Research Article

Interpolation Polynomials of Entire Functions for Erdős-Type Weights

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Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \rightarrow [0, \infty)$ be an even function. In this paper, we consider some Lagrange interpolation polynomials and the Gauss-Jacobi quadrature formula of entire functions associated with Erdős-type weights $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$, and we will estimate the error terms.

1. Introduction

For a weight w , we suppose that $\int_0^\infty x^n w^2(x) dx < \infty$ for all $n = 0, 1, 2, \dots$. Then we define a unique sequence of orthonormal polynomials $\{p_n(w; x)\}$ as follows:

$$\int_{-\infty}^{\infty} p_n(w; x) p_m(w; x) w(x) dx = \delta_{n,m} \text{ (the Kronecker symbol)}, \quad (1)$$

where $p_n(w; x) := \gamma_n x^n + \dots$, $\gamma_n > 0$ is a polynomial of degree n . Furthermore, we denote the zeros of $p_n(w; x)$ by $x_{k,n}$ ($k = 1, 2, \dots, n$), $x_{1,n} > x_{2,n} > \dots > x_{n,n}$. Hence,

$$p_n(w; x) = \gamma_n (x - x_{n,n}) \cdots (x - x_{2,n}) (x - x_{1,n}). \quad (2)$$

For a given function f , the Lagrange interpolation polynomial $L_n(w; f)$ with the weight w is defined to be a unique polynomial of degree at most $n - 1$ which coincides with f at $x_{k,n}$, $k = 1, 2, \dots, n$. Then we have the representation

$$L_n(w; f; \xi) = \sum_{k=1}^n f(x_{k,n}) l_{k,n}(\xi), \quad (3)$$

where

$$l_{k,n}(\xi) = \frac{p_n(w; \xi)}{p'_n(w; x_{k,n}) (\xi - x_{k,n})}, \quad k = 1, 2, \dots, n. \quad (4)$$

The Gauss-Jacobi quadrature formula to the function f is defined by

$$Q_n(w; f) := \sum_{k=1}^n \lambda_n(w; x_{k,n}) f(x_{k,n}), \quad (5)$$

where the coefficients $\lambda_n(w; x_{k,n})$ are the Christoffel numbers, which are given by

$$\lambda_n(w; x) := \left(\sum_{k=1}^{n-1} p_k(w; x)^2 \right)^{-1}. \quad (6)$$

We denote by W a class of all weight functions of the form $w = \exp(-2Q)$ such that Q is an even differentiable function on $\mathbb{R}$, except possibly at $x = 0$, increasing for $x > 0$, and there exists $\rho < 1$ such that $x^\rho Q'(x)$ is increasing. In addition, the unique positive sequence $\{q_n\}$ determined by $q_n Q'(q_n) = n$ satisfies the condition $q_{2n}/q_n \geq C > 1$, $n = 1, 2, \dots$, for some constant C .

For an entire function f and $R > 0$, we set

$$M_f(R) := \max_{|z|=R} |f(z)|, \quad z \in \mathbb{C}. \quad (7)$$

In [1, 2], Al-Jarrah showed the following results (see also [3–5]).

Theorem 1. Let $w \in W$ and f be an entire function.

- (1) There exists a constant $A \in (0, 1)$, depending on Q only, such that if f satisfies

$$\limsup_{R \rightarrow \infty} \frac{\log M_f(R)}{2Q(R)} \leq A, \quad (8)$$

then, uniformly on compact subsets of $\mathbb{C}$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} |f(\xi) - L_n(w; f; \xi)|^{1/n} &< 1, \\ \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f(x) w(x) dx - Q_n(w; f) \right|^{1/n} &< 1. \end{aligned} \quad (9)$$

- (2) If the entire function f satisfies

$$\lim_{R \rightarrow \infty} \frac{\log M_f(R)}{2Q(R)} = 0, \quad (10)$$

then, uniformly on compact subsets of $\mathbb{C}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} |f(\xi) - L_n(w; f; \xi)|^{1/n} &= 0, \\ \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f(x) w(x) dx - Q_n(w; f) \right|^{1/n} &= 0. \end{aligned} \quad (11)$$

In this paper, we extend the above results to Erdős-type weights. Note that every weight in W is not Erdős-type (see Remark 4). In Section 2, we recall the definition of $\mathcal{F}(\text{lip}(1/2)+)$, which is the class of weights we discuss here. Our theorems are stated in Section 3.

