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Research Article

On Paranorm Zweier I-Convergent Sequence Spaces

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In this paper, we introduce the paranorm Zweier I-convergent sequence spaces $\mathcal{Z}^I(q)$, $\mathcal{Z}_I^0(q)$, and $\mathcal{Z}_\infty^I(q)\mathcal{Z}_\infty^I(q)$ for $q = (q_k)$, a sequence of positive real numbers. We study some topological properties, prove the decomposition theorem, and study some inclusion relations on these spaces.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all natural, real, and complex numbers, respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}, \tag{1}$$

the space of all real or complex sequences.

Let l_∞, c, and c_0 denote the Banach spaces of bounded, convergent, and null sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$.

The following subspaces of ω were first introduced and discussed by Maddox [1]:

$$l(p) := \{x \in \omega : \textstyle\sum_k |x_k|^{p_k} < \infty\},$$

$$l_\infty(p) := \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\},$$

$$c(p) := \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\},$$

$$c_0(p) := \{x \in \omega : \lim_k |x_k|^{p_k} = 0\},$$

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After that Lascarides [2, 3] defined the following sequence spaces:

$$\begin{aligned} l_\infty\{p\} &= \{x \in \omega : \text{ there exists } r > 0 \\ &\qquad \text{such that } \sup_k |x_k r|^{p_k} t_k < \infty\}, \\ c_0\{p\} &= \{x \in \omega : \text{ there exists } r > 0 \\ &\qquad \text{such that } \lim_k |x_k r|^{p_k} t_k = 0\}, \\ l\{p\} &= \left\{x \in \omega : \text{ there exists } r > 0 \right. \\ &\qquad \left. \text{such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\right\}, \end{aligned} \tag{2}$$

where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$, is called a sequence space.

A sequence space λ with linear topology is called a K-space provided each map $p_i \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

A K-space λ is called an FK-space provided λ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$.

Then we say that A defines a matrix mapping from λ to μ, and we denote it by writing $A : \lambda \to \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{3}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \to \mu$.

Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (3) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method has been recently employed by Altay et al. [4], Başar and Altay [5], Malkowsky [6], Ng and Lee [7], and Wang [8].

Şengönül [9] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$, that is,

$$y_i = px_i + (1-p)\, x_{i-1}, \tag{4}$$

where $x_{-1} = 0$, $p \neq 1$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$\mathcal{Z}_{ik} = \begin{cases} p, & (i = k), \\ 1-p, & (i-1 = k); (i, k \in \mathbb{N}), \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Following Başar and Altay [5], Şengönül [9] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows:

$$\begin{aligned} \mathcal{Z} &= \left\{x = (x_k) \in \omega : Z^p x \in c\right\}, \\ \mathcal{Z}_0 &= \left\{x = (x_k) \in \omega : Z^p x \in c_0\right\}. \end{aligned} \tag{6}$$

Here we quote below some of the results due to Şengönül [9] which we will need in order to establish the results of this paper.

Theorem 1 (see [9, Theorem 2.1]). *The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the coordinate wise addition and scalar multiplication which are the BK-spaces with the norm*

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \left\|Z^p x\right\|_c. \tag{7}$$

Theorem 2 (see [9, Theorem 2.2]). *The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0, respectively, that is, $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.*

Theorem 3 (see [9, Theorem 2.3]). *The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.*

Theorem 4 (see [9, Theorem 2.6]). *$\mathcal{Z}_0$ is solid.*

Theorem 5 (see [9, Theorem 3.6]). *$\mathcal{Z}$ is not a solid sequence space.*

The concept of statistical convergence was first introduced by Fast [10] and also independently by Buck [11] and Schoenberg [12] for real and complex sequences. Further this concept was studied by Connor [13, 14], Connor et al. [15], and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be statistically convergent to L if for a given $\epsilon > 0$ as

$$\lim_k \frac{1}{k} \left|\{i : |x_i - L| \geq \epsilon, i \leq k\}\right| = 0. \tag{8}$$

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage, it was studied by Kostyrko et al. [16]. Later on, it was studied by Šalát et al. [17, 18], Demirci [19], Tripathy and Hazarika [20, 21], and Khan et al. [22–24].

Here we give some preliminaries about the notion of I-convergence.

Let X be a nonempty set. Then a family of sets $I \subseteq 2^X$ (denoting the power set of X) is said to be an ideal if I is additive, that is, $A, B \in I \Rightarrow A \cup B \in I$, and hereditary, that is, $A \in I, B \subseteq A \Rightarrow B \in I$.

