called finite volume method [1], marker and cell (MAC) method [2], or cell-centered method [3], and the latter one is called finite volume element method (FVE) [4–9], covolume method [10], or vertex-centered method [11, 12]. We refer to the monographs [13, 14] for general presentation of these methods. The most important property of FVE method is that it can preserve the conservation laws (mass, momentum, and heat flux) on each control volume. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field.

Recently, the authors in [8, 15] studied FVE method for general self-adjoint elliptic problems. The authors in [16] presented and analyzed the semidiscrete and full discrete symmetric finite volume schemes for a class of parabolic problems. In [6, 7] the authors have studied FVE for one- and

two-dimensional parabolic integrodifferential equations and have obtained an optimal-order estimate in the L^2 -norm. The regularity required on the exact solution u is $W^{3,p}$ for $p > 1$ which is higher when compared to that for finite element methods.

The aim of this paper is to study the convergence of FVE discretization for a nonself-adjoint parabolic integrodifferential problem (1). Both spatially discrete scheme and discrete-in-time scheme are analyzed, and optimal error estimates in L^2 and H^1 norms are proved using only energy method. We also explore and generalize that idea to develop the lumped mass modification and L^p estimates, $2 \leq p < \infty$. Our analysis avoids the use of semigroup theory, and the regularity requirement on the solution is the same of that of finite element method. Furthermore, based on the Crank-Nicolson method the fully discrete scheme is analyzed and the related optimal error estimates are established.

This paper is organized as follows. In Section 2, we introduce some notations and present some preliminary materials to be used later. The Ritz-Volterra projection to finite volume element spaces is introduced and related estimates are carried out in Section 3. In Section 4, we estimate the error of the finite volume element approximations derived in the previous section. In Section 5, the lumped mass is presented and optimal estimates in L^2 and H^1 norms are obtained. Finally, the Crank-Nicolson scheme is studied in Section 6.

2. Finite Volume Element Scheme

In this section, we introduce some material which will be used repeatedly hereafter. Throughout this paper, C (with or without index) denotes a generic positive constant which does not depend on the spatial and time discretization parameters h and k , respectively.

2.1. Notations. We will use $\|\cdot\|_m$ and $|\cdot|_m$ (resp., $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$) to denote the norm and seminorm of the Sobolev space $H^m(\Omega)$ (resp., $W^{m,p}(\Omega)$). The scalar product and norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let $H_1^0(\Omega)$ be the standard Sobolev subspace of $H^1(\Omega)$ of functions vanishing on $\partial\Omega$.

The weak form of (1) is used to find $u(\cdot, t) : [0, T] \rightarrow H_1^0(\Omega)$, such that

$$\begin{aligned} (u_t, v) + A(t; u, v) + \int_0^t B(t, s; u(s), v) ds \\ = (f, v), \quad \forall v \in H_1^0(\Omega), \end{aligned} \quad (2)$$

$$u(0) = u_0,$$

where

$$\begin{aligned} A(t; u, v) &= \int_{\Omega} A(x, t) \nabla u \cdot \nabla v, \\ B(t, s; u(s), v) &= \int_{\Omega} B(x, t, s) \nabla u(s) \cdot \nabla v. \end{aligned} \quad (3)$$

Let $\mathcal{T}_h$ be a decomposition of Ω into triangles (for the 2D case) or tetrahedral (for the 3D case) with $h = \max h_K$, where h_K is the diameter of the element $K \in \mathcal{T}_h$.

In order to describe the FVE method for solving problem (1), we will introduce a dual partition $\mathcal{T}_h^*$ based upon the original partition $\mathcal{T}_h$ whose elements are called control volumes. We construct the control volumes in the same way as in [7, 17]. Let z_K be a point of $K \in \mathcal{T}_h$. In the 2D case, on each edge e of K a point q_e is selected; then we connect z_K with line segments to q_e ; thus, partitioning K into three quadrilaterals K_z , $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K . Then with each vertex $z \in Z_h = \cup_{K \in \mathcal{T}_h} Z_h(K)$ we associate a control volume V_z , which consists of the union of the subregions K_z , sharing the vertex z (see Figure 1).

Similarly, in the 3D case, on each of the four faces S_i , $i = 1, \dots, 4$, a point q_{S_i} , $i = 1, \dots, 4$, is selected, and on each of the six edges e a point q_e is selected. On each of the two faces S_1 and S_2 of K sharing an edge e , we connect q_{S_i} , $i = 1, 2$, with q_e and with z_K by line segments, thus, partitioning K into twelve tetrahedron K_z , $z \in Z_h(K)$ (see Figure 2). Then for $z \in Z_h$ the control volume V_z consists of the union of the subregions K_z sharing the vertex z . Thus, we finally obtain a group of control volumes covering the domain Ω , which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$. We denote by Z_h^0 the set of interior vertices and $N_h = \#Z_h^0$. For a vertex $z_i \in Z_h^0$, let $\Pi(i)$ be the index set of those vertices that along with z_i are in some element of T_h (Figure 2).

There are various ways to introduce a regular dual partition $\mathcal{T}_h^*$. In this paper, we will also use the construction of the control volumes in which we let z_K be the barycenter of $K \in \mathcal{T}_h$. In the 2D case, we choose q_e to be the midpoint of the edge e (see Figure 3).

In the 3D case, we choose q_e to be the midpoint of the edge e and q_{S_i} to be the barycenter of the face S_i (Figure 4).

We call the partition $\mathcal{T}_h^*$ regular or quasiuniform, if there exists a positive $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(V_z) \leq Ch^2, \quad \forall V_z \in \mathcal{T}_h^*. \quad (4)$$

If the finite element triangulation $\mathcal{T}_h$ is quasiuniform, that is, there exists a positive $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(K) \leq Ch^2, \quad \forall K \in \mathcal{T}_h, \quad (5)$$

then the dual partition $\mathcal{T}_h^*$ is also quasiuniform.

Based on the triangulation T_h , let S_h be the standard conforming finite element space of piecewise linear functions, defined on the triangulation T_h as follows:

$$S_h = \{v \in \mathcal{C}(\Omega) : v|_K \text{ is linear } \forall K \in T_h, \text{ and } v|_{\Gamma} = 0\}. \quad (6)$$

Let $I_h : \mathcal{C}(\Omega) \rightarrow S_h$ be the standard interpolation operators, such that

$$I_h u = \sum_{z \in Z_h^0} v_z(t) \varphi_z(x), \quad \forall v \in S_h, \quad (7)$$

where $\{\varphi_z\}_{z \in Z_h^0}$ are the standard basis functions of S_h and $v_z(t) = v(t; z)$.

FIGURE 1: (a) A sample region with blue lines indicating the corresponding control volume V_z . (b) A triangle K partitioned into three subregions K_z .

FIGURE 2: A tetrahedron K partitioned into twelve subregions K_z .

