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Research Article

Rate Estimation of Identical Synchronization by Designing Controllers

Mitul Islam,[1] Bipul Islam,[2] and Nurul Islam[3]

[1] *Department of Mathematics, Jadavpur University, Jadavpur, Kolkata 700032, India*
[2] *Indian Statistical Institute, 203 B. T. Road, Barrackpore, Kolkata 700108, India*
[3] *Ramakrishna Mission Residential College (Autonomous), Narendrapur, Kolkata 700103, India*

Correspondence should be addressed to Mitul Islam; mitul.islam@gmail.com

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This paper investigates the synchronization rate for identical synchronization of chaotic dynamical systems, achieved by using controllers. The paper stresses on the hybrid feedback control technique and the tracking control technique and determines their corresponding maximum, minimum, and average synchronization rates. The results obtained are applied on the Shimizu-Morioka chaotic system, and some necessary and sufficient conditions for synchronization are obtained. Comparison of the two controllers is undertaken on the basis of their synchronization rates, in the context of the Shimizu-Morioka system. The results are analyzed both theoretically and numerically. Moreover, a method of graphical analysis is proposed to completely characterize the set of hybrid controllers for a given system.

1. Introduction

Chaos control and synchronization has got much attention of the scientists and engineers due to its applicability to various disciplines after the pioneering works by Pecora and Carroll [1] and Ott et al. [2]. It is applied in various fields like information processing, secure communication, neural networks, chemical reactions, biological systems, and so on. Sun and Cao [3] proposed the synchronization between two different noise perturbed chaotic systems. Researchers have proposed different synchronization methods and applied them successfully to control chaos and obtain synchronized chaotic system. Notable methods among them are non-linear feedback control [4], active control [5], adaptive control [6], backstepping design [7], hybrid feedback control [8], and so forth. When the two identical chaotic oscillators are mutually coupled or when one of them drives the other, the synchronization that occurs in this case is known as identical synchronization. Let $(x_1(t), x_2(t), \ldots, x_n(t))$ and $(y_1(t), y_2(t), \ldots, y_n(t))$ denote the variables describing the states of the first and second identical oscillators, respectively, depending on time t. For a set of initial conditions $(x_1(0), x_2(0), \ldots, x_n(0))$ and $(y_1(0), y_2(0), \ldots, y_n(0))$ of the two systems, if $\lim_{t\to\infty}[y_i(t) - x_i(t)] = 0$, for $i = 1, 2, 3, \ldots, n$, then the identical synchronization occurs.

In this paper, two synchronization schemes for chaotic dynamical systems, using two different controllers [9], are primarily discussed in Section 2. Section 3 deals with measurement of the rate of synchronization [10], when synchronization is achieved using controllers. A quantitative estimate of the rate of convergence of the two systems towards each other in the phase space is obtained. In the following section, the methods stated earlier are applied on coupled Shimizu-Morioka chaotic dynamical system. Synchronization of the chaotic system is achieved both via hybrid feedback control and tracking control, and the results are discussed both analytically and numerically. Finally, the corresponding rates of synchronization are determined and compared. Notably, Section 4.3 explores a method of graphical analysis for determining the possible controller parameters for effective synchronization. It proves to be an efficient and elegant tool for determination of possible controllers where traditional mathematical treatment is rendered inadequate. In fact, it enables us to completely characterize the set of all hybrid controllers for any given problem.

2. Description of the Controllers

2.1. Description of the Hybrid Controller. System of ordinary differential equations can be expressed as

$$\dot{X} = AX + B\Phi(X), \tag{1}$$

where $X \in \mathfrak{R}^n$, $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times n}$, and $\Phi : \mathfrak{R}^n \to \mathfrak{R}^n$ is the nonlinear part of the system.

In the following way, a new system which is identical with the system (1) has been constructed as

$$\dot{Y} = AY + B[\Phi(Y) + U]. \tag{2}$$

U is known as the controller which controls the motion of system (2). Feedback controller U should be chosen appropriately to get the identical synchronization of the systems (1) and (2). The controller U is said to be hybrid if it is a combination of linear controller and non-linear controller.

Here the controller is chosen as $U = u_1 + u_2$, where

$$u_1 = \Phi(X) - \Phi(Y) \tag{3}$$

is the non-linear controller and

$$u_2 = K(X - Y) \tag{4}$$

is the linear controller, with $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$ as the feedback matrix.

The synchronization error between the systems (1) and (2) is defined as $e = (e_1, e_2, \ldots, e_n)^T = X - Y$, where $X = (x_1, x_2, \ldots, x_n)^T$ and $Y = (y_1, y_2, \ldots, y_n)^T$. It is easy to observe that the time evolution of the synchronization error obeys the dynamical equation given by

$$\dot{e} = (A - BK)e, \tag{5}$$

where K is the feedback matrix.