A nonempty family of sets $\pounds \subseteq 2^X$ is said to be a filter on X if and only if $\phi \notin \pounds$, for $A, B \in \pounds$ we have $A \cap B \in \pounds$ and for each $A \in \pounds$ and $A \subseteq B$ implies $B \in \pounds$.

An ideal $I \subseteq 2^X$ is called nontrivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\pounds(I)$ corresponding to I. that is, $\pounds(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

Definition 6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if $\{k \in N : |x_k - L| \geq \epsilon\} \in I$ for every $\varepsilon > 0$. In this case we write $I - \lim x_k = L$.

The space c^I of all I-convergent sequences converging to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{C}\}. \tag{9}$$

Definition 7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 8. A sequence $(x_k) \in \omega$ is said to be I-Cauchy if for every $\epsilon > 0$ there exists a number $k_0 \in \mathbb{N}$ such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$ for all $k, m \geq k_0$.

Definition 9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in N : |x_k| > M\} \in I$.

Definition 10. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for *almost all k relative to I* (*a.a.k.r.I*), if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$.

The following lemma will be used for establishing some results of this paper.

Lemma 11. *If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$ (see [20, 21]) cf. ([17, 18, 20–24]).*

Recently Khan and Ebadullah [25] introduced the following classes of sequence spaces:

$$\begin{aligned}\mathcal{Z}^I &= \left\{k \in \mathbb{N} : \left\{x = (x_k) \in \omega : I - \lim Z^p x = L \right.\right. \\ &\qquad \left.\left. \text{for some } L\right\}\right\} \in I, \\ \mathcal{Z}_0^I &= \left\{k \in \mathbb{N} : \left\{x = (x_k) \in \omega : I - \lim Z^p x = 0\right\}\right\} \in I, \\ \mathcal{Z}_\infty^I &= \left\{k \in \mathbb{N} : \left\{x = (x_k) \in \omega : \sup_k \left|Z^p x\right| < \infty\right\}\right\} \in I,\end{aligned} \tag{10}$$

We also denote by

$$m_{\mathcal{Z}}^I = \mathcal{Z}_\infty^I \cap \mathcal{Z}^I, \qquad m_{\mathcal{Z}_0}^I = \mathcal{Z}_\infty^I \cap \mathcal{Z}_0^I. \tag{11}$$

In this paper we introduce the following classes of sequence spaces:

$$\begin{aligned}\mathcal{Z}^I(q) &= \left\{x = (x_k) \in \omega : \left\{k \in \mathbb{N} : \left|Z^p x - L\right|^{q_k} \geq \epsilon\right\} \in I, \right. \\ &\qquad \left. \text{for some } L \in \mathbb{C}\right\}; \\ \mathcal{Z}_0^I(q) &= \left\{x = (x_k) \in \omega : \left\{k \in \mathbb{N} : \left|Z^p x\right|^{q_k} \geq \epsilon\right\} \in I\right\}; \\ \mathcal{Z}_\infty^I(q) &= \left\{x = (x_k) \in \omega : \sup_k \left|Z^p x\right|^{q_k} < \infty\right\}.\end{aligned} \tag{12}$$

We also denote by

$$\begin{aligned}m_{\mathcal{Z}}^I(q) &= \mathcal{Z}_\infty^I(q) \cap \mathcal{Z}^I(q), \\ m_{\mathcal{Z}_0}^I(q) &= \mathcal{Z}_\infty^I(q) \cap \mathcal{Z}_0^I(q),\end{aligned} \tag{13}$$

where $q = (q_k)$, is a sequence of positive real numbers.

Throughout the paper, for the sake of convenience now we will denote by $Z^p x = x^/, Z^p y = y^/, Z^p z = z^/$ for all $x, y, z \in \omega$.

2. Main Results

Theorem 12. *The classes of sequences $\mathcal{Z}^I(q), \mathcal{Z}_0^I(q), m_{\mathcal{Z}}^I(q)$ and $m_{\mathcal{Z}_0}^I(q)$ are linear spaces.*

Proof. We shall prove the result for the space $\mathcal{Z}^I(q)$.