2. Preliminaries

For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exists a constant $C > 1$ such that $C^{-1}g(x) \leq f(x) \leq Cg(x)$ for all x . Throughout this paper, C denotes a positive constant independent of n, x, t . The same symbol does not necessarily denote the same constant in different occurrences. We say that $f : \mathbb{R}^+ := [0, \infty) \rightarrow \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$, $0 < x < y$.

First, we recall a class of weights.

Definition 2 (see cf. [6, Definition 1.2]). Let $Q : \mathbb{R} \rightarrow [0, \infty)$ be a continuous even function. We write $w = \exp(-Q) \in \mathcal{F}(\text{lip}(1/2)+)$ if Q satisfies the following properties.

- (a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.
- (b) $Q'(x)$ is nondecreasing in $\mathbb{R}$.
- (c) $\lim_{x \rightarrow \infty} Q(x) = \infty$.
- (d) The function $T(x) := xQ'(x)/Q(x)$, $x \neq 0$, is quasi-increasing in $(0, \infty)$ and $T(x) \geq \Lambda > 1$, $x \in \mathbb{R} \setminus \{0\}$.
- (e) There exists $\varepsilon_0 \in (0, 1)$ such that for $x \in \mathbb{R} \setminus \{0\}$, $T(x) \sim T(x(1 - (\varepsilon_0/T(x))))$.

- (f) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in \mathbb{R} \setminus \{0\}$,

$$\int_{x-\delta|x|/T(x)}^{x+\delta|x|/T(x)} \frac{Q'(s) - Q'(x)}{|s - x|^{3/2}} ds \leq \varepsilon |Q'(x)| \sqrt{\frac{T(x)}{|x|}}. \quad (12)$$

Example 3. Let $w = \exp(-Q) \in \mathcal{F}(\text{lip}(1/2)+)$.

- (1) If $T(x)$ is bounded, then the weight $w(x)$ is called a Freud-type weight (see [5]). We see that $w(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, is a Freud-type weight.
- (2) If $T(x)$ is unbounded, then $w(x)$ is called an Erdős-type weight. The following Q s give Erdős-type weights $w = \exp(-Q)$ (see [7, Example 1.2], [8, Theorem 3.1]). Let $\exp_l(x) := \exp(\exp(\exp \cdots \exp x) \cdots)$ (l times) for $l = 1, 2, \dots$. Then for $\alpha + \beta > 1$, $\alpha > 0$, and $\beta \geq 0$, we set

$$Q(x) = Q_{l,\alpha,\beta}(x) := |x|^\beta (\exp_l(|x|^\alpha) - \exp_l(0)). \quad (13)$$

Also, for $\alpha > 1$, put $Q(x) = Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1$.

Remark 4. Every weight w in W is Freud type. In fact, by definition, $q_{2^k} \geq C^k q_1$. On the other hand, if we assume that w is Erdős type, then for any given $M > 0$, there exists a constant $C > 0$ such that $Q'(x)/Q(x) = T(x)/x \geq (M+1)/x$ for $x > C$. By integration, we see

$$x^{M+1} \leq CQ(x) \quad (14)$$

for $x > C$. Since $q_n Q(q_n) = n$, $q_n^{M+1} \leq CQ(q_n) = Cn/q_n$; that is, $q_n \leq Cn^{1/(M+2)}$. If we take $M > 0$ sufficiently large, this contradicts the fact that $q_{2^k} \geq C^k q_1$ (cf. [7, Lemma 3.2]).

For $u > 0$, we define the Mhaskar-Rakhmanov-Saff number a_u by

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{(1 - t^2)^{1/2}} dt. \quad (15)$$

Lemma 5. Let $w = \exp(-Q) \in \mathcal{F}(\text{lip}(1/2)+)$.