Yang et al. [8] derived the sufficient condition that if the eigenvalues of the matrix $A-BK$ have negative real parts, then the error dynamical system (5) will be asymptotically stable at origin and the synchronization between the systems (1) and (2) will occur.

It is noted that the error dynamical system is of the form

$$\dot{e} = He, \tag{6}$$

where $H \in \mathfrak{R}^{n\times n}$.

2.2. Description of the Tracking Controller. As already noted in Section 2.1, any system of first-order non-linear differential equation can be represented as (1). In the context of synchronization of chaotic systems through introduction of tracking controller, a chaotic system of the form (1) is termed as the drive system. Introducing the control vector $U \in \mathfrak{R}^n$, the controlled response system is taken as

$$\dot{Y} = AY + B\Phi(Y) + U, \tag{7}$$

where $Y \in \mathfrak{R}^n$ is the state vector of the response system. The aim of this technique is to design the controller U which will synchronize the states of both the drive and the response systems.

Thus, the error dynamical system has a representation

$$\dot{e} = Ae + B(\Phi(Y) - \phi(X)) + U, \tag{8}$$

where $e = Y - X$ is the error vector. With a proper choice of controller U, the previous system can be put in the form

$$\dot{e} = He, \tag{9}$$

where $H \in \mathfrak{R}^{n\times n}$. Section 4.5 illustrates a method of choosing the controller suitably.

Let us now construct a Lyapunov function $V(e)$ of the form $V(e) = (1/2)e^T e$. Evidently, $V(e)$ is positive definite. The controller U has to be chosen suitably so that $\dot{V} < 0$ which implies that $\lim_{t\to\infty}\|e(t)\| = 0$, and, hence, global synchronization between the states of the drive and response system is achieved asymptotically via the tracking controller U.

3. Theoretical Analysis: Rate Measure for Identical Synchronization via Controllers

As observed already in Section 2, the proper choice of controllers for the coupled chaotic dynamical systems produces an error dynamical systems of the form

$$\dot{e} = He, \tag{10}$$

where $e \in \mathfrak{R}^n$ and $H \in \mathfrak{R}^{n\times n}$. Let us define the Euclidean norm for all $n\times 1$ real matrices G as $\sqrt{G^T G}$, to be represented, henceforth, as $\|G\|$.

Definition 1. If there exist positive constants C_1, C_2, r_1, and r_2 such that $C_1 \exp(-r_1 t) \le \|e(t)\| \le C_2 \exp(-r_2 t)$, then r_1 is defined as the minimum rate of synchronization, r_2 as the maximum rate of synchronization, and their mean $r = (r_1 + r_2)/2$ as the average rate of synchronization.

If the Euclidean norm of error, that is, $\|e\|$, can be written in the form $C_1 \exp(-r_1 t) \le \|e(t)\| \le C_2 \exp(-r_2 t)$, where C_1, C_2 are positive constants and r_1, r_2 are positive, then $\|e(t)\| \to 0$ as $t \to \infty$. Clearly, $\|e(t)\|$ tends to zero at least as fast as $\exp(-r_2 t)$, but not faster than $\exp(-r_1 t)$. Clearly, for large r_2, the rate of convergence of $\|e\|$ to zero is very rapid while it is much slower for smaller values of r_2. The trend for r_1 is also the same. It is noted that if r_1 and r_2 are close to each other, then $\|e\|$ lies within a narrow window between two exponentially decaying curves. Moreover, its value is approximately the same as that of $\exp(-rt)$, where $r = (r_1 + r_2)/2$. When r_1 and r_2 are far apart, then $\exp(-rt)$ determines a mean curve and, hence, is a reasonable approximation for $\|e(t)\|$. In this sense, we define r_1 as the minimum rate of synchronization, r_2 as the maximum rate of synchronization, and $r = (r_1 + r_2)/2$ as the mean rate of synchronization.

Theorem 2. *The average rate of synchronization r is given by $r = (r_1 + r_2)/2$, where $r_1 \ge \sup\{-\sigma(t) \mid t \ge 0\}$ and $r_2 \le \inf\{-\Sigma(t) \mid t \ge 0\}$, $\sigma(t)$ and $\Sigma(t)$ being the minimum and the maximum eigenvalues of the symmetric matrix $G(t) = (H(t) + H(t)^T)/2$; $H(t)$ is defined in* (10).

Proof. Simple mathematical calculation yields

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 = e^T He = e^T H^T e. \tag{11}$$

It yields an estimate of $\|e(t)\|$ as

$$\begin{aligned}\|e(t_0)\| \exp\left(\int_{t_0}^{t} \sigma(u)\,du\right) &\le \|e(t)\| \\ &\le \|e(t_0)\| \exp\left(\int_{t_0}^{t} \Sigma(u)\,du\right),\end{aligned} \tag{12}$$

where $\sigma(t)$ and $\Sigma(t)$ are the minimum and the maximum eigenvalues of the symmetric matrix $G = (H + H^T)/2$, respectively.

