Research Article

Lacunary Statistical Convergence of Sequences of Real-Valued Functions

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We introduce the concepts of the lacunary statistical convergence of sequences of real-valued functions. We also give the relation between this convergence and strongly lacunary and pointwise statistical convergence. Furthermore we introduce the concept of a lacunary statistical Cauchy sequence for functional sequences and prove that it is equivalent to lacunary statistical convergence of sequences of real-valued functions.

1. Introduction

The idea of the statistical convergence of sequence of real and complex numbers was introduced by Fast [1] and also independently by Buck [2] and Shoenberg [3]. Gökhan and Güngör [4] defined pointwise statistical convergence of sequences of real-valued functions by saying that $\text{st} - \lim f_k(x) = f(x)$ or $f_k \xrightarrow{\text{st}} f$ on S if and only if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| = 0. \quad (1)$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be abbreviated by q_r .

Quite recently Gökhan et al. [5] defined the strongly lacunary convergence of sequences of real-valued functions in the following way.

For any lacunary sequence $\theta = (k_r)$, a sequence of functions (f_k) is said to be strongly lacunary convergent to f on the set $S \subset \mathbb{R}$ if there exists $f(x)$ such that

$$\frac{1}{h_r} \sum_{I_r} |f_k(x) - f(x)| \longrightarrow 0 \quad (2)$$

for every $x \in S$. We denote this symbolically by writing $f_k \xrightarrow{N_\theta} f$ on S .

Fridy and Orhan [6] defined lacunary statistical convergence by saying that $S_\theta - \lim x = L$ or $x_k \rightarrow L(S_\theta)$ if and only if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0. \quad (3)$$

Furthermore, Fridy and Orhan [7] introduced the lacunary statistical Cauchy sequence.

2. Lacunary Statistical Convergence

Now, we introduce the concepts of the lacunary statistical convergence of sequences of real valued functions.

Definition 1. Let θ be a lacunary sequence, and the sequence (f_k) is θ -lacunary statistically convergent to $f(x)$ on S provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| = 0. \quad (4)$$

In this case we write $S_\theta - \lim f_k(x) = f(x)$ or $f_k \xrightarrow{S_\theta} f$ on S .

Now, we first give some relations between strongly lacunary convergence and lacunary statistical convergence and show that they are equivalent for bounded sequences. We also study $f_k \xrightarrow{\text{st.}} f$ on $S \Rightarrow f_k \xrightarrow{S_\theta} f$ on S and $f_k \xrightarrow{S_\theta} f \Rightarrow f_k \xrightarrow{\text{st.}} f$ on S under certain restrictions on $\theta = (k_r)$.

Theorem 2. Let $\theta = (k_r)$ be a lacunary sequence; then

$$(i) f_k \xrightarrow{N_\theta} f \text{ implies } f_k \xrightarrow{S_\theta} f \text{ on } S;$$

$$(ii) \sup_k |f_k(x)| < \infty \text{ for every } x \in S \text{ and } f_k \xrightarrow{S_\theta} f \text{ implies } f_k \xrightarrow{N_\theta} f \text{ on } S;$$

$$(iii) \text{ let } \sup_k |f_k(x)| < \infty \text{ for every } x \in S. \text{ Then } f_k \xrightarrow{S_\theta} f \text{ if and only if } f_k \xrightarrow{N_\theta} f \text{ on } S.$$

Proof. (i) If $\varepsilon > 0$ and $f_k \xrightarrow{N_\theta} f$ on S we can write

$$\begin{aligned} & \sum_{k \in I_r} |f_k(x) - f(x)| \\ & \geq \sum_{\substack{k \in I_r \\ |f_k(x) - f(x)| \geq \varepsilon}} |f_k(x) - f(x)| \\ & \geq \varepsilon |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| \end{aligned} \quad (5)$$

which yields the result.