- (1) ([6, Lemma 3.4 (3.18)]) For $t > 0$, $Q(a_t) \sim t/\sqrt{T(a_t)}$.
- (2) ([6, Theorem 1.19 (f)]) There exists $n_0 > 0$ such that for $n \geq n_0$, $1 - x_{1,n}/a_n \sim 1/(nT(a_n))^{2/3}$. In particular $x_{1,n} \leq a_n$ holds.
- (3) ([6, Theorem 15.2 (15.8)]) There exists $C > 0$ such that $\gamma_{n-1}/\gamma_n \leq Ca_n$ for all $n = 1, 2, \dots$

For an entire function f and $n = 1, 2, \dots$, we set

$$E_n(\xi) := f(\xi) - L_n(w; f; \xi). \quad (16)$$

Then by the residue theorem, for $\xi \in D_n$, we have

$$E_n(\xi) = \frac{p_n(w; \xi)}{2\pi i} \int_{C_n} \frac{f(z) dz}{p_n(w; z)(z - \xi)}, \quad (17)$$

where D_n is a simply connected domain containing all zeros of $p_n(w)$ and C_n is its boundary. Note that for any $\xi \in \mathbb{C}$, there exists n such that $\xi \in D_n$. For each $m = 1, 2, \dots$ we also set

$$\begin{aligned} L_{n,m}(w; f; \xi) &:= \sum_{j=1}^n \sum_{s=0}^{m-1} \frac{1}{(m-1-s)!} \left(\frac{f(z)}{\gamma_n^m \prod_{k \neq j} (z - x_{k,n})^m} \right)^{(m-1-s)} \bigg|_{z=x_{j,n}} \\ &\quad \times \frac{p_n(w; \xi)^m}{(\xi - x_{j,n})^{s+1}}, \\ E_{n,m}(\xi) &:= f(\xi) - L_{n,m}(w; f; \xi). \end{aligned} \quad (18)$$

Then $L_{n,m}(w; f; \xi)$ is a polynomial of degree at most $nm - 1$ such that

$$\begin{aligned} L_{n,m}^{(j)}(w; f; x_{k,n}) &= f^{(j)}(x_{k,n}), \quad k = 1, 2, \dots, n, \\ j &= 1, 2, \dots, m-1 \end{aligned} \quad (19)$$

holds. By the residue theorem again, we see

$$E_{n,m}(\xi) = \frac{p_n(w; \xi)^m}{2\pi i} \int_{C_n} \frac{f(z) dz}{p_n(w; z)^m (z - \xi)} \quad (20)$$

for $\xi \in D_n$.

For the Gauss-Jacobi quadrature, we set

$$E_{n,*} := \int_{\mathbb{R}} f(x) w(x) dx - Q_n(w; f). \quad (21)$$

Then, we also see

$$E_{n,*} = \sum_{k=n}^{\infty} \frac{\gamma_{k+1}}{\gamma_k} \frac{1}{2\pi i} \int_{C_k} \frac{f(z) dz}{p_k(w; z) p_{k+1}(w; z)}, \quad (22)$$

(cf. [1–5]).

3. Theorems and Proofs

We state our theorems. Throughout this section, we assume $w = \exp(-Q) \in \mathcal{F}(\text{lip}(1/2)+)$, and let

$$\eta := \limsup_{t \rightarrow \infty} \frac{Q(a_t) \sqrt{T(a_t)}}{t}. \quad (23)$$

Note that η is determined finitely by Lemma 5(1).

Theorem 6. (1) If an entire function f satisfies

$$\limsup_{R \rightarrow \infty} \frac{\log M_f(5R)}{\sqrt{T(R)}Q(R)} \leq \frac{1}{\eta}, \quad (24)$$

then

$$\limsup_{n \rightarrow \infty} |f(\xi) - L_n(w; f; \xi)|^{1/n} < 1 \quad (25)$$

uniformly on compact subsets; that is, for any compact subset K in $\mathbb{C}$, there exists a constant $0 < c < 1$ such that

$$\limsup_{n \rightarrow \infty} |f(\xi) - L_n(w; f; \xi)|^{1/n} \leq c \quad (26)$$

for $\xi \in K$. Moreover, for any $m = 1, 2, \dots$,

$$\limsup_{n \rightarrow \infty} |f(\xi) - L_{n,m}(w; f; \xi)|^{1/n} < 1 \quad (27)$$

holds also uniformly on compact subsets.

(2) If an entire function f satisfies

$$\limsup_{R \rightarrow \infty} \frac{\log M(10R)}{\sqrt{T(R)}Q(R)} \leq \frac{1}{\eta}, \quad (28)$$

then for any $j = 1, 2, 3, \dots$,

$$\limsup_{n \rightarrow \infty} |f^{(j)}(\xi) - L_n(w; f^{(j)}; \xi)|^{1/n} < 1 \quad (29)$$

uniformly on compact subsets. Moreover, for any $m = 1, 2, \dots$,

$$\limsup_{n \rightarrow \infty} |f^{(j)}(\xi) - L_{n,m}(w; f^{(j)}; \xi)|^{1/n} < 1 \quad (30)$$

holds also uniformly on compact subsets.