Choosing r_1 and r_2 such that

$$-r_1 \le \sigma(t) \le \Sigma(t) \le -r_2 \tag{13}$$

for all $t \ge 0$, we obtain a fairly accurate estimate of the average rate of synchronization $r = (r_1 + r_2)/2$. □

Note. In case G becomes a constant matrix, it is easy to obtain the least upper bound and the greatest lower bound of the eigenvalues of G as follows:

$$r_1 = \sup\{-\sigma(t) \mid t \ge 0\} \le \inf\{-\Sigma(t) \mid t \ge 0\} = r_2, \tag{14}$$

whereby we obtain the average rate of synchronization $r = (r_1 + r_2)/2$.

4. Application of the Results on the Shimizu-Morioka Chaotic System

4.1. The Shimizu-Morioka Chaotic System. Shimizu-Morioka dynamical system is as follows:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - \lambda x_2 - x_1 x_3, \\ \dot{x}_3 &= -\alpha x_3 + x_1^2,\end{aligned} \tag{15}$$

where x_1, x_2, x_3 are the state variables and λ, α are the parameters. Studies by Shil'nikov [11] revealed that the system exhibits Lorentz-like attractors for $\alpha = 0.608$, $\lambda = 1.0499$ and for $\alpha = 0.549$, $\lambda = 0.605$.

The previous system of (15) can be written as $\dot{X} = AX + B\Phi(X)$ where

$$\begin{aligned}X &= (x_1, x_2, x_3)^T, \\ A &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\alpha \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \Phi(X) &= \left(0, -x_1 x_3, x_1^2\right)^T.\end{aligned} \tag{16}$$

4.2. Synchronization of Shimizu-Morioka Chaotic System Using Hybrid Controller. For the feedback matrix $K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$, we have

$$A - BK = \begin{pmatrix} -k_1 & 0 & 0 \\ 0 & -\lambda - k_2 & 0 \\ 0 & 0 & -\alpha - k_3 \end{pmatrix} = H. \tag{17}$$

Characteristic equation of the matrix $A - BK$ is

$$\begin{aligned}&\mu^3 + (k_1 + k_2 + k_3 + \lambda + \alpha)\mu^2 \\ &\quad + \{k_1(k_2 + \lambda) - (\lambda + k_1 + k_2)(-\alpha - k_3) - 1\}\mu \\ &\quad + \{-\alpha - k_3 - k_1(\lambda + k_2)(-\alpha - k_3)\} = 0.\end{aligned} \tag{18}$$

By the previous Routh-Hurwitz criterion, the roots of the above equation in μ will have negative real parts if and only if