(ii) Suppose that $f_k \xrightarrow{S_\theta} f$ and $\sup_k |f_k(x)| < \infty$ on S , say

$$|f_k(x) - f(x)| \leq M_x \quad (6)$$

for all k and every $x \in S$. Given $\varepsilon > 0$ we get

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} |f_k(x) - f(x)| \\ & = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |f_k(x) - f(x)| \geq \varepsilon}} |f_k(x) - f(x)| \\ & \quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |f_k(x) - f(x)| < \varepsilon}} |f_k(x) - f(x)| \\ & \leq \frac{M_x}{h_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| + \varepsilon, \end{aligned} \quad (7)$$

for every $x \in S$, from which the result follows.

Let θ be given and define $f_k(x)$ to be $|x|, 2|x|, \dots, [\sqrt{h_r}] \cdot |x|$ at the first $[\sqrt{h_r}]$ integers in I_r , and $x_k = 0$ otherwise on $\mathbb{R}$. Note that (f_k) is not bounded. We have, for every $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{h_r} |\{k \in I_r : |f_k(x) - 0| \geq \varepsilon \text{ for every } x \in \mathbb{R}\}| \\ & = \frac{[\sqrt{h_r}]}{h_r} \longrightarrow 0, \quad \text{as } r \longrightarrow \infty \end{aligned} \quad (8)$$

that is, $f_k \xrightarrow{S_\theta} f = 0$. On the other hand,

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} |f_k(x) - 0| \\ & = \frac{1}{h_r} \frac{[\sqrt{h_r}] \left([\sqrt{h_r}] + 1 \right) \cdot |x|}{2} \longrightarrow \frac{|x|}{2} \neq 0 \end{aligned} \quad (9)$$

on $\mathbb{R} - \{0\}$; hence $f_k \not\xrightarrow{N_\theta} f = 0$.

We remark that the example given in (ii) shows that the boundedness condition cannot be omitted from the hypothesis of Theorem 2(ii).

(iii) This is an immediate consequence of (i) and (ii). $\square$

Lemma 3. For any lacunary sequence θ , $f_k \xrightarrow{\text{st.}} f$ implies $f_k \xrightarrow{S_\theta} f$ on S if and only if $\liminf_r q_r > 1$.

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r , which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}. \quad (10)$$

If $f_k \xrightarrow{\text{st.}} f$ on S , then for every $\varepsilon > 0$ and for sufficiently large r , we have

$$\begin{aligned} & \frac{1}{k_r} |\{k \leq k_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| \\ & \geq \frac{1}{k_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| \\ & \geq \frac{\delta}{1 + \delta} \frac{1}{h_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}|. \end{aligned} \quad (11)$$

This proves the sufficiency.

Conversely, suppose that $\liminf_r q_r > 1$. Proceeding as in Lemma 2.1 [5] we can select a subsequence $(k_{r(j)})$ of the lacunary sequence $\theta = (k_r)$ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j}, \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \quad (12)$$

where $r(j) \geq r(j-1) + 2$.

Now define a bounded sequence (f_k) by $f_i(x) = x$ if $i \in I_{r(j)}$ for some $j = 1, 2, \dots$ and $f_i(x) = 0$ otherwise on $\mathbb{R}$. It is shown in Lemma 2.1 [5] that (f_k) is not strongly lacunarily convergent on $\mathbb{R}$, but (f_k) is strongly Cesàro summability. Theorem 2(ii) implies that (f_k) is not lacunarily statistically convergent, but it follows from Theorem 2.3. (i) of [5] that (f_k) is pointwise statistically convergent on S . Hence we obtain that (f_k) is pointwise statistical convergent on S , but it is not lacunarily statistically convergent on S . This contradicts the assumption, whence $\liminf_r q_r > 1$. $\square$

Lemma 4. For any lacunary sequence θ , $f_k \xrightarrow{S_\theta} f$ on S implies $f_k \xrightarrow{st.} f$ on S if and only if $\limsup_r q_r < \infty$.