The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I(q)$, and let α, β be scalars. Then for a given $\epsilon > 0$: we have

$$\begin{aligned}&\left\{k \in \mathbb{N} : \left|x_k^/ - L_1\right|^{q_k} \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C}\right\} \in I \\ &\left\{k \in \mathbb{N} : \left|y_k^/ - L_2\right|^{q_k} \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\right\} \in I,\end{aligned} \tag{14}$$

where

$$\begin{aligned}M_1 &= D \cdot \max\left\{1, \sup_k |\alpha|^{q_k}\right\}, \\ M_2 &= D \cdot \max\left\{1, \sup_k |\beta|^{q_k}\right\}, \\ D &= \max\left\{1, 2^{H-1}\right\} \quad \text{where } H = \sup_k q_k \geq 0.\end{aligned} \tag{15}$$

Let

$$\begin{aligned}A_1 &= \left\{k \in \mathbb{N} : \left|x_k^/ - L_1\right|^{q_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C}\right\} \in I \\ A_2 &= \left\{k \in \mathbb{N} : \left|y_k^/ - L_2\right|^{q_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\right\} \in I\end{aligned} \tag{16}$$

be such that $A_1^c, A_2^c \in I$.

Then

$$\begin{aligned}A_3 &= \left\{k \in \mathbb{N} : \left|\left(\alpha x_k^/ + \beta y_k^/\right) - \left(\alpha L_1 + \beta L_2\right)\right|^{q_k} < \epsilon\right\} \\ &\supseteq \left\{k \in \mathbb{N} : |\alpha|^{q_k}\left|x_k^/ - L_1\right|^{q_k} < \frac{\epsilon}{2M_1}|\alpha|^{q_k} \cdot D\right\} \\ &\cap \left\{k \in \mathbb{N} : |\beta|^{q_k}\left|y_k^/ - L_2\right|^{q_k} < \frac{\epsilon}{2M_2}|\beta|^{q_k} \cdot D\right\}.\end{aligned} \tag{17}$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(q)$. Therefore $\mathcal{Z}^I(q)$ is a linear space. The rest of the result follows similarly. □

Theorem 13. *Let $(q_k) \in l_\infty$. Then $m_{\mathcal{Z}}^I(q)$ and $m_{\mathcal{Z}_0}^I(q)$ are paranormed spaces, paranormed by $g(x) = \sup_k |x_k|^{q_k/M}$ where $M = \max\{1, \sup_k q_k\}$.*

Proof. Let $x = (x_k), y = (y_k) \in m_{\mathcal{Z}}^I(q)$.

(1) Clearly, $g(x) = 0$ if and only if $x = 0$.

(2) $g(x) = g(-x)$ is obvious.

(3) Since $q_k/M \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$\sup_k |x_k + y_k|^{q_k/M} \leq \sup_k |x_k|^{q_k/M} + \sup_k |y_k|^{q_k/M} \tag{18}$$

(4) Now for any complex λ, we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.

Let $x_k \in m_{\mathcal{Z}}^I(q)$ such that $|x_k - L|^{q_k} \geq \epsilon$.

Therefore, $g(x - Le) = \sup_k |x_k - L|^{q_k/M} \leq \sup_k |x_k|^{q_k/M} + \sup_k |L|^{q_k/M}$, where $e = (1, 1, 1, \ldots)$.

Hence $g((\lambda_n x_k - \lambda L)) \leq g((\lambda_n x_k)) + g(\lambda L) = \lambda_n g(x) + \lambda g(L)$ as $(k \to \infty)$.

Hence $m_{\mathcal{Z}}^I(q)$ is a paranormed space.

The rest of the result follows similarly. □

Theorem 14. *$m^I_{\mathcal{Z}}(q)$ is a closed subspace of $l_\infty(q)$.*

Proof. Let $(x_k^{(n)})$ be a Cauchy sequence in $m^I_{\mathcal{Z}}(q)$ such that $x^{(n)} \to x$.

We show that $x \in m^I_{\mathcal{Z}}(q)$.

Since $(x_k^{(n)}) \in m^I_{\mathcal{Z}}(q)$, then there exists a_n such that

$$\left\{k \in \mathbb{N} : \left|x^{(n)} - a_n\right| \geq \epsilon\right\} \in I. \tag{19}$$

We need to show that

(1) (a_n) converges to a,

(2) if $U = \{k \in \mathbb{N} : |x_k - a| < \epsilon\}$, then $U^c \in I$.

(1) Since $(x_k^{(n)})$ is a Cauchy sequence in $m^I_{\mathcal{Z}}(q)$ then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k \left|x_k^{(n)} - x_k^{(i)}\right| < \frac{\epsilon}{3}, \quad \forall n, i \geq k_0. \tag{20}$$

For a given $\varepsilon > 0$, we have

$$\begin{aligned} B_{ni} &= \left\{k \in \mathbb{N} : \left|x_k^{(n)} - x_k^{(i)}\right| < \frac{\epsilon}{3}\right\}, \\ B_i &= \left\{k \in \mathbb{N} : \left|x_k^{(i)} - a_i\right| < \frac{\epsilon}{3}\right\}, \\ B_n &= \left\{k \in \mathbb{N} : \left|x_k^{(n)} - a_n\right| < \frac{\epsilon}{3}\right\}. \end{aligned} \tag{21}$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$.