2.2. Construction of the FVE Scheme. We formulate the FVE method for the problem (1) as follows: Given a $z \in Z_h^0$, integrating (1)¹ over the associated control volume V_z and applying Green's formula, we obtain an integral conservation as follows form:

$$\begin{aligned} \int_{V_z} u_t - \int_{\partial V_z} A(x, t) \nabla u \cdot n ds - \int_{\partial V_z} B(x, t, s) \nabla u \cdot n ds \\ = \int_{V_z} f(x, t), \end{aligned} \quad (8)$$

where n denotes the unit outer normal vector to ∂V_z .

Let $I_h^* : \mathcal{C}(\Omega) \rightarrow S_h^*$ be the transfer operator defined by

$$I_h^* v = \sum_{z \in Z_h^0} v(z) \chi_z, \quad \forall v \in S_h, \quad (9)$$

where

$$S_h^* = \{v \in L^2(\Omega) : v|_{V_z} \text{ is constant}, \forall z \in Z_h^0\}, \quad (10)$$

and χ_z is the characteristic function of the control volume V_z .

Now for $t > 0$ and for an arbitrary $I_h^* v$, we multiply (8) by $v(z)$, and sum over all $z \in Z_h^0$. Then the semidiscrete FVE approximation u_h of (1) is a solution to the following problem: find $u_h(t) \in S_h$ for $t > 0$ such that

$$\begin{aligned} (u_{ht}, v_h) + A(t; u_h, v_h) + \int_0^t B(t, s; u_h(s), v_h) ds \\ = (f, v_h) \quad v_h \in S_h^*, \\ u_h(0) = u_{0h} \in S_h. \end{aligned} \quad (11)$$

Here the bilinear forms $A(t; u, v)$ and $B(t, s; u, v)$ are defined by

$$\begin{aligned} A(t; u, v) &= \begin{cases} -\sum_{z \in Z_h^0} v_i \int_{\partial V_z} A(x, t) \nabla u \cdot n ds, & (u, v) \in ((H_0^1 \cap H^2) \cup S_h) \times S_h^*, \\ \int_{\Omega} A(x) \nabla u \cdot \nabla v dx, & (u, v) \in H_0^1 \times H_0^1, \end{cases} \\ B(t, s; u, v) &= \begin{cases} -\sum_{z \in Z_h^0} v_z \int_{\partial V_z} B(x, t, s) \nabla u \cdot n ds, & (u, v) \in ((H_0^1 \cap H^2) \cup S_h) \times S_h^*, \\ \int_{\Omega} B(x, t, s) \nabla u \cdot \nabla v dx, & (u, v) \in H_0^1 \times H_0^1. \end{cases} \end{aligned} \quad (12)$$

Let

$$u_h = \sum_{j=1}^{N_h} \alpha_z(t) \varphi_z(x), \quad (13)$$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_{N_h}(t))^T.$$

Then, we can rewrite scheme (11)¹ as systems of ordinary differential equations as follows:

$$M_h \alpha'(t) + A_h(t) \alpha(t) + \int_0^t B_h(t, s) \alpha(s) ds = F_h(t). \quad (14)$$

FIGURE 3: z_K is the barycenter of K , and q_e is to be the midpoint of the edge e .

FIGURE 4: q_e is the midpoint of the edge e , and q_{S_i} is the barycenter of the face S_i .

Here $F_h(t) = (f_1(t), f_2(t), \dots, f_{N_h}(t))^T$, the mass matrix $M_h = \{M_{h_{ij}}\} = \{(\varphi_i, \chi_j)\}$ is tridiagonal, and both $A_h(t) = \{A(t; \varphi_i, \chi_j)\}$ and $B_h(t, s) = \{B(t, s; \varphi_i, \chi_j)\}$ are positive definites.

In order to describe features of the bilinear forms defined in (11), we introduce some discrete norms on S_h in the same way as in [7]:

$$\begin{aligned} \|v_h\|_{0,h}^2 &= (v_h, v_h)_{0,h} = (I_h^* v_h, I_h^* v_h), \\ |v_h|_{1,h}^2 &= \sum_{x_i \in Z_h^0} \sum_{x_j \in \Pi(i)} \text{meas}(V_i) \left(\frac{v_i - v_j}{d_{ij}} \right)^2, \end{aligned} \quad (15)$$

$$\|v_h\|_{1,h}^2 = \|v_h\|_{0,h}^2 + |v_h|_{1,h}^2, \quad |||v_h|||^2 = (v_h, I_h^* v_h),$$

where $d_{ij} = d(x_i, x_j)$, the distance between x_i and x_j . Obviously, these norms are well defined for $v_h \in S_h$ as well and $\|v_h\|_{0,h} = |||v_h|||$.

Hereafter we state the equivalence of discrete norms $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ with usual norms $\|\cdot\|$ and $\|\cdot\|_1$ on S_h , respectively.

Lemma 1 (see [7]). *There exist two positive constants C_0 and C_1 such that for all $v_h \in S_h$, we have*

$$\begin{aligned} C_0 \|v_h\|_{0,h} &\leq \|v_h\| \leq C_1 \|v_h\|_{0,h}, \quad \forall v_h \in S_h, \\ C_0 |||v_h||| &\leq \|v_h\| \leq C_1 |||v_h|||, \quad \forall v_h \in S_h, \\ C_0 \|v_h\|_{1,h} &\leq \|v_h\|_1 \leq C_1 \|v_h\|_{1,h}, \quad \forall v_h \in S_h. \end{aligned} \quad (16)$$

Next we recall some properties of the bilinear forms (see [7, 18])

Lemma 2 (see [7]). *There exist two positive constants C and C_0 such that for all $u_h, v_h \in S_h$, we have*

$$\begin{aligned} A(u_h, I_h^* v_h) &\leq C \|u_h\|_1 \|v_h\|_1, \quad \forall u_h, v_h \in S_h, \\ A(v_h, I_h^* v_h) &\geq C_0 \|v_h\|_1^2, \quad \forall v_h \in S_h. \end{aligned} \quad (17)$$

The following lemmas are proved in [3, 7], which give the key feature of the bilinear forms in the FVE method.

Lemma 3 (see [3]). *Assume that $\varphi \in W_0^{1,p}$. Then, one has*

$$\begin{aligned} &A(t; \varphi, v_h) - A(t; \varphi, I_h^* v_h) \\ &= \sum_{K \in \tau_h} \int_{\partial K} (A(t) \nabla \varphi \cdot \mathbf{n}) (v_h - I_h^* v_h) ds \\ &\quad - \sum_{K \in \tau_h} \int_K (\nabla \cdot A(t) \nabla \varphi) (v_h - I_h^* v_h) ds, \quad \forall v_h \in S_h. \end{aligned} \quad (18)$$

The aforementioned identity holds true when $A(\cdot, \cdot)$ is replaced by $B(t, s; \cdot, \cdot)$.