Proof. If $\limsup_r q_r < \infty$, then there is an $H > 0$ such that $q_r < H$ for all r . Suppose that $f_k \xrightarrow{S_\theta} f$ on S , and let

$$N_{r,x} = |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}|. \quad (13)$$

By (4) given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that

$$\frac{N_{r,x}}{h_r} < \varepsilon \quad \text{for all } r > r_0 \text{ and every } x \in S. \quad (14)$$

Now let $M_x = \{\max N_{r,x} : 1 \leq r \leq r_0\}$ for every $x \in S$ and let n be any integer satisfying $k_{r-1} < n \leq k_r$, then we can write

$$\begin{aligned} & \frac{1}{n} |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| \\ & \leq \frac{1}{k_{r-1}} |\{k \leq k_r : |f_k(x) - f(x)| \geq \varepsilon \text{ for every } x \in S\}| \\ & = \frac{1}{k_{r-1}} \{N_{1,x} + N_{2,x} + \dots + N_{r_0,x} + N_{r_0+1,x} + \dots + N_{r,x}\} \\ & \leq \frac{M_x}{k_{r-1}} \cdot r_0 + \frac{1}{k_{r-1}} \left\{ h_{r_0+1} \frac{N_{r_0+1,x}}{h_{r_0+1}} + \dots + h_r \frac{N_{r,x}}{h_r} \right\} \\ & \leq \frac{r_0 \cdot M_x}{k_{r-1}} + \frac{1}{k_{r-1}} \left(\sup_{r > r_0} \frac{N_{r,x}}{h_r} \right) \{h_{r_0+1} + \dots + h_r\} \\ & \leq \frac{r_0 \cdot M_x}{k_{r-1}} + \varepsilon \cdot \frac{k_r - k_{r_0}}{k_{r-1}} \quad \text{by (14)} \\ & \leq \frac{r_0 \cdot M_x}{k_{r-1}} + \varepsilon \cdot q_r \leq \frac{r_0 \cdot M_x}{k_{r-1}} + \varepsilon H \end{aligned} \quad (15)$$

for every $x \in S$, and the sufficiency follows immediately.

Conversely, suppose that $\limsup_r q_r = \infty$. Following the idea in Lemma 2.2 [5] we can select a subsequence $(k_{r(j)})$ of the lacunary sequence $\theta = (k_r)$ such that $q_{r(j)} > j$, and define a bounded sequence by $f_i(x) = x$ if $k_{r(j)-1} < i \leq 2k_{r(j)-1}$ for some $j = 1, 2, \dots$, and $f_i(x) = 0$ otherwise on $\mathbb{R}$. It is shown in Lemma 2.2 [5] that $f_k \xrightarrow{N_\theta} f = 0$ on $\mathbb{R}$ but $f_k \not\xrightarrow{|\sigma_1|} f$ on $\mathbb{R}$. By Theorem 2(i) we conclude that $f_k \xrightarrow{S_\theta} f = 0$ on $\mathbb{R}$ but

Theorem 2.3 of [5] implies that $f_k \xrightarrow{st.} f = 0$ on $\mathbb{R}$. Hence $f_k \xrightarrow{S_\theta} f = 0$ does not imply $f_k \xrightarrow{st.} f = 0$ on $\mathbb{R}$.

Combining Lemmas 3 and 4 we get the following. $\square$

Theorem 5. Let θ be a lacunary sequence and let (f_k) be a sequence of functions on a set S ; then $f_k \xrightarrow{st.} f$ and $f_k \xrightarrow{S_\theta} f$ if and only if

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty. \quad (16)$$

3. Cauchy Criterion

Now we introduce the lacunary statistical analog of the Cauchy convergence criterion which is, as we shall see, equivalent to the lacunary statistical convergence.

Definition 6. Let $\theta = (k_r)$ be a lacunary sequence and let (f_k) be a sequence of functions on a set S . The sequence (f_k) is said to be a θ lacunary statistical Cauchy sequence if there is a subsequence $(f_{k'(r)})$ of (f_k) such that $k'(r) \in I_r$ for each r , $\lim_r f_{k'(r)}(x) = f(x)$ on S , and for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |f_k(x) - f_{k'(r)}(x)| \geq \varepsilon \text{ for every } x \in S\}| = 0. \quad (17)$$

Theorem 7. The sequence (f_k) is lacunarily statistically convergent if and only if (f_k) is a lacunary statistical Cauchy sequence.