Let $B^c = B_{ni}^c \cap B_i^c \cap B_n^c$, where $B = \{k \in \mathbb{N} : |a_i - a_n| < x\epsilon\}$.

Then $B^c \in I$.

We choose $k_0 \in B^c$, then for each $n,\ i \geq k_0$, we have

$$\begin{aligned} \{k \in \mathbb{N} : |a_i - a_n| < \epsilon\} \supseteq &\left\{k \in \mathbb{N} : \left|x_k^{(i)} - a_i\right| < \frac{\epsilon}{3}\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|x_k^{(n)} - x_k^{(i)}\right| < \frac{\epsilon}{3}\right\} \\ &\cap \left\{k \in \mathbb{N} : \left|x_k^{(n)} - a_n\right| < \frac{\epsilon}{3}\right\}. \end{aligned} \tag{22}$$

Then (a_n) is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $a_n \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then we show that if $U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\}$, then $U^c \in I$.

Since $x^{(n)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \left\{k \in \mathbb{N} : \left|x^{(q_0)} - x\right| < \left(\frac{\delta}{3D}\right)^M\right\} \tag{23}$$

which implies that $P^c \in I$.

The number q_0 can be so chosen that together with (23), we have

$$Q = \left\{k \in \mathbb{N} : \left|a_{q_0} - a\right|^{q_k} < \left(\frac{\delta}{3D}\right)^M\right\} \tag{24}$$

such that $Q^c \in I$.

Since $\{k \in \mathbb{N} : |x^{(q_0)} - a_{q_0}|^{q_k} \geq \delta\} \in I$. Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{k \in \mathbb{N} : \left|x^{(q_0)} - a_{q_0}\right|^{q_k} < \left(\frac{\delta}{3D}\right)^M\right\}. \tag{25}$$

Let $U^c = P^c \cap Q^c \cap S^c$, where $U = \{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\}$.

Therefore for each $k \in U^c$, we have

$$\begin{aligned} &\{k \in \mathbb{N} : |x_k - a|^{q_k} < \delta\} \\ &\quad \supseteq \left\{k \in \mathbb{N} : \left|x^{(q_0)} - x\right|^{q_k} < \left(\frac{\delta}{3D}\right)^M\right\} \\ &\quad \cap \left\{k \in \mathbb{N} : \left|x^{(q_0)} - a_{q_0}\right|^{q_k} < \left(\frac{\delta}{3D}\right)^M\right\} \\ &\quad \cap \left\{k \in \mathbb{N} : \left|a_{q_0} - a\right|^{q_k} < \left(\frac{\delta}{3D}\right)^M\right\}. \end{aligned} \tag{26}$$

Then the result follows. □

Since the inclusions $m^I_{\mathcal{Z}}(q) \subset l_\infty(q)$ and $m^I_{\mathcal{Z}_0}(q) \subset l_\infty(q)$ are strict, so in view of Theorem 14 we have the following result.

Theorem 15. *The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are nowhere dense subsets of $l_\infty(q)$.*

Theorem 16. *The spaces $m^I_{\mathcal{Z}}(q)$ and $m^I_{\mathcal{Z}_0}(q)$ are not separable.*

Proof. We shall prove the result for the space $m^I_{\mathcal{Z}}(q)$.

The proof for the other spaces will follow similarly.

Let M be an infinite subset of $\mathbb{N}$ of increasing natural numbers such that $M \in I$.

Let

$$q_k = \begin{cases} 1, & \text{if } k \in M, \\ 2, & \text{otherwise.} \end{cases} \tag{27}$$

Let $P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{for } k \in M \text{ and } x_k = 0, \text{otherwise}\}$.

Clearly P_0 is uncountable.

Consider the class of open balls $B_1 = \{B(z, 1/2) : z \in P_0\}$.

Let C_1 be an open cover of $m^I_{\mathcal{Z}}(q)$ containing B_1.

Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m^I_{\mathcal{Z}}(q)$.

Thus $m^I_{\mathcal{Z}}(q)$ is not separable. □

Theorem 17. *Let $G = \sup_k q_k < \infty$ and I an admissible ideal. Then the following is equivalent.*

(a) $(x_k) \in \mathcal{Z}^I(q)$;

(b) *there exists $(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I;*

(c) *there exists $(y_k) \in \mathcal{Z}(q)$ and $(x_k) \in \mathcal{Z}_0^I(q)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L|^{q_k}) \geq \epsilon\} \in I$;*

(d) *there exists a subset $K = \{k_1 < k_2 < \cdots\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty}|x_{k_n} - L|^{q_{k_n}} = 0$.*

Proof. (a) implies (b).