Lemma 4 (see [3]). *Assume that $\varphi \in S_h$. Then, one has*

$$A(t; \varphi, \chi) - A(t; \varphi, I_h^* \chi) \leq Ch |\varphi|_{1,p} |\chi|_{1,q}. \quad (19)$$

Furthermore, for $\varphi \in W_0^{1,p} \cap W^{2,p}$, we have

$$A(t; \varphi, \chi) - A(t; \varphi, I_h^* \chi) \leq Ch \|\varphi\|_{2,p} \|\chi\|_{1,q}. \quad (20)$$

3. Ritz-Volterra Projection and Related Estimates

Following [7, 19, 20], we define the Ritz-Volterra projection $V_h(t) : H_0^1 \rightarrow S_h$ as follows:

$$A(t; u - V_h u, I_h^* v_h) + \int_0^t B(t, s; u(s) - V_h u(s), I_h^* v_h) ds = 0, \quad t > 0, \quad \forall v_h \in S_h. \quad (21)$$

This $V_h(t)$ is an elliptic projection with memory of u into S_h^* . It is easy to see that (21) is actually a system of integral equations of Volterra type. In fact, if $V_h(t)u = \sum_{j=1}^{N_h} \alpha_j(t) \varphi_j(x)$, then (21) can be rewritten as

$$A_h(t) \alpha(t) + \int_0^t B_h(t, s) \alpha(s) ds = F_h(t), \quad (22)$$

where $A_h(t)$, $B_h(t, s)$ are matrices and $\alpha(t)$, $F_h(t)$ are vectors, defined via

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_{N_h}(t))^T,$$

$$F_{hk}(t) = A(t; u, \chi_k) + \int_0^t B(t, s; u(s), \chi_k) ds,$$

$$k = 1, 2, \dots, N_h,$$

$$A_h(t) = A(t, \varphi_k(x), \chi_l), \quad B_h(t, s) = B(t, s; \varphi_k(x), \chi_l). \quad (23)$$

From the positivity of A (Lemma 2) and the linearity of (22), we see that the system (22) possesses a unique solution $\alpha(t)$. Consequently, $V_h(t)u$ in (21) is well defined.

Set $\rho = u - V_h(t)u$. The following lemma was proved in [7], which shows the H^1 error estimate for ρ and its temporal derivative.

Lemma 5 (see [7]). *Assume that $D_t^n u \in L^\infty(H_0^1 \cap H^2)$ for all $0 \leq n \leq k$, for some integer $k \geq 0$. Then, for $T > 0$ fixed there is a constant $C = C(T; k) > 0$, independent of h and u , such that for all $0 \leq n \leq k$ and $0 < t < T$,*

$$\begin{aligned} \|\rho(t)\|_1 &\leq Ch \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right), \\ \|D_t^n \rho(t)\|_1 &\leq Ch \left(\sum_{i=0}^n \|D_t^i u\|_2 + \int_0^t \|u\|_2 ds \right). \end{aligned} \quad (24)$$

Now we establish L^2 error estimate for ρ and its temporal derivative which improves Theorem 2.2 in [7]. This estimate is optimal with respect to the order.

Lemma 6. *Assume that, for some integer $k \geq 0$, $D_t^n u \in L^\infty(H_0^1 \cap H^2)$ for all $0 \leq n \leq k$. Then, for $T > 0$ fixed there is*

a constant $C = C(T; k) > 0$, independent of h and u , such that for all $0 \leq n \leq k$ and $0 < t < T$,

$$\|\rho(t)\| \leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right), \quad (25)$$

$$\|D_t^n \rho(t)\| \leq Ch^2 \left(\sum_{i=0}^n \|D_t^i u\|_2 + \int_0^t \|u\|_2 ds \right).$$

Proof. The proof will proceed by duality argument. Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned} A^*(t) \psi &= \rho, \quad \text{in } \Omega \\ \psi &= 0, \quad \text{in } \partial\Omega. \end{aligned} \quad (26)$$

The solution $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the following regularity estimate:

$$\|\psi\|_2 \leq C \|\rho\|. \quad (27)$$

Multiplying this equation by ρ and then taking L^2 innerproduct over Ω , we obtain the following:

$$\begin{aligned} \|\rho\|^2 &= A(t; \rho, \psi) \\ &= A(t; \rho, \psi - R_h \psi) + A(t; \rho, R_h \psi - I_h^*(R_h \psi)) \\ &\quad - \int_0^t B(t, s; \rho(s), I_h^* R_h \psi - R_h \psi) ds \\ &\quad - \int_0^t B(t, s; \rho(s), R_h \psi - \psi) ds \\ &\quad - \int_0^t B(t, s; \rho(s), \psi) ds = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (28)$$

We have

$$|I_1| + |I_4| \leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) \|\psi\|_2. \quad (29)$$

Applying Lemma 4, we obtain

$$|I_2| + |I_3| \leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) \|\psi\|_2. \quad (30)$$

Finally, we have

$$|I_5| \leq \int_0^t (\rho(s), B^*(t, s) \psi) ds \leq C \left(\int_0^t \|\rho\| ds \right) \|\psi\|_2; \quad (31)$$

then, we have

$$\|\rho\| \leq Ch^2 \left(\|u\|_2 + \int_0^t \|u\|_2 ds \right) + C \left(\int_0^t \|\rho\| ds \right). \quad (32)$$

Finally, an application of Gronwall's lemma yields the first estimate.

The second inequality follows in a similar fashion. $\square$

Lemma 7. *There exists a constant C independent of h such that*

$$\|\rho\|_{0,p} + h\|\rho\|_{1,p} \leq Ch^2 \left(\|u\|_{2,p} + \int_0^t \|u\|_{2,p} ds \right). \quad (33)$$

Proof. Let ρ_x be an arbitrary component of $\nabla \rho$, with p and q conjugate indices; we have $\|\rho_x\|_p = \sup\{(\rho_x, \varphi); \varphi \in \mathcal{C}_0^\infty(\Omega), \|\varphi\|_q = 1\}$.

For any such φ , let ψ be the solution of

$$\begin{aligned} A^*(t; \psi, v) &= -(\varphi_x, v) \quad \forall v \in H_0^1(\Omega), \\ \psi &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (34)$$

It follows from the regularity theory for the elliptic problem that

$$\|\psi\|_{1,q} \leq C_p \|\varphi\|_q = C_p. \quad (35)$$

We then have by application of (21) that

$$\begin{aligned} (\rho_x, \varphi) &= A(t; \rho, \psi) = A(t; \rho, \psi - R_h \psi) \\ &\quad + A(t; \rho, R_h \psi - I_h^*(R_h \psi)) \\ &\quad + \int_0^t B(t, s; \rho(s), I_h^*(R_h \psi)) ds \\ &= I_1 + I_2 + I_3, \\ A(t; \rho, \psi - R_h \psi) &= A(t; R_h u - u, \psi) \\ &= -((R_h u - u)_x, \varphi) \leq Ch \|u\|_{2,p}. \end{aligned} \quad (36)$$

Applying Lemma 4, we have

$$\begin{aligned} I_2 &= A(t; u, R_h \psi - I_h^*(R_h \psi)) - A(t; V_h u, R_h \psi - I_h^*(R_h \psi)) \\ &\leq Ch \|u\|_{2,p}. \end{aligned} \quad (37)$$

Finally, I_3 is estimated as follows:

$$I_3 = \int_0^t B(t, s; \rho(s), I_h^*(R_h \psi)) ds \leq C_p \int_0^t \|\rho\|_{1,p} ds. \quad (38)$$