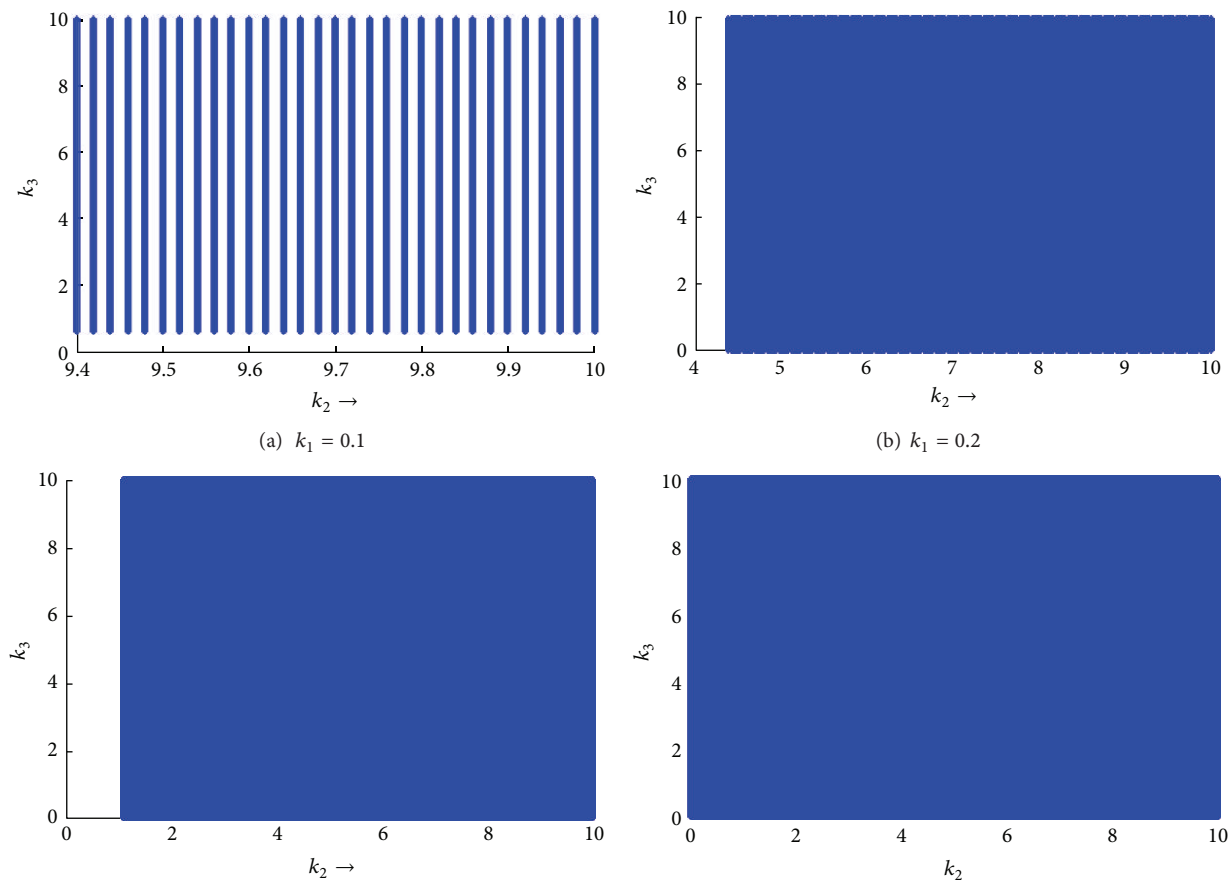
$$\begin{aligned}&k_1 + k_2 + k_3 + \lambda + \alpha > 0, \\ &k_1(k_2 + \lambda) - (\lambda + k_1 + k_2)(-\alpha - k_3) - 1 > 0, \\ &-\alpha - k_3 + k_1(\lambda + k_2)(\alpha + k_3) > 0, \\ &(k_1 + k_2 + k_3 + \lambda + \alpha) \\ &\quad \times \{k_1(k_2 + \lambda) - (\lambda + k_1 + k_2)(-\alpha - k_3) - 1\} \\ &\quad + \alpha + k_3 - k_1(\lambda + k_2)(\alpha + k_3) > 0.\end{aligned} \tag{19}$$

Using (3) and (4), the hybrid controller for the Shimizu-Morioka system is given as $U = \begin{pmatrix} k_1(x_1 - y_1) \\ -x_1 x_3 + y_1 y_3 + k_2(x_2 - y_2) \\ x_1^2 - y_1^2 + k_3(x_3 - y_3) \end{pmatrix}$. Therefore, the response system is

$$\begin{aligned}\dot{y}_1 &= y_2 + k_1(x_1 - y_1), \\ \dot{y}_2 &= y_1 - \lambda y_2 - x_1 x_3 + k_2(x_2 - y_2), \\ \dot{y}_3 &= -\alpha y_3 + x_1^2 + k_3(x_3 - y_3).\end{aligned} \tag{20}$$

4.3. Complete Characterization of Control Parameters for Hybrid Controller. The necessary and sufficient conditions for synchronization are given in (19), involving the three control parameters k_1, k_2, and k_3. Due to the complicated nature of the expressions involved, we adopt an alternate path of graphical analysis for explicit determination of "feasible" control parameters, that is, the set of values of (k_1, k_2, k_3) for which the inequalities in (19) are simultaneously satisfied. In this sense, we can claim to have completely characterized the set of hybrid controllers for the synchronization of chaotic coupled Shimizu-Morioka system. The same procedure can be applied to any coupled dynamical system, thus completely characterizing the set of hybrid controllers for it. The method presented also has potential practical applications in controller design.

The set of figures presented in Figures 1 and 2 illustrate the effectiveness of the method of graphical analysis. Since the problem involves three parameters, a three-dimensional



(a) $k_1 = 0.1$ (b) $k_1 = 0.2$ (c) $k_1 = 0.6$ (d) $k_1 = 2.0$

Figure 1: Region of "controllability" for different fixed values of k_1.

graphical analysis would have been most general. But for the sake of clarity and understandability, we keep one parameter fixed and generate the two-dimensional graphs. The blue regions in all the graphs are the regions of "controllability." For any set (k_1, k_2, k_3), values within the blue region produce a hybrid controller that drives the response into synchrony with the drive system. Figure 1 fixes k_1 at different values and shows the possible values of k_2 and k_3. Clearly, $k_3 > -\alpha$ always for a successful controller design. However, the lower limit on values of k_2 decreases with increase of k_1. In Figure 2, when k_2 is fixed, again $k_3 > -\alpha$ always and k_1 assumes much lower values as k_2 is increased. Thus, high values of k_1 and k_2 always increases the region of "controllability," while k_3 is constrained to lie above $-\alpha$ always.