Let $(x_k) \in \mathcal{Z}^I(q)$. Then there exists $L \in \mathbb{C}$ such that

$$\left\{k \in \mathbb{N} : \left|x_k^{/} - L\right|^{q_k} \geq \epsilon\right\} \in I. \tag{28}$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\left\{k \leq m_t : \left|x_k^{/} - L\right|^{q_k} \geq t^{-1}\right\} \in I. \tag{29}$$

Define a sequence (y_k) as

$$y_k = x_k, \quad \forall k \leq m_1. \tag{30}$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } \left|x_k^{/} - L\right|^{q_k} < t^{-1}, \\ L, & \text{otherwise.} \end{cases} \tag{31}$$

Then $(y_k) \in \mathcal{Z}(q)$ and form the following inclusion:

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \left\{k \leq m_t : \left|x_k^{/} - L\right|^{q_k} \geq \epsilon\right\} \in I, \tag{32}$$

we get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c).

For $(x_k) \in \mathcal{Z}^I(q)$. Then there exists $(y_k) \in \mathcal{Z}(q)$ such that $x_k = y_k$, for a.a.k.r.I.

Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $k \in I$.

Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

Then $z_k \in \mathcal{Z}_0^I(q)$ and $y_k \in \mathcal{Z}(q)$.

(c) implies (d).

Suppose (c) holds.

Let $\epsilon > 0$ be given.

Let $P_1 = \{k \in \mathbb{N} : |z_k^{/}|^{q_k} \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < \cdots\} \in \pounds(I). \tag{34}$$

Then we have $\lim_{n\to\infty}|x_{k_n}^{/} - L|^{q_{k_n}} = 0$.

(d) implies (a).

Let $K = \{k_1 < k_2 < k_3 < \cdots\} \in \pounds(I)$ and $\lim_{n\to\infty}|x_{k_n}^{/} - L|^{q_{k_n}} = 0$.

Then for any $\epsilon > 0$, and Lemma 11, we have

$$\left\{k \in \mathbb{N} : \left|x_k^{/} - L\right|^{q_k} \geq \epsilon\right\} \subseteq K^c \cup \left\{k \in K : \left|x_k^{/} - L\right|^{q_k} \geq \epsilon\right\}. \tag{35}$$

Thus $(x_k) \in \mathcal{Z}^I(q)$. □

Theorem 18. *Let $h = \inf_k q_k$ and $G = \sup_k q_k$. Then the following results are equivalent.*

(a) $G < \infty$ *and* $h > 0$.

(b) $\mathcal{Z}_0^I(q) = \mathcal{Z}_0^I$.

Proof. Suppose that $G < \infty$ and $h > 0$, then the inequalities $\min\{1, s^h\} \leq s^{q_k} \leq \max\{1, s^G\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$.

Therefore the equivalence of (a) and (b) is obvious. □

Theorem 19. *Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m_{\mathcal{Z}_0}^I(q) \supseteq m_{\mathcal{Z}_0}^I(r)$ if and only if $\lim_{k\in K} \inf(q_k/r_k) > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. Let $\lim_{k\in K} \inf(q_k/r_k) > 0$ and $(x_k) \in m_{\mathcal{Z}_0}^I(r)$. Then there exists $\beta > 0$ such that $q_k > \beta r_k$, for all sufficiently large $k \in K$.

Since $(x_k) \in m_{\mathcal{Z}_0}^I(r)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : |x_k|^{r_k} \geq \epsilon\} \in I. \tag{36}$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$.

Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : |x_k|^{q_k} \geq \epsilon\} \subseteq \left\{k \in \mathbb{N} : |x_k|^{\beta r_k} \geq \epsilon\right\} \in I. \tag{37}$$

Therefore $(x_k) \in m_{\mathcal{Z}_0}^I(q)$.

The converse part of the result follows obviously. □

Theorem 20. *Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m_{\mathcal{Z}_0}^I(r) \supseteq m_{\mathcal{Z}_0}^I(q)$ if and only if $\lim_{k\in K} \inf(r_k/q_k) > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. The proof follows similarly as the proof of Theorem 19. □

Theorem 21. *Let (q_k) and (r_k) be two sequences of positive real numbers. Then $m_0^I(r) = m_0^I(q)$ if and only if $\lim_{k\in K} \inf(q_k/r_k) > 0$, and $\lim_{k\in K} \inf(r_k/q_k) > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

Proof. By combining Theorems 19 and 20, we get the required result. □

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