Combining these estimates, we get

$$\|\rho\|_{1,p} \leq Ch \|u\|_{2,p} + C_p \int_0^t \|\rho\|_{1,p} ds, \quad (39)$$

hence by Gronwall's lemma

$$\|\rho\|_{1,p} \leq Ch \left(\|u\|_{2,p} + \int_0^t \|u\|_{2,p} ds \right). \quad (40)$$

The derivation of the error estimate in L^p is similar to the case when $p = 2$. $\square$

4. Error Estimates for Semidiscrete Approximations

We split the error $e(t) = u(t) - u_h(t)$ as follows:

$$e(t) = (u(t) - V_h u(t)) + (V_h u(t) - u_h(t)) = \rho + \theta. \quad (41)$$

It is easy to see that $\theta = V_h u(t) - u_h(t) \in S_h$ satisfies an error equation of the form

$$\begin{aligned} (\theta_t, I_h^* v_h) + A(t; \theta, I_h^* v_h) + \int_0^t B(t, s; \theta(s), I_h^* v_h) ds \\ = -(\rho_t, I_h^* v_h), \quad v_h \in S_h. \end{aligned} \quad (42)$$

Since the estimates of ρ are already known, it is enough to have estimates for θ .

We will prove a sequence of lemmas which lead to the following result.

Lemma 8. *There is a positive constant C independent of h such that*

$$|||\theta(t)||| \leq C \left(|||\theta(0)|||^2 + \int_0^t \|\rho_t\| ds \right). \quad (43)$$

Proof. Since $\theta \in S_h$ we may take $v_h = \theta$ in (42) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |||\theta(t)|||^2 + c \|\theta\|_1^2 &\leq \|\rho_t\| \|\theta\| + C \int_0^t \|\theta\|_1 ds \|\theta\|_1 \\ &\leq \|\rho_t\| \|\theta\| + \frac{1}{2} c \|\theta\|_1^2 + C \int_0^t \|\theta\|_1^2 ds, \end{aligned} \quad (44)$$

and hence by integration and Lemma 1, we have

$$\begin{aligned} |||\theta(t)|||^2 + \int_0^t \|\theta\|_1^2 ds \\ \leq C \left(|||\theta(0)|||^2 + \int_0^t \|\rho_t\| \|\theta\| ds + \int_0^t \int_0^s \|\theta(\tau)\|_1^2 d\tau ds \right). \end{aligned} \quad (45)$$

Gronwall's lemma now implies the following:

$$\begin{aligned} |||\theta(t)|||^2 + \int_0^t \|\theta\|_1^2 ds &\leq C \left(|||\theta(0)|||^2 + \int_0^t \|\rho_t\| \|\theta\| ds \right) \\ &\leq C |||\theta(0)|||^2 + \frac{1}{2} \sup_{s \leq t} \|\theta(s)\|^2 \\ &\quad + \left(\int_0^t \|\rho_t\| ds \right)^2. \end{aligned} \quad (46)$$

Since this holds for all $t \in J$, we may conclude that

$$|||\theta(t)||| \leq C \left(|||\theta(0)||| + \int_0^t \|\rho_t\| ds \right). \quad (47)$$

$\square$

Remark 9. If the initial value was chosen so that $\|u_{0h} - u_0\| \leq Ch^2 \|u_0\|_2$, then $\|\theta(0)\| \leq \|u_{0h} - u_0\| + \|V_h u_0 - u_0\| \leq Ch^2 \|u_0\|_2$. One can derive

$$\|\theta(t)\| \leq Ch^2 \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right). \quad (48)$$

Lemma 10. There is a positive constant C independent of h such that

$$\int_0^t \|\theta_t\|^2 ds + \|\theta\|_1^2 \leq C \left(\|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 ds \right). \quad (49)$$

Proof. Set $v_h = \theta_t$ in (42) to get

$$\begin{aligned} & \|\theta_t\|^2 + \frac{1}{2} \frac{d}{dt} A(t; \theta, I_h^* \theta) \\ &= -(\rho_t, I_h^* \theta_t) - \int_0^t B(t, s; \theta(s), I_h^* \theta_t(t)) ds \\ & \quad + \frac{1}{2} A_t(t; \theta, I_h^* \theta) \\ & \quad + \frac{1}{2} [A(t; \theta_t, I_h^* \theta) - A(t; \theta, I_h^* \theta_t)] \\ &\leq \frac{1}{2} \|\rho_t\|^2 + \frac{1}{2} \|\theta_t\|^2 + A_t(t; \theta, I_h^* \theta) \\ & \quad + \frac{1}{2} [A(t; \theta_t, I_h^* \theta) - A(t; \theta, I_h^* \theta_t)] \\ & \quad - \frac{d}{dt} \int_0^t B(t, s; \theta(s), I_h^* \theta(t)) ds \\ & \quad + B(t, t; \theta(t), I_h^* \theta(t)) + \int_0^t B_t(t, s; \theta(s), I_h^* \theta(t)) ds. \end{aligned} \quad (50)$$

Then

$$\begin{aligned} & \|\theta_t\|^2 + \frac{d}{dt} A(t; \theta, I_h^* \theta) \\ &\leq \|\rho_t\|^2 - \frac{d}{dt} \int_0^t B(t, s; \theta(s), I_h^* \theta) ds \\ & \quad + C \left(\|\theta\|_1^2 + \int_0^t \|\theta(s)\|_1^2 ds \right) \\ & \quad + \frac{1}{2} [A(t; \theta_t, I_h^* \theta) - A(t; \theta, I_h^* \theta_t)]. \end{aligned} \quad (51)$$

In addition, recall that

$$A(t; u_h, I_h^* v_h) - A(t; v_h, I_h^* u_h) \leq Ch \|u_h\|_1 \|v_h\|_1, \quad (52)$$

$$\forall u_h, v_h \in S_h;$$

then applying an inverse inequality and using kickback argument, we obtain

$$\begin{aligned} [A(t; \theta_t, I_h^* \theta) - A(t; \theta, I_h^* \theta_t)] &\leq Ch \|\theta_t\|_1 \|\theta\|_1 \leq C \|\theta_t\| \|\theta\|_1 \\ &\leq \varepsilon \|\theta_t\|^2 + C \|\theta\|_1^2. \end{aligned} \quad (53)$$

Combining these estimates, we derive

$$\begin{aligned} & \|\theta_t\|^2 + \frac{d}{dt} A(t; \theta, I_h^* \theta) \\ &\leq \|\rho_t\|^2 - \frac{d}{dt} \int_0^t B(t, s; \theta(s), I_h^* \theta) ds \\ & \quad + C \left(\|\theta\|_1^2 + \int_0^t \|\theta(s)\|_1^2 ds \right). \end{aligned} \quad (54)$$

So after integration in time and using the weak coercivity of $A(t; \theta, I_h^* \theta)$, we get

$$\begin{aligned} & \int_0^t \|\theta_t\|^2 ds + c_0 \|\theta\|_1^2 \\ &\leq c_0 \|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 ds \\ & \quad + \int_0^t B(t, s; \theta(s), I_h^* \theta) ds + C \int_0^t \|\theta(s)\|_1^2 ds \\ &\leq c_0 \|\theta(0)\|_1^2 + \frac{c}{2} \|\theta\|_1^2 + C \left(\int_0^t \|\rho_t\|^2 ds + \|\theta(s)\|_1^2 ds \right), \end{aligned} \quad (55)$$

and by Gronwall's lemma,

$$\int_0^t \|\theta_t\|^2 ds + c \|\theta\|_1^2 \leq C \left(\|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 ds \right). \quad (56)$$

□

Remark 11. If $\theta(0) = 0$, then

$$\int_0^t \|\theta_t\|^2 ds + c \|\theta\|_1^2 \leq Ch^2 \left(\int_0^t \|u_t\|_2^2 ds \right). \quad (57)$$

Theorem 12 (error estimates in L^2 and H^1 -norms). Let u, u_h be the solutions of (2) and (11), respectively. Assume that $u, u_t \in L^\infty(H_0^1 \cap H^2)$.