As for the Gauss-Jacobi quadrature, we have the following theorem.

Theorem 7. Let f be an entire function. If

$$\limsup_{R \rightarrow \infty} \frac{\log M_f(\lambda_0 R)}{\sqrt{T(R)}Q(R)} \leq \frac{1}{\eta} \quad (31)$$

holds for some $\lambda_0 > 1$ large enough, then

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f(x) w(x) dx - Q_n(w; f) \right|^{1/n} \leq c < 1, \quad (32)$$

for a constant $c > 0$. Further, if

$$\limsup_{R \rightarrow \infty} \frac{\log M_f(2\lambda_0 R)}{\sqrt{T(R)}Q(R)} \leq \frac{1}{\eta} \quad (33)$$

holds, then for any $j = 1, 2, 3, \dots$,

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f^{(j)}(x) w(x) dx - Q_n(w; f^{(j)}) \right|^{1/n} \leq c < 1. \quad (34)$$

Theorem 8. Let f be an entire function. Suppose that

$$\lim_{R \rightarrow \infty} \frac{\log M_f(\lambda R)}{\sqrt{T(R)}Q(R)} = 0, \quad (35)$$

for every $\lambda > 0$ large enough. Then

$$\lim_{n \rightarrow \infty} |f(\xi) - L_n(w; f; \xi)|^{1/n} = 0 \quad (36)$$

holds uniformly on compact subsets of $\mathbb{C}$ and for any $j = 1, 2, 3, \dots$,

$$\lim_{n \rightarrow \infty} |f^{(j)}(\xi) - L_{n,m}(w; f^{(j)}; \xi)|^{1/n} = 0 \quad (37)$$

also holds uniformly on compact subsets of $\mathbb{C}$. Moreover for any $j = 1, 2, 3, \dots$,

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f^{(j)}(x) w(x) dx - Q_n(w; f^{(j)}) \right|^{1/n} = 0 \quad (38)$$

holds true.

Remark 9. The order of an entire function f is given by

$$\mu(f) := \limsup_{R \rightarrow \infty} \frac{\log \log M_f(R)}{\log R}. \quad (39)$$

Also the type of f with order $\mu < \infty$ is given by

$$\tau(f) := \tau_\mu(f) = \limsup_{R \rightarrow \infty} \frac{\log M_f(R)}{R^\mu}. \quad (40)$$

If $w = \exp(-Q)$ is an Erdős-type weight and $\mu(f), \tau(f) < \infty$, then (35) holds for every $\lambda > 0$. In fact, by (14),

$$\lim_{R \rightarrow \infty} \frac{\log M_f(\lambda R)}{\sqrt{T(R)Q(R)}} = \lim_{R \rightarrow \infty} \frac{\log M_f(\lambda R)}{(\lambda R)^\mu} \frac{(\lambda R)^\mu}{\sqrt{T(R)Q(R)}} = 0. \quad (41)$$

The following lemma was proved in [2, Lemmas 3.3 and 3.4]. Its proof is available for our case.

Lemma 10. Let $w = \exp(-Q) \in \mathcal{F}(\text{lip}(1/2)+)$, and let $p_n(w; z)$ be the n th orthonormal polynomial generated by w . Then

$$|p_n(w; z)| \leq \gamma_n 2^{n/2} x_{1,n}^n, \quad (42)$$

for all z such that $|z| \leq x_{1,n}$, and

$$|p_n(w; z)|^{-1} \leq \frac{1}{\gamma_n |z|^n} \exp\left(\frac{nx_{1,n}^2}{2(|z|^2 - x_{1,n}^2)}\right) \quad (43)$$

for all z such that $|z| > x_{1,n}$. Also we have

$$\sum_{j=1}^{[n/2]} x_{j,n}^2 = \sum_{j=1}^{n-1} \left(\frac{\gamma_{j-1}}{\gamma_j} \right)^2. \quad (44)$$