Proof. Let $f_k \xrightarrow{S_\theta} f$ on S , and write

$$K_x^{(j)} = \left\{ k \in \mathbb{N} : |f_k(x) - f(x)| < \frac{1}{j} \text{ for every } x \in S \right\} \quad (18)$$

for each $j \in \mathbb{N}$. Hence, for each j , $K_x^{(j)} \supseteq K_x^{(j+1)}$ for every $x \in S$ and

$$\frac{|K_x^{(j)} \cap I_r|}{h_r} \longrightarrow 1 \quad \text{as } r \longrightarrow \infty. \quad (19)$$

Choose $m(1)$ such that $r \geq m(1)$ implies $|K_x^{(1)} \cap I_r|/h_r > 0$, that is, $K_x^{(1)} \cap I_r \neq \emptyset$ for every $x \in S$. Next choose $m(2) > m(1)$ so that $r \geq m(2)$ implies $|K_x^{(2)} \cap I_r|/h_r > 0$ for every $x \in S$. Then for each r satisfying $m(1) \leq r < m(2)$, choose $k'(r) \in I_r$ such that $k'(r) \in K_x^{(1)}$, that is, $|f_{k'(r)}(x) - f(x)| < 1$ for every $x \in S$. In general, choose $m(p+1) > m(p)$ such that $r > m(p+1)$ implies $|K_x^{(p+1)} \cap I_r|/h_r > 0$ for every $x \in S$. Then for all r satisfying $m(p) \leq r < m(p+1)$, choose $k'(r) \in I_r \cap K_x^{(p)}$ for every $x \in S$, that is:

$$|f_{k'(r)}(x) - f(x)| < \frac{1}{p} \quad (20)$$

for every $x \in S$. Hence, we get $k'(r) \in I_r$ for every r , and (20) implies that $\lim_r f_{k'(r)}(x) = f(x)$ for every $x \in S$. Furthermore, we have, for every $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{h_r} \left| \left\{ k \in I_r : \left| f_k(x) - f_{k'(r)}(x) \right| \geq \varepsilon \text{ for every } x \in S \right\} \right| \\ & \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \left| f_k(x) - f(x) \right| \geq \frac{\varepsilon}{2} \text{ for every } x \in S \right\} \right| \\ & \quad + \frac{1}{h_r} \left| \left\{ k \in I_r : \left| f_{k'(r)}(x) - f(x) \right| \geq \frac{\varepsilon}{2} \text{ for every } x \in S \right\} \right|. \end{aligned} \quad (21)$$

Using the assumption that $f_k \xrightarrow{S_\theta} f$ on S and $\lim_r f_{k'(r)}(x) = f(x)$ on S , we infer (17), whence $(f_k(x))$ is a θ lacunary statistical Cauchy sequence on S .

Conversely, suppose that $(f_k(x))$ is a θ lacunary statistical Cauchy sequence. For every $\varepsilon > 0$, we have

$$\begin{aligned} & \left| \left\{ k \in I_r : \left| f_k(x) - f(x) \right| \geq \varepsilon \text{ for every } x \in S \right\} \right| \\ & \leq \left| \left\{ k \in I_r : \left| f_k(x) - f_{k'(r)}(x) \right| \geq \frac{\varepsilon}{2} \text{ for every } x \in S \right\} \right| \\ & \quad + \left| \left\{ k \in I_r : \left| f_{k'(r)}(x) - f(x) \right| \geq \frac{\varepsilon}{2} \text{ for every } x \in S \right\} \right|, \end{aligned} \quad (22)$$

from which it follows that $f_k \xrightarrow{S_\theta} f$ on S . $\square$

4. Conclusion

It is known that several different types of convergence may define sequences (f_k) of which terms are real-valued functions having a common domain on the real line $\mathbb{R}$. The present paper gives the lacunary statistical convergence of a sequence of real-valued functions (f_k) and gives some relations between this convergence and strongly lacunary convergence and shows that they are equivalent for a bounded sequence of real-valued functions. Furthermore, the paper proves that this convergence is equivalent to lacunary statistical Cauchy sequence.

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