Research Article

On Paranorm Zweier I-Convergent Sequence Spaces

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In this paper, we introduce the paranorm Zweier I-convergent sequence spaces $\mathcal{Z}_I(q, q_0_I(q)$, and $\mathcal{Z}_\infty(q_I(q_I(q$ for $q = (q_k)$, a sequence of positive real numbers. We study some topological properties, prove the decomposition theorem, and study some inclusion relations on these spaces.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all natural, real, and complex numbers, respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}, \quad (1)$$

the space of all real or complex sequences.

Let $l_{\infty}$, $c$, and $c_0$ denote the Banach spaces of bounded, convergent, and null sequences, respectively, normed by $\|x\|_{\infty} = \sup_k |x_k|$. The following subspaces of $\omega$ were first introduced and discussed by Maddox [1]:

$$l(p) := \{x \in \omega : \sum_k |x_k|^p < \infty\},$$

$$l_{\infty}(p) := \{x \in \omega : \sup_k |x_k|^p < \infty\},$$

$$c(p) := \{x \in \omega : \lim_k |x_k - l|^p = 0, \text{ for some } l \in \mathbb{C}\},$$

$$c_0(p) := \{x \in \omega : \lim_k |x_k|^p = 0\},$$

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After that Lascarides [2, 3] defined the following sequence spaces:

$$l_{\infty} \{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k|^p t_k < \infty\},$$

$$c_0 \{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k|^p t_k = 0\}, \quad (2)$$

$$l \{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^\infty |x_k|^p t_k < \infty\},$$

where $t_k = \frac{p_k}{\sum_{k=1}^\infty p_k}$, for all $k \in \mathbb{N}$.

Each linear subspace of $\omega$, for example, $\lambda, \mu \subset \omega$, is called a sequence space.

A sequence space $\lambda$ with linear topology is called a $K$-space provided each map $p_i \rightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

A $K$-space $\lambda$ is called an FK-space provided $\lambda$ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let $\lambda$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$.
Then we say that $A$ defines a matrix mapping from $\lambda$ to $\mu$, and we denote it by writing $A : \lambda \rightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = (Ax_n)_n$, the $A$ transform of $x$ is in $\mu$, where

\[(Ax)_n = \sum_{k=1}^{n} a_{nk}x_k, \quad (n \in \mathbb{N}). \tag{3}\]

By $(\lambda : \mu)$, we denote the class of matrices $A$ such that $A : \lambda \rightarrow \mu$.

Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (3) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method has been recently employed by Altay et al. [4], Başar and Altay [5], Malkowsky [6], Ng and Lee [7], and Wang [8].

Sengönül [9] defined the sequence $y = (y_i)$ which is frequently used as the $Z^p$ transform of the sequence $x = (x_i)$, that is,

\[y_i = px_i + (1 - p)x_{i-1}, \tag{4}\]

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and $Z^p$ denotes the matrix $Z^p = (z_{ik})$ defined by

\[z_{ik} = \begin{cases} p, & (i = k), \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, & \text{otherwise} \end{cases} \tag{5}\]

Following Başar and Altay [5], Şengönül [9] introduced the Zweier sequence spaces $Z$ and $Z_0$ as follows:

\[Z = \left\{ x = (x_k) \in \omega : Z^p x \in \ell \right\}, \tag{6}\]

\[Z_0 = \left\{ x = (x_k) \in \omega : Z^p x \in \ell_0 \right\}. \tag{7}\]

Here we quote below some of the results due to Şengönül [9] which we will need in order to establish the results of this paper.

**Theorem 1** (see [9, Theorem 2.1]). The sets $Z$ and $Z_0$ are the linear spaces with the coordinate wise addition and scalar multiplication which are the BK-spaces with the norm

\[\|x\|_Z = \|x\|_X_0 = \left\|Z^p x\right\|_c. \tag{8}\]

**Theorem 2** (see [9, Theorem 2.2]). The sequence spaces $Z$ and $Z_0$ are linearly isomorphic to the spaces $c$ and $c_0$, respectively, that is, $Z \cong c$ and $Z_0 \cong c_0$.