Proof of Theorem 6. (1) We estimate the error form $E_n(\xi)$ given by the formula (17). For a given compact subset K , choose a number n such that $|\xi| \leq x_{1,n}$ for $\xi \in K$. We may take the path of integration C_n to be the circle $|z| = 5a_n$ because $x_{1,n} \leq a_n$. By (42), we see $|p_n(w; \xi)| \leq \gamma_n 2^{n/2} a_n^n$. From the definition of η and (24), for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that for $n \geq N_\varepsilon$,

$$\begin{aligned} Q(a_n) \sqrt{T(a_n)} &\leq (\eta + \varepsilon)n, \\ \log M_f(5a_n) &\leq \left(\frac{1}{\eta} + \varepsilon\right) \sqrt{T(a_n)Q(a_n)}. \end{aligned} \quad (45)$$

Hence by (43), for n large enough,

$$\begin{aligned} |E_n(\xi)| &\leq C \gamma_n 2^{n/2} a_n^n \frac{1}{\gamma_n |5a_n|^n} \\ &\quad \times \exp\left(\frac{nx_{1,n}^2}{2((5a_n)^2 - x_{1,n}^2)}\right) \frac{M_f(5a_n)}{2\pi} \int_{C_n} \frac{|dz|}{4a_n} \\ &\leq C 2^{n/2} \frac{1}{5^n} \exp\left\{\frac{n}{48}\right\} \exp\left(\left(\frac{1}{\eta} + \varepsilon\right) Q(a_n) \sqrt{T(a_n)}\right) \\ &\leq C 2^{n/2} \frac{1}{5^n} \exp\left\{\frac{n}{48}\right\} \exp\left(\left(\frac{1}{\eta} + \varepsilon\right)(\eta + \varepsilon)n\right). \end{aligned} \quad (46)$$

Here we select $\varepsilon > 0$ such as

$$\frac{1}{48} + \left(\frac{1}{\eta} + \varepsilon\right)(\eta + \varepsilon) \leq \frac{25}{24}. \quad (47)$$

Then we have for $n \geq N_\varepsilon$ and any $\xi \in K$,

$$|E_n(\xi)| \leq C \left(\frac{\sqrt{2}}{5}\right)^n \exp\left(\frac{25n}{24}\right) = C \left(\left(\frac{\sqrt{2}}{5}\right) \exp\left(\frac{25}{24}\right)\right)^n. \quad (48)$$

Since $(\sqrt{2}/5) \exp(25/24) =: c < 1$, we conclude (26). In the above proof, we only exchange (48) for

$$\begin{aligned} |E_{n,m}(\xi)| &\leq C \left(\frac{\sqrt{2}}{5}\right)^{nm} \exp\left(\frac{49n + nm}{48}\right) \\ &\leq C \left(\left(\frac{\sqrt{2}}{5}\right) \exp\left(\frac{25}{24}\right)\right)^{nm}. \end{aligned} \quad (49)$$

Then (27) follows.

(2) Since

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_{C_{2R}} \frac{f(\zeta)}{(\zeta - z)^{j+1}} d\zeta, \quad (50)$$

where $C_{2R} = \{\zeta; |\zeta| = 2R\}$, we see $M_{f^{(j)}}(R) \leq 2(j!)M_f(2R)/R^j$. Hence, (28) gives us

$$\begin{aligned} &\limsup_{R \rightarrow \infty} \frac{\log M_{f^{(j)}}(5R)}{\sqrt{T(R)Q(R)}} \\ &\leq \limsup_{R \rightarrow \infty} \frac{\log(2(j!)/R^j) + \log(M_f(10R))}{\sqrt{T(R)Q(R)}} \\ &\leq \limsup_{R \rightarrow \infty} \frac{\log M_f(10R)}{\sqrt{T(R)Q(R)}} \leq \frac{1}{\eta}. \end{aligned} \quad (51)$$

Thus, (29) and (30) follow from (26) and (27), respectively. $\square$

Proof of Theorem 7. First, we show the case $j = 0$. From Lemma 5(3) and (44), we have

$$\sum_{k=1}^{[n/2]} x_{k,n}^2 \leq C \sum_{k=1}^{n-1} a_k^2 \leq C a_{n-1}^2 (n-1), \quad (52)$$

where $C > 0$ is an absolutely constant. Also by Lemma 5(3), we see

$$\frac{1}{\gamma_n} \leq C \frac{a_n}{\gamma_{n-1}} \leq \dots \leq C^n (a_n a_{n-1} \dots a_1) \leq (Ca_n)^n. \quad (53)$$