- (a) Let u_{0h} be chosen so that $\|u_{0h} - u_0\| \leq Ch^2 \|u_0\|_2$. Then for $T > 0$ fixed there is a constant $C = C(T)$ independent of h , such that for all $0 < t < T$,

$$\|u_h(t) - u(t)\| \leq Ch^2 \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right). \quad (58)$$

- (b) Let u_{0h} be chosen so that $\|u_{0h} - u_0\|_1 \leq Ch \|u_0\|_2$. Then for $T > 0$ fixed there is a constant $C = C(T)$ independent of h , such that for all $0 < t < T$,

$$\|u_h(t) - u(t)\|_1 \leq Ch \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right). \quad (59)$$

We now prove error estimates for FVE approximations in L^p and $W^{1,p}$ -norms.

Theorem 13 (error estimates in L^p and $W^{1,p}$ -norms). Let u, u_h be the solutions of (2) and (11), respectively and $u_{0h} = V_h u_0$. Assume that $u, u_t \in L^\infty(H_0^1 \cap W^{2,p})$. For h sufficiently small, we have

$$\|u - u_h\|_{0,p} \leq Ch^2 \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right), \quad (60)$$

$$\|u - u_h\|_{1,p} \leq Ch \left(\|u_0\|_2 + \|u\|_{2,p} + \int_0^t \|u_t\|_2 ds \right).$$

Proof. If $2 \leq p < \infty$, by the following Sobolev embedding inequality

$$\|\theta\|_{0,p} \leq C\|\theta\|_1, \quad (61)$$

then the first desired estimate follows from Lemmas 7 and 10. Given $\varphi \in \mathcal{C}_0^\infty(\Omega)$, find $\psi \in H_0^1(\Omega)$ such that

$$\begin{aligned} A(t)^* \psi &= -\varphi_x, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \partial\Omega, \\ \|\psi\|_{1,q} &\leq \|\varphi\|_{0,q}. \end{aligned} \quad (62)$$

We have

$$\begin{aligned} ((u - u_h)_x, \varphi) &= A(t; u - u_h, \psi) = A(t; u - u_h, \psi - R_h \psi) \\ &\quad + A(t; u - u_h, R_h \psi - I_h^* R_h \psi) \\ &\quad - \int_0^t B(t, s; (u - u_h)(s), I_h^* R_h \psi) ds \\ &\quad - ((u - u_h)_t, I_h^* R_h \psi) \\ &= I_1 + I_2 + I_3 + I_4, \\ |I_1| &\leq |A(t; u - R_h u, \psi)| \leq C \|u - R_h u\|_{1,p} \|\psi\|_{1,q} \\ &\leq Ch \|u\|_{2,p} \|\psi\|_{1,q}. \end{aligned} \quad (63)$$

By Lemma 4

$$\begin{aligned} |I_2| &\leq A(t; u - u_h, R_h \psi - I_h^* R_h \psi) \\ &\leq Ch \left(\|u - u_h\|_{1,p} + \|u\|_{2,p} \right) \|\psi\|_{1,q}, \\ |I_3| &\leq \int_0^t \|u - u_h\|_{1,p} ds \|\psi\|_{1,q}, \\ |I_4| &\leq (\|u - u_h\|) \|\psi\| \leq Ch^2 \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right) \|\psi\|_{1,q}, \end{aligned} \quad (64)$$

where we have used the fact $\|\psi\| \leq \|\psi\|_{1,r}$, $r > 1$. Combining these estimates, we get

$$\begin{aligned} |((u - u_h)_x, \varphi)| &\leq Ch \left(\|u_0\|_2 + \|u\|_{2,p} + \int_0^t \|u_t\|_2 ds \right) \|\psi\|_{1,q}, \\ \|(u - u_h)_x\|_{0,p} &= \sup \frac{|((u - u_h)_x, \varphi)|}{\|\varphi\|_{0,q}} \\ &\leq Ch \|u - u_h\|_{1,p} \\ &\quad + Ch \left(\|u_0\|_2 + \|u\|_{2,p} + \int_0^t \|u_t\|_2 ds \right). \end{aligned} \quad (65)$$

Hence using the Poincaré inequality, we have for h sufficiently small

$$\|u - u_h\|_{1,p} \leq Ch \left(\|u_0\|_2 + \|u\|_{2,p} + \int_0^t \|u_t\|_2 ds \right). \quad (66)$$

□

We compare the relationship between covolume solution and the Galerkin finite element solution.

Corollary 14. Let $\tilde{u}_h$ be the finite element solution to (2); that is,

$$\begin{aligned} (\tilde{u}_h, v_h) + A(t; \tilde{u}_h, v_h) \\ + \int_0^t B(t, s; \tilde{u}_h(s), v_h) ds &= (f, v_h), \quad v_h \in S_h, \\ \tilde{u}_h(0) &= R_h u_0. \end{aligned} \quad (67)$$

For h sufficiently small, we have

$$\begin{aligned} \|(\tilde{u}_h - u_h)\|_{1,p} \\ \leq C \left(h \|u - u_h\|_{1,p} + \|(u - u_h)_t\| + \|(\tilde{u}_h - u)_t\| \right) \\ + \int_0^t (\|(u - u_h)(s)\|_{1,p} + \|(\tilde{u}_h - u)(s)\|_{1,p}) ds \\ \leq C(u)h. \end{aligned} \quad (68)$$

Proof. By (2) and (67),

$$\begin{aligned} ((\tilde{u}_h - u)_t, v_h) + A(t; \tilde{u}_h - u, v_h) \\ + \int_0^t B(t, s; (\tilde{u}_h - u)(s), v_h) ds &= 0, \quad v_h \in S_h. \end{aligned} \quad (69)$$