4.4. Rate of Synchronization Using Hybrid Controller. The symmetric matrix G is given by $\begin{pmatrix} -k_1 & 1 & 0 \\ 1 & -(\lambda+k_2) & 0 \\ 0 & 0 & -\alpha-k_3 \end{pmatrix}$. It is clear that the problem has become largely simplified because the matrix G has become constant because of a suitable controller choice. The characteristic equation for this matrix is

$$(\alpha + k_3 + \mu)\{(k_1 + \mu)(\lambda + k_2 + \mu) - 1\} = 0 \tag{21}$$

which has roots $-\mu_1$, $-\mu_2$, and $-\mu_3$, where

$$\begin{aligned} \mu_1 &= k_3 + \alpha, \\ \mu_2 &= \frac{1}{2}\left\{\lambda + k_1 + k_2 + \sqrt{(\lambda + k_2 - k_1)^2 + 4}\right\}, \\ \mu_3 &= \frac{1}{2}\left\{\lambda + k_1 + k_2 - \sqrt{(\lambda + k_2 - k_1)^2 + 4}\right\}. \end{aligned} \tag{22}$$

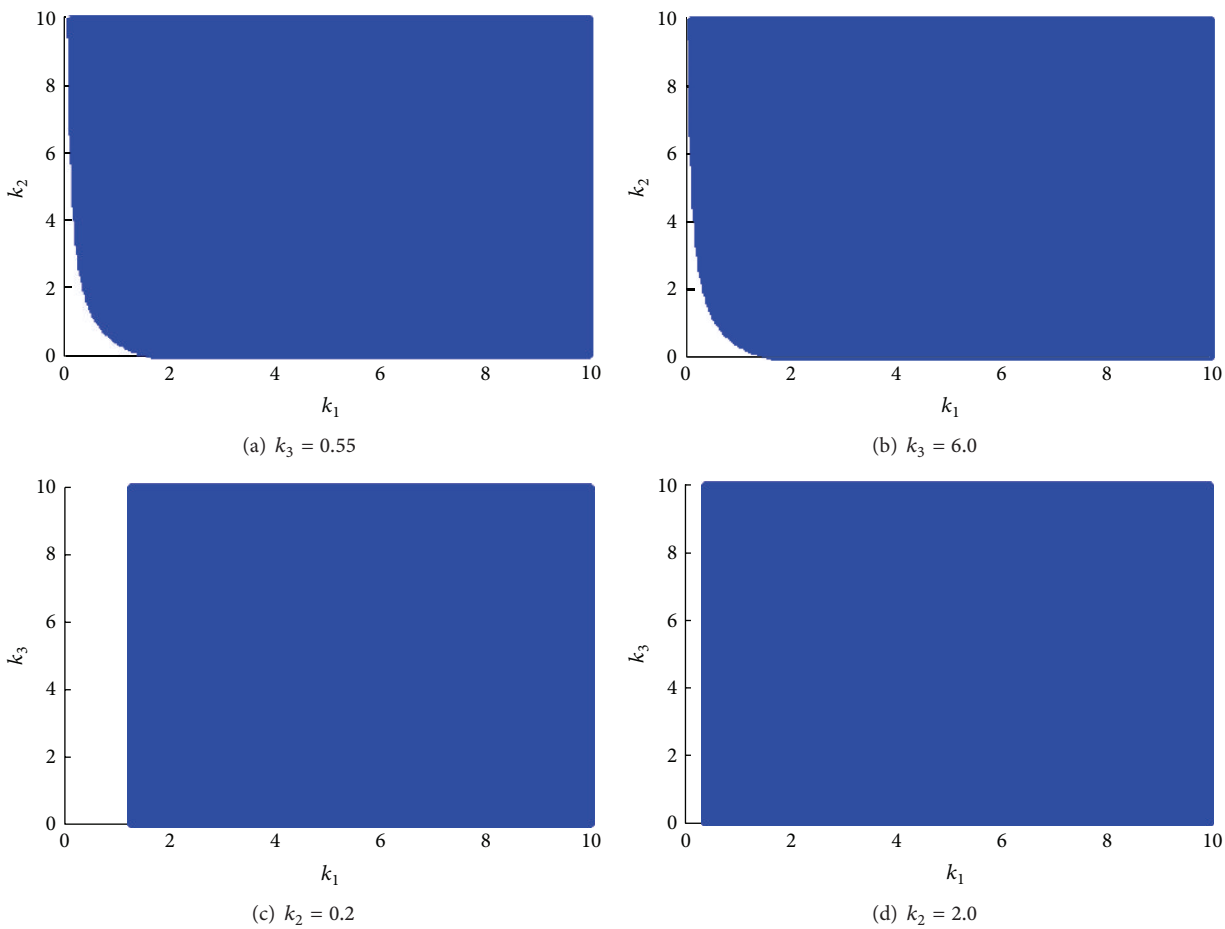
With the conditions $k_3 + \alpha > 0$ and $k_1(\lambda + k_2) > 1$, it is observed that $\mu_2 > \mu_3$. Thus, $\Sigma(t) = -\min\{\mu_1, \mu_3\}$ and $\sigma(t) = -\max\{\mu_1, \mu_2\}$ for all $t \geq 0$. Using Theorem 2,

(1) minimum rate of synchronization $= r_1 = \max\{\mu_1, \mu_2\}$;

(2) maximum rate of synchronization $= r_2 = \min\{\mu_1, \mu_3\}$;

(3) mean rate $= r = (r_1 + r_2)/2$.