Now let $|z| > x_{1,n}$. Since

$$|p_n(w; z)|^{-1} \leq \frac{1}{\gamma_n |z|^n} \exp \left(\frac{1}{|z|^2 - x_{1,n}^2} \sum_{j=1}^{[n/2]} x_{j,n}^2 \right), \quad (54)$$

(see [3, page 131]), (52) gives us

$$|p_n(w; z)|^{-1} \leq \frac{1}{\gamma_n |z|^n} \exp \left(\frac{Ca_{n-1}^2 (n-1)}{|z|^2 - x_{1,n}^2} \right). \quad (55)$$

Therefore, for $|z| > x_{1,n+1}$, we conclude that

$$\begin{aligned} & |p_n(w; z) p_{n+1}(w; z)|^{-1} \\ & \leq \frac{1}{\gamma_n \gamma_{n+1} |z|^{2n+1}} \exp \left(\frac{2Ca_n^2 n}{|z|^2 - x_{1,n+1}^2} \right). \end{aligned} \quad (56)$$

We consider the circle $C_n = \{z; |z| = \lambda_0 a_n\}$, where we will choose $\lambda_0 > 0$ large enough later. Now we estimate

$$I_n := \frac{\gamma_{n+1}}{\gamma_n} \frac{1}{2\pi i} \int_{C_n} \frac{f(z) dz}{p_n(w; z) p_{n+1}(w; z)}. \quad (57)$$

From Lemma 5(1), (53), (56), and our assumption (31), we have for $\delta > 0$ small enough,

$$\begin{aligned} |I_n| & \leq \frac{1}{\gamma_n^2 |\lambda_0 a_n|^{2n+1}} M_f(\lambda_0 a_n) \exp \left(\frac{2Ca_n^2 n}{|\lambda_0 a_n|^2 - x_{1,n}^2} \right) \lambda_0 a_n \\ & \leq (Ca_n)^{2n} \frac{\exp \{((1/\eta + \delta)(\eta + \delta)n)\}}{(\lambda_0 a_n)^{2n}} \exp \left(\frac{2Cn}{\lambda_0^2 - 1} \right) \\ & \leq \left(\left(\frac{C}{\lambda_0} \right)^2 \exp \left(\left(\frac{1}{\eta} + \delta \right) (\eta + \delta) \right) \exp \left(\frac{C}{\lambda_0^2 - 1} \right) \right)^n. \end{aligned} \quad (58)$$

Here we select λ_0 as

$$0 < c := \left(\frac{C}{\lambda_0} \right)^2 \exp \left(\left(\frac{1}{\eta} + \delta \right) (\eta + \delta) + \frac{C}{\lambda_0^2 - 1} \right) < 1. \quad (59)$$

Then we have

$$|E_{n,*}| \leq \sum_{j=n}^{\infty} |I_j| \leq \frac{c^n}{1-c}, \quad (60)$$

which shows that

$$\limsup_{n \rightarrow \infty} |E_{n,*}|^{1/n} \leq c \limsup_{n \rightarrow \infty} \left(\frac{1}{1-c} \right)^{1/n} = c < 1. \quad (61)$$

For the case of $j \geq 1$, as in the proof of (2) of Theorem 6, (33) gives us

$$\limsup_{R \rightarrow \infty} \frac{\log M_{f^{(j)}}(\lambda_0 R)}{\sqrt{T(R)} Q(R)} \leq \limsup_{R \rightarrow \infty} \frac{M_f(2\lambda_0 R)}{\sqrt{T(R)} Q(R)} \leq \frac{1}{\eta}. \quad (62)$$

Hence, (34) follows from (32). $\square$

Proof of Theorem 8. In (17), we take the path of integration C_n to be the circle $|z| = \lambda a_n$ ($\lambda > 0$). Then we have

$$|E_n(\xi)| \leq C \left(\left(\frac{\sqrt{2}}{\lambda} \right) \exp \left(\frac{\lambda^2}{\lambda^2 - 1} \right) \right)^n. \quad (63)$$

In this right-hand side, we can take λ as large as we like because of our assumption (35). Hence, we have (36). Similarly as above, (37) follows from (36). Also Theorem 7 shows (38). $\square$

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