Consider the following auxiliary problem. For any such φ , let ψ be the solution of the following:

$$\begin{aligned} A(t)^* \psi &= -\varphi_x, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (70)$$

with

$$\begin{aligned}
& \| \psi \|_{1,q} \leq \| \varphi \|_{0,q}, \\
& ((\tilde{u}_h - u_h)_x, \varphi) \\
& = A(t; \tilde{u}_h - u_h, \psi) \\
& = A(t; \tilde{u}_h - u_h, \psi - R_h \psi) + A(t; u - u_h, R_h \psi) \\
& \quad - A(t; u - u_h, I_h^* R_h \psi) - ((u - u_h)_t, I_h^* R_h \psi) \\
& \quad - \int_0^t B(t, s; (u - u_h)(s), I_h^* R_h \psi) ds \\
& \quad + A(t; \tilde{u}_h - u, R_h \psi) \\
& = [A(t; u - u_h, R_h \psi) - A(t; u - u_h, I_h^* R_h \psi)] \\
& \quad - ((u - u_h)_t, I_h^* R_h \psi) - ((\tilde{u}_h - u)_t, R_h \psi) \\
& \quad - \int_0^t B(t, s; (u - u_h)(s), I_h^* R_h \psi) ds \\
& \quad - \int_0^t B(t, s; (\tilde{u}_h - u)(s), R_h \psi) ds \\
& = I_1 + I_2 + I_3.
\end{aligned} \tag{71}$$

On the other hand,

$$\begin{aligned}
|I_1| & \leq Ch \|u - u_h\|_{1,p} \| \psi \|_{1,q}, \\
|I_2| & \leq C (\| (u - u_h)_t \| + \| (\tilde{u}_h - u)_t \|) \| \psi \| \\
& \leq C (\| (u - u_h)_t \| + \| (\tilde{u}_h - u)_t \|) \| \psi \|_{1,q},
\end{aligned} \tag{72}$$

where we have used the fact $\| \psi \| \leq \| \psi \|_{1,r}$, $r > 1$

$$\begin{aligned}
|I_3| & \leq \int_0^t (\| (u - u_h)(s) \|_{1,p} + \| (u - \tilde{u}_h)(s) \|_{1,p}) ds \| \psi \|_{1,q} \\
& \| (\tilde{u}_h - u_h)_x \|_{0,p} \\
& = \sup_{\varphi \in \mathcal{C}_0^\infty} \frac{((\tilde{u}_h - u_h)_x, \varphi)}{\| \varphi \|_{0,q}} \\
& \leq C \left(h \|u - u_h\|_{1,p} + \| (u - u_h)_t \| + \| (\tilde{u}_h - u)_t \| \right. \\
& \quad \left. + \int_0^t (\| (u - u_h)(s) \|_{1,p} + \| (u - \tilde{u}_h)(s) \|_{1,p}) ds \right).
\end{aligned} \tag{73}$$

We deduce the result from the known finite element estimates. $\square$

Remark 15. In order to estimate $\| (u - u_h)_t \|$, by differentiating (42) with respect to t , we obtain

$$\begin{aligned}
& (\theta_{tt}, I_h^* v_h) + A(t; \theta_t, I_h^* v_h) + A_t(t; \theta_t, I_h^* v_h) \\
& + B(t, t; \theta, I_h^* v_h) + \int_0^t B_t(t, s; \theta(s), I_h^* v_h) ds \\
& = -(\rho_{tt}, I_h^* v_h).
\end{aligned} \tag{74}$$

Setting $v_h = \theta_t$, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \| \theta_t \|^2 + c \| \theta_t \|_1^2 \leq \| \rho_{tt} \| \| \theta_t \| \\
& + \frac{1}{2} c \| \theta_t \|_1^2 + C \| \theta_t \|_1^2 + \int_0^t \| \theta_t \|_1^2 ds \\
& \leq \| \rho_{tt} \| \| \theta_t \| + \frac{1}{2} c \| \theta_t \|_1^2 + C \int_0^t \| \theta_t \|_1^2 ds.
\end{aligned} \tag{75}$$

Using kickback argument, integrating and applying Gronwall's lemma, we deduce

$$\begin{aligned}
\| \theta_t \| & \leq C \left(\| \theta_t(0) \| + \int_0^t \| \rho_{tt} \|_1 ds \right) \\
& \leq Ch^2 \left(\| u_0 \|_2 + \| u_1 \|_2 + \int_0^t \| u_{tt} \|_2 ds \right).
\end{aligned} \tag{76}$$

5. The Lumped Mass Finite Volume Element Method

In this section, we restrict our study to the 2D case. A simple way to define the lumped mass scheme [21] is to replace the mass matrix M_h in (14) by the diagonal matrix $\bar{M}_h$ obtained by taking for its diagonal elements the numbers $\bar{M}_{hii} = \sum_{j=1}^{N_h} M_{hij}$ or by lumping all masses in one row into the diagonal entry. This makes the inversion of the matrix in front of $\alpha'(t)$ a triviality. We will therefore study the matrix problem

$$\bar{M}_h \alpha'(t) + A_h(t) \alpha(t) + \int_0^t B_h(t, s) \alpha(s) ds = F_h(t). \tag{77}$$

We know that the lumped mass method defined by (77) above is equivalent to

$$\begin{aligned}
& (I_h^* u_{ht}, I_h^* v_h) + A(t; u_h, I_h^* v_h) + \int_0^t B(t, s; u_h(s), I_h^* v_h) ds \\
& = (f, I_h^* v_h), \quad v_h \in S_h.
\end{aligned} \tag{78}$$

Our alternative interpretation of this procedure will be to think of (77) as being obtained by evaluating the first term in (78) by numerical quadrature. Let K be a triangle of the triangulation T_h , let x_j , $j = 1, 2, 3$, be its vertices, and consider the quadrature formula

$$Q_{K,h}(f) = \frac{1}{3} \text{area } K \sum_{j=1}^3 f(x_j) \approx \int_K f dx. \tag{79}$$

We may then define the associated bilinear form in $S_h \times S_h^*$, using the quadrature scheme, by the following:

$$\begin{aligned}
(v_h, \eta_h)_h & = \sum_{K \in T_h} Q_{K,h}(v_h \eta_h) = \sum_{x_i \in N_h^a} v_h(x_i) \eta_h(x_i) |V_{x_i}|, \\
& \quad \forall v_h \in S_h, \quad \eta_h \in S_h^*.
\end{aligned} \tag{80}$$

We note that $\|v_h\|_h^2 = (v_h, I_h^* v_h)_h$ is a norm in S_h which is equivalent to the L^2 -norm uniformly in h ; that is, there exist two positive constants C_1 and C_2 such that for all $v_h \in S_h$, we have

$$C_0 \|v_h\| \leq \|v_h\|_h \leq C_1 \|v_h\|, \quad \forall v_h \in S_h. \quad (81)$$

We note that the aforementioned definition $(v_h, \eta_h)_h$ may be used also for $\eta_h \in S_h$ and that $(v_h, w_h)_h = (v_h, I_h^* w_h)_h$ for $v_h, w_h \in S_h$.