Simple calculations yield the maximum possible value of r given by $r^H_{\max} = (\mu_1 + \mu_2)/2$. It is possible to attain $r = r^H_{\max}$ if the following inequality holds: $\mu_2 > \mu_3 > \mu_1$.



(a) $k_3 = 0.55$ (b) $k_3 = 6.0$

(c) $k_2 = 0.2$ (d) $k_2 = 2.0$

Figure 2: Region of "controllability" for different fixed values of k_3 (a), (b) and k_2 (c), (d).

4.5. Synchronization of Shimizu-Morioka Chaotic System via Tracking Controller. The dynamical system (15) is the drive system. Its identical controlled response system is taken as

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= y_1 - \lambda y_2 - y_1 y_3 + u_2, \\ \dot{y}_3 &= -\alpha y_3 + y_1^2 + u_3, \end{aligned} \tag{23}$$

where $U = (u_1, u_2, u_3)^T$ is the controller to be determined and $Y = (y_1, y_2, y_3)^T$ is the response system. Let us define $e_i = y_i - x_i$ $(i = 1, 2, 3)$. Motivated by the observation stated in the outline of this controller design, let us choose the controller as

$$\begin{aligned} u_1 &= -e_1 - e_2, \\ u_2 &= -e_1 + y_1 y_3 - x_1 x_3, \\ u_3 &= -(-\alpha + 1) e_3 - y_1^2 + x_1^2. \end{aligned} \tag{24}$$

Using systems (15), (23), and (24), the error dynamical system takes the form $\dot{e} = He$, where

$$H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{25}$$

Some trite calculations yield $\dot{V} = -(e_1^2 + \lambda e_2^2 + e_3^2) < 0$. Hence, the tracking controller given by (24) leads to global synchronization between the drive and the response systems.

4.6. Rate of Synchronization Using Tracking Controller. The symmetric matrix G is given by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1 \end{pmatrix}$. It is clear that the problem has again become largely simplified because the matrix G is a constant diagonal matrix due to proper controller choice. Here, $\Sigma(t) = -\min\{1, \lambda\}$ and $\sigma(t) = -\max\{1, \lambda\}$ for all $t \geq 0$. By Theorem 2,

(i) Case I: $0 < \lambda < 1$, $r_1 = \lambda$, and $r_2 = 1$;

(ii) Case II: $\lambda = 1$, $r_1 = 1$, and $r_2 = 1$;

(iii) Case III: $\lambda > 1$, $r_1 = 1$, and $r_2 = \lambda$.

In all the cases, average rate of synchronization is always a constant, where $r = (1 + \lambda)/2$; that is, $r = r_{\max}^T = (1 + \lambda)/2$.

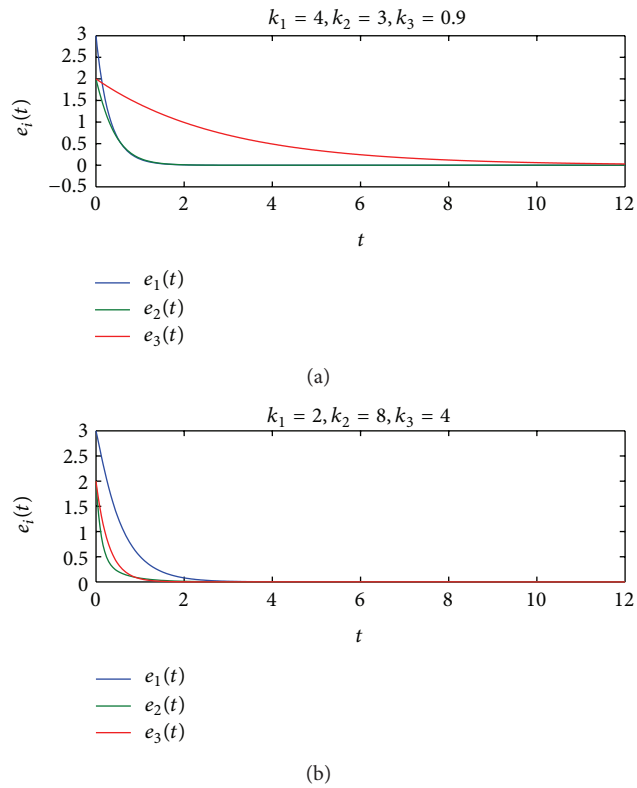


Figure 3: Convergence of $\|e(t)\|$ for different parameter choices of the hybrid controller.

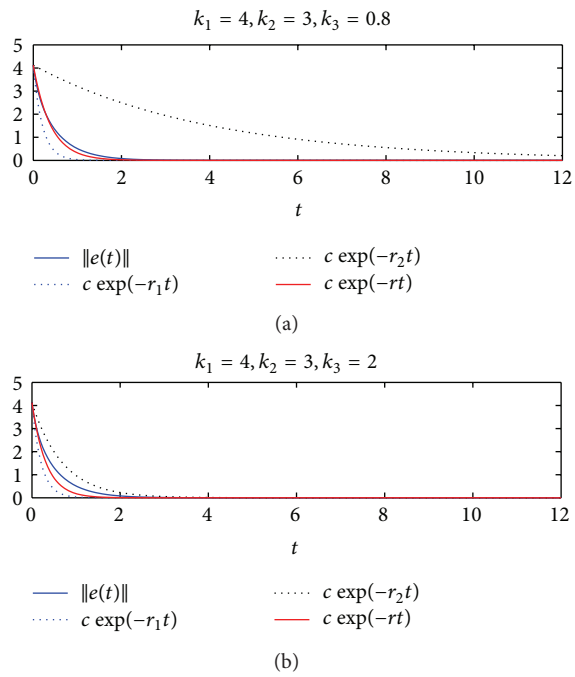


Figure 4: Comparison of the proposed "rate" curves with $\|e(t)\|$ for synchronization using the hybrid control technique ($c = \|e(0)\|$).

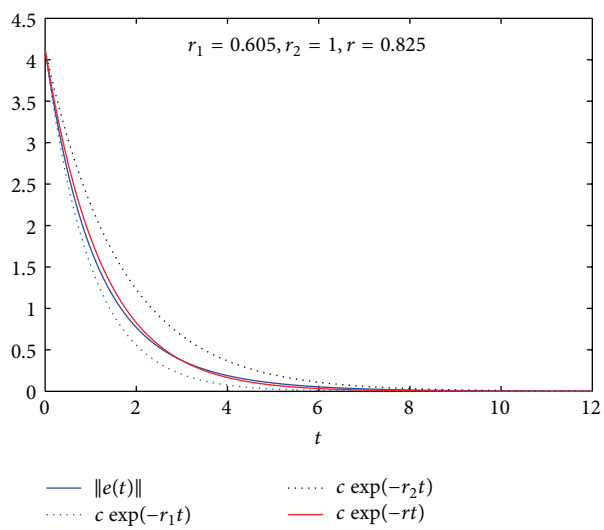


Figure 5: Comparison of the proposed "rate" curves with $\|e(t)\|$ for synchronization using the tracking control technique ($c = \|e(0)\|$).

5. Discussion and Conclusion

Numerical simulations are done with the parameter values $\alpha = 0.549$ and $\lambda = 0.605$, and the initial conditions are taken as $X(0) = (1, 3, 4)^T$ and $Y(0) = (-2, 1, 2)^T$. In hybrid control technique, the control parameters k_1, k_2, k_3 play an important role for the synchronization of the drive and the response systems. Time evolution of the synchronization errors is plotted in Figure 3. All the errors vanish with time but the rapidity of vanishing of errors depend on the values of k_i ($i = 1, 2, 3$). The synchronization rates are depicted in Figures 4 and 5 for hybrid and tracking controllers, respectively. In both figures, the minimum, maximum, and the average rates of synchronization are shown in addition to $\|e(t)\|$. In hybrid control technique, the rate of synchronization is highly dependent on the control parameters k_i ($i = 1, 2, 3$) as depicted in Figures 3 and 4.

It is seen from extensive numerical experiments that the average synchronisation rate function, that is, $\|e(0)\| \exp(-rt)$, lies extremely close to $\|e(t)\|$ (refer to Figures 4 and 5). Clearly, the rate estimation is highly accurate for both the tracking controller and the hybrid controller, illustrating the power of our method of rate estimation. Thus, by and large, the average synchronisation rate function $\|e(0)\| \exp(-rt)$, where r = average synchronisation rate, can be used as a good approximation to $\|e(t)\|$ for practical purposes.