The lumped mass method defined by (78) is equivalent to

$$\begin{aligned} & (u_{ht}, I_h^* v_h)_h + A(t; u_h, I_h^* v_h) + \int_0^t B(t, s; u_h(s), I_h^* v_h) ds \\ &= (f, I_h^* v_h), \quad v_h \in S_h. \end{aligned} \quad (82)$$

We introduce the quadrature error

$$\varepsilon_h(v_h, w_h) = (v_h, w_h)_h - (v_h, w_h), \quad (83)$$

Lemma 16 (see [21]). *Let $v_h, w_h \in S_h$. Then*

$$|\varepsilon_h(v_h, w_h)| \leq Ch^2 \|\nabla v_h\| \|\nabla w_h\|. \quad (84)$$

Theorem 17. *Let u_h and u be the solutions of (82) and (2), respectively, and assume $u_h(0) = R_h u_0$. Then we have for the error in the lumped mass semidiscrete method, for $t \geq 0$, the following:*

$$\|u_h(t) - u(t)\| \leq Ch^2 \left(\|u_0\|_2 + \|u\|_2 + \int_0^t \|u_t\|_2 ds \right). \quad (85)$$

Proof. In order to estimate $\|\theta\|$, we write

$$\begin{aligned} & (\theta_t, I_h^* v_h)_h + A(t; \theta, I_h^* v_h) + \int_0^t B(t, s; \theta(s), I_h^* v_h) ds \\ &= (u_{ht}, I_h^* v_h)_h + A(t; u_h, I_h^* v_h) \\ &+ \int_0^t B(t, s; u_h(s), I_h^* v_h) ds \\ &- ((V_h u)_t, I_h^* v_h)_h - A(t; V_h u, I_h^* v_h) \\ &- \int_0^t B(t, s; V_h u(s), I_h^* v_h) ds \\ &= (f, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h - A(t; u, I_h^* v_h) \\ &- \int_0^t B(t, s; u(s), I_h^* v_h) ds \\ &= (u_t, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h \\ &= -(\rho_t, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h + ((V_h u)_t, I_h^* v_h). \end{aligned} \quad (86)$$

We rewrite

$$\begin{aligned} & ((V_h u)_t, I_h^* v_h)_h - ((V_h u)_t, I_h^* v_h) \\ &= ((V_h u)_t, I_h^* v_h)_h - ((V_h u)_t, v_h) \\ &+ ((V_h u)_t, v_h) - ((V_h u)_t, I_h^* v_h) \\ &= \varepsilon_h((V_h u)_t, v_h) + ((V_h u)_t, v_h) - ((V_h u)_t, I_h^* v_h). \end{aligned} \quad (87)$$

$$\begin{aligned} & (\theta_t, I_h^* v_h)_h + A(t; \theta, I_h^* v_h) + \int_0^t B(t, s; \theta(s), I_h^* v_h) ds \\ &= -(\rho_t, I_h^* v_h) + \varepsilon_h((V_h u)_t, v_h) \\ &+ ((V_h u)_t, v_h) - ((V_h u)_t, I_h^* v_h). \end{aligned}$$

Setting $v_h = \theta$ in (87), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\theta\|_h^2 + c_0 \|\theta\|_1^2 \\ & \leq \|\rho_t\| \|\theta\| + \frac{1}{2} c_0 \|\theta\|_1^2 + C \int_0^t \|\theta\|_1^2 ds \\ & + \varepsilon_h((V_h u)_t, \theta) + ((V_h u)_t, \theta) - ((V_h u)_t, I_h^* \theta). \end{aligned} \quad (88)$$

Using Lemma 16 and the inverse estimate, we get

$$\begin{aligned} |\varepsilon_h(V_h u_t, \theta)| & \leq Ch^2 \|\nabla(V_h u)_t\| \|\nabla \theta\| \\ & \leq Ch^2 \|\nabla u_t\| \|\nabla \theta\| \\ & \leq Ch \|\nabla u_t\| \|\theta\|, \end{aligned} \quad (89)$$

we have

$$|((V_h u)_t, \theta) - ((V_h u)_t, I_h^* \theta)| \leq Ch \|\nabla u_t\| \|\theta\|. \quad (90)$$

Using Young's inequality and Gronwall's lemma to eliminate $\|\theta\|_1$ on the right-hand side and using integration in t , we get the result.

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_h^2 + c_0 \|\theta\| \leq \|\rho_t\| \|\theta\| + Ch \|\nabla u_t\| \|\theta\|. \quad (91)$$

Using Young's inequality to eliminate $\|\theta\|$ on the right hand side it becomes.

Using integration in t , we get the result. $\square$

We will now show that the H^1 -norm error bound of theorem remains valid for the lumped mass method (82).

Theorem 18. *Let u_h and u be the solutions of (82) and (2), respectively, and assume*

$$u_h(0) = R_h u_0, \quad \|u_{1h}(0) - u_1\| \leq Ch^2 \|u_1\|_2. \quad (92)$$

Then, we have for the error in the lumped mass semidiscrete method, for $t \geq 0$, the following:

$$\|u_h(t) - u(t)\|_1 \leq Ch^2 \left(\|u_0\|_2 + \|u_1\|_2 + \int_0^t \|u_{tt}\|_2 ds \right). \quad (93)$$

Proof. Setting $v_h = \theta_t$ in (87), we obtain

$$\begin{aligned} & \|\theta_t\|_h^2 + \frac{1}{2} \frac{d}{dt} A(t; \theta, I_h^* \theta) \\ &= \frac{1}{2} A_t(t; \theta, I_h^* \theta) + \frac{1}{2} [A(t; \theta_t, I_h^* \theta) - A(t; \theta, I_h^* \theta_t)] \\ & B(t, t; \theta(t), I_h^* \theta(t)) + \int_0^t B_t(t, s; \theta(s), I_h^* \theta(s)) ds \\ & - \frac{d}{dt} \int_0^t B(t, s; \theta(s), I_h^* \theta(s)) ds - (\rho_t, I_h^* \theta_t) \\ & - \varepsilon_h((V_h u)_t, \theta_t) + ((V_h u)_t, \theta) - ((V_h u)_t, I_h^* \theta). \end{aligned} \quad (94)$$

It follows, thus, that using integration in t and Gronwall's lemma, we have

$$\begin{aligned} \int_0^t \|\theta_t\|_h^2 + \|\theta\|_1^2 &\leq C \|\nabla \theta(0)\|^2 + C \int_0^t \|\rho_t\| \|\theta_t\| ds \\ &+ Ch^2 \int_0^t \|u_t\|_1^2 ds. \end{aligned} \quad (95) \quad \square$$

6. Full Discretization

Let $\bar{\partial}U^n = (U^n - U^{n-1})/k$ be the backward difference quotient of U^n ; assume that $A_h = P_h A$ is a discrete analogue of A (similarly $B_h = P_h B$), where $P_h : L^2(\Omega) \rightarrow S_h^*$ the L^2 projection is defined by

$$(P_h v, I_h^* v_h) = (v, I_h^* v_h), \quad v \in L^2(\Omega), \quad v_h \in S_h. \quad (96)$$

In order to define fully discrete approximation of (11), we discretize the time by taking $t_n = nk$, $k > 0$, $n = 1, 2, \dots$ and use the numerical quadrature

$$\int_0^{t_{n-1/2}} g(s) ds \approx \sum_{k=1}^n \omega_{n,k} g(t_{k-1/2}), \quad t_{n-1/2} = \left(n - \frac{1}{2}\right)k. \quad (97)$$

Here $\{\omega_{n,k}\}$ are the integration weights and we assume that the following error estimate is valid:

$$\begin{aligned} q^n(g) &= \int_0^{t_{n-1/2}} g(s) ds - \sum_{k=1}^n \omega_{n,k} g(t_{k-1/2}) \\ &\leq Ck^2 \int_0^{t_n} (|g'| + |g''|) ds. \end{aligned} \quad (98)$$

Now, define our complete discrete FVE approximation of (11) by the following: find $U^n \in S_h$ for $n = 1, 2, \dots$, such that for all $v_h \in S_h$

$$\begin{aligned} & (\bar{\partial}U^n, I_h^* v_h) + A(t_{n-1/2}; U^{n-1/2}, I_h^* v_h) \\ & + \sum_{k=1}^n \omega_{n,k} B(t_{n-1/2}, t_{k-1/2}, U^{k-1/2}, I_h^* v_h) \\ & = (f^{n-1/2}, I_h^* v_h), \\ & U^0 \text{ in } S_h, \end{aligned} \quad (99)$$

where $U^{n-1/2} = (U^n + U^{n-1})/2$.

Theorem 19. Let $u(t)$ and U^n be the solutions of problem (2) and its complete discrete scheme (99), respectively. Then for any $T > 0$ there exists a positive constant $C = C(T) > 0$, independent of h , such that for $0 < t_n \leq T$

$$\begin{aligned} & \|u(t_n) - U^n\| \\ & \leq Ch^2 \left(\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 ds \right) \\ & + Ck^2 \left(\int_0^{t_n} (\|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2 + \|u_{ttt}\|) ds \right). \end{aligned} \quad (100)$$

Proof. Let us split the error into two parts: $u(t_n) - U^n = \rho^n + \theta^n$, where $\rho^n = u(t_n) - V_h u(t_n)$ and $\theta^n = V_h u(t_n) - U^n$, and let $W = V_h u(t) \in S_h$ be the Ritz-Volterra projection of u . Then from (2) and (99) we have for all $v_h \in S_h$ the following:

$$\begin{aligned} & (\bar{\partial}\theta^n, I_h^* v_h) + A(t_{n-1/2}; \theta^{n-1/2}, I_h^* v_h) \\ & + \sum_{k=1}^n \omega_{n,k} B(t_{n-1/2}, t_{k-1/2}, \theta^{k-1/2}, I_h^* v_h) \end{aligned} \quad (101)$$

$$= -(r_n, I_h^* v_h), \quad \forall v_h \in S_h,$$

where

$$\begin{aligned} r_n &= r_n^1 + r_n^2 + r_n^3 + r_n^4, \\ r_n^1 &= \bar{\partial}\rho^n, \\ r_n^2 &= \bar{\partial}u(t_n) - u_t(t_{n-1/2}), \\ r_n^3 &= A\left(t_{n-1/2}; \frac{(u(t_n) + u(t_{n-1})))}{2} - u(t_{n-1/2})\right), \\ r_n^4 &= q^n(B_h W) \end{aligned} \quad (102)$$

$$\begin{aligned} & = \sum_{k=1}^n \omega_{n,k} B_h(t_{n-1/2}, t_{k-1/2}, W^{k-1/2}, I_h^* v_h) \\ & - \int_0^{t_{n-1/2}} B(t_n, s, W(s), I_h^* v_h) ds. \end{aligned}$$

In fact, by Taylor expansion,

$$\begin{aligned} u^{n+1} &= u^n + ku'(t_n) + \int_{t_n}^{t_{n+1}} u''(s)(t_{n+1} - s) ds \\ &= u^n + ku'(t_n) + \frac{k^2}{2} u''(t_n) + \frac{k^3}{6} u^{(3)}(t_n) \\ &+ \frac{1}{6} \int_{t_n}^{t_{n+1}} u^{(4)}(s)(t_{n+1} - s)^3 ds, \end{aligned} \quad (103)$$

we have

$$\begin{aligned}
 \|r_n^1\| &= \|\bar{\partial}\rho^n\| \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t\| ds \leq C \frac{h^2}{k} \int_{t_{n-1}}^{t_n} \|u_{tt}\|_2 ds, \\
 \|r_n^2\| &= \|\bar{\partial}u(t_n) - u_t(t_{n-1/2})\| \\
 &= \frac{1}{k} \left\| \int_{t_{n-1}}^{t_n} (u_t(s) - u_t(t_{n-1/2})) ds \right\| \\
 &\leq Ck \int_{t_{n-1}}^{t_n} \|u^{(3)}(s)\| ds, \\
 \|r_n^3\| &= \left\| A \left(t_{n-1/2}, \frac{u(t_n) + u(t_{n-1})}{2} - u(t_{n-1/2}), I_h^* v_h \right) \right\| \\
 &\leq Ck \int_{t_{n-1}}^{t_n} \|Au_{tt}(s)\| ds \leq Ck \int_{t_{n-1}}^{t_n} \|u_{tt}\|_2 ds.
 \end{aligned} \tag{104}$$

In addition, the quadrature error satisfies

$$\begin{aligned}
 \|r_n^4\| &= q^{n-1/2} (B_h W) \\
 &= \sum_{k=1}^n \omega_{n,k} B(t_{n-1/2}, t_{k-1/2}, W^{k-1/2}, I_h^* v_h) \\
 &\quad - \int_0^{t_{n-1/2}} B(t_n, s, W(s), I_h^* v_h) ds \\
 &\leq Ck^2 \int_0^{t_n} \|(B_h W)_{ss}\| ds \\
 &\leq Ck^2 \int_0^{t_n} (\|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2) ds. \\
 k \sum_{n=1}^N \|r_n\| &\leq Ch^2 \int_0^{t_n} \|u_{tt}\|_2 ds \\
 &\quad + Ck^2 \int_0^{t_n} (\|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2 + \|u^{(3)}\|) ds.
 \end{aligned} \tag{105}$$

Taking $v_h = \theta^{n-1/2}$ in (101) and noting that $(\bar{\partial}\theta^n, I_h^* \theta^{n-1/2}) = (1/2)\bar{\partial}|||\theta^n|||^2$, there is

$$\begin{aligned}
 |||\theta^n|||^2 - |||\theta^{n-1}|||^2 + 2kc |||\theta^{n-1/2}|||^2 \\
 \leq Ck^2 \sum_{k=1}^n \|\theta^{k-1/2}\|_1 \|\theta^{n-1/2}\|_1 + Ck \|r_n\| \|\theta^{n-1/2}\| \\
 \leq kc |||\theta^{n-1/2}|||^2 + Ck^2 \sum_{k=1}^n \|\theta^{k-1/2}\|_1^2 + Ck \|r_n\| \|\theta^{n-1/2}\|.
 \end{aligned} \tag{106}$$

Summing from $n = 1$ to N and then, after cancelling the common factor and using Gronwall's lemma, we obtain

$$|||\theta^N|||^2 \leq C |||\theta^0|||^2 + Ck \sum_{k=1}^N \|r_n\| (\|\theta^k\| + \|\theta^{k-1/2}\|), \tag{107}$$

and then

$$|||\theta^N||| \leq C |||\theta^0||| + Ck \sum_{n=1}^N \|r_n\|, \tag{108}$$

the theorem follows from the estimates of ρ^n and r^n . $\square$

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