Figure 6 compares the rate of synchronization for hybrid control and the tracking control techniques. It is evident that the hybrid controller may be made better than the tracking controller by adjusting the parameters of the hybrid controller. It is seen from numerical experiments that if k_1, k_2 and k_3 have significantly large values, then hybrid controller is by and large better than the tracking controller in the context of synchronizing the chaotic Shimizu-Morioka system. But choice of arbitrarily low values of k_1 and k_2 from the region of "controllability" (refer to Section 4.3) causes the tracking controller to work better than the hybrid controller. Thus, we can safely conclude that the choice of comparatively larger parameter values is a sufficient condition for the hybrid controller to be better than the tracking controller. The average

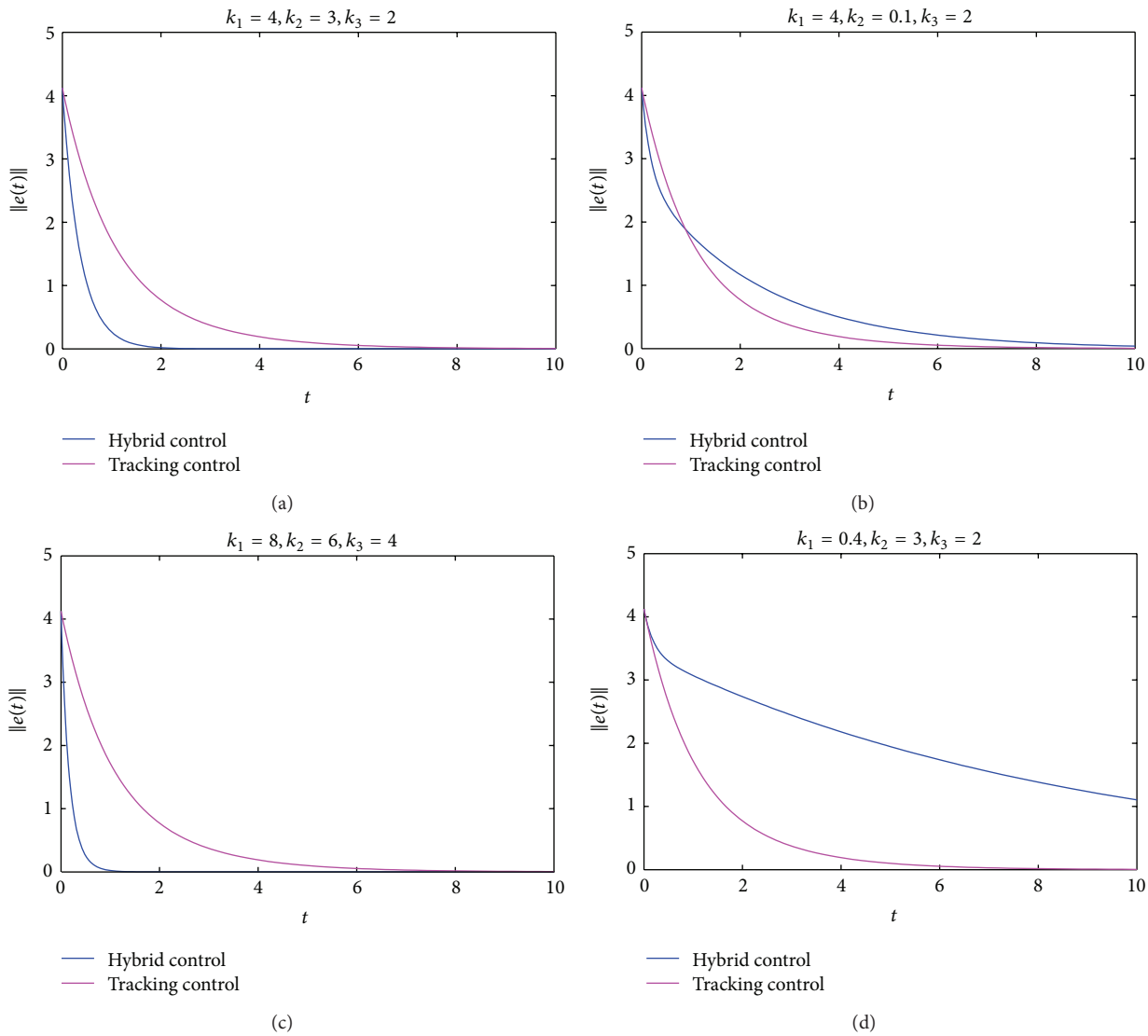


Figure 6: Comparison of the hybrid and the tracking control techniques in the context of chaotic Shimizu-Morioka system.

rate of synchronization for hybrid controller can be modified at will through parameter adjustments. It is this flexibility that makes hybrid control technique important in chaos synchronization. This is in contrast to tracking controller whose average rate of synchronisation is fairly constant.

As already noted, maximum possible average rate of hybrid control synchronization is $r_{\max}^{H} = (\mu_1 + \mu_2)/2$ and that of tracking control synchronization is $r_{\max}^{T} = (1 + \lambda)/2$, which is a constant. Thus, a more rigorous sufficient condition for the hybrid controller to be made superior over tracking controller is

$$\mu_1 + \mu_2 > 1 + \lambda. \tag{26}$$

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