**Theorem 3** (see [9, Theorem 2.3]). The inclusions $Z_0 \subset Z$ strictly hold for $p \neq 1$.

**Theorem 4** (see [9, Theorem 2.6]). $Z_0$ is solid.

**Theorem 5** (see [9, Theorem 3.6]). $Z$ is not a solid sequence space.

The concept of statistical convergence was first introduced by Fast [10] and also independently by Buck [11] and Schoenberg [12] for real and complex sequences. Further this concept was studied by Connor [13, 14], Connor et al. [15], and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for a given $\epsilon > 0$ as

\[\lim_{k \to \infty} \frac{1}{k} \left| \left| \{i : |x_i - L| \geq \epsilon, i \leq k \} \right| \right| = 0. \tag{9}\]

The notion of $I$-convergence is a generalization of the statistical convergence. At the initial stage, it was studied by Kostyrko et al. [16]. Later on, it was studied by Šalát et al. [17, 18], Demirci [19], Tripathy and Hazarika [20, 21], and Khan et al. [22–24].

Here we give some preliminaries about the notion of $I$-convergence.

Let $X$ be a nonempty set. Then a family of sets $I \subseteq 2^X$ (denoting the power set of $X$) is said to be an ideal if $I$ is additive, that is, $A, B \in I \Rightarrow A \cup B \in I$, and hereditary, that is, $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\mathcal{E} \subseteq 2^X$ is said to be a filter on $X$ if and only if $\phi \notin \mathcal{E}$, for $A, B \in \mathcal{E}$ we have $A \cap B \in \mathcal{E}$ and for each $A \in \mathcal{E}$ and $A \subseteq B$ implies $B \in \mathcal{E}$.

An ideal $I \subseteq 2^X$ is called nontrivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{x : x \in X\} \subseteq I$.

A non-trivial ideal $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

For each ideal $I$, there is a filter $\mathcal{E}(I)$ corresponding to $I$. that is, $\mathcal{E}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

**Definition 1**. A sequence $(x_k) \in \omega$ is said to be $I$-convergent to a number $L$ if $\{k \in N : |x_k - L| \geq \epsilon\} \in I$ for every $\epsilon > 0$. In this case we write $I \lim x_k = L$.

The space $c_I$ of all $I$-convergent sequences converging to $L$ is given by

\[c_I = \{ (x_k) \in \omega : \{k \in N : |x_k - L| \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{C} \}. \tag{10}\]

**Definition 7**. A sequence $(x_k) \in \omega$ is said to be $I$-null if $L = 0$.

In this case we write $I \lim x_k = 0$.

**Definition 8**. A sequence $(x_k) \in \omega$ is said to be $I$-Cauchy if for every $\epsilon > 0$ there exists a number $k_0 \in \mathbb{N}$ such that $k \in N : |x_k - x_m| \geq \epsilon\} \in I$ for all $k, m \geq k_0$.

**Definition 9**. A sequence $(x_k) \in \omega$ is said to be $I$-bounded if there exists $M > 0$ such that $\{k \in N : |x_k| > M\} \in I$.

**Definition 10**. Let $(x_k, y_k)$ be two sequences. We say that $(x_k - y_k)$ for all $k$ relative to $I$ (a.a.k.r.I), if $(k \in \mathbb{N} : x_k \neq y_k) \in I$.

The following lemma will be used for establishing some results of this paper.

**Lemma 11**. If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$ (see [20, 21]) cf. ([17, 18, 20–24]).
Recently Khan and Ebadullah [25] introduced the following classes of sequence spaces:

\[ \mathcal{X}^I = \{ x = (x_k) \in \omega : I - \lim Z^p x = L \} \]

for some \( L \in \mathbb{C} \},

\[ \mathcal{X}^0 = \{ x = (x_k) \in \omega : I - \lim Z^p x = 0 \} \}

\[ \mathcal{X}_{\infty}^I = \{ x = (x_k) \in \omega : \sup_k |Z^p x| < \infty \} \}

(10)

We also denote by

\[ m^I_x = \mathcal{X}_{\infty}^I \cap \mathcal{X}^I, \quad m^0_x = \mathcal{X}_{\infty}^I \cap \mathcal{X}^0. \]

(11)

In this paper, we introduce the following classes of sequence spaces:

\[ \mathcal{X}^I(q) = \{ x = (x_k) \in \omega : \{ k \in \mathbb{N} : |Z^p x - L|^q_k \geq \epsilon \} \in I, \]

for some \( L \in \mathbb{C} \},

\[ \mathcal{X}^0(q) = \{ x = (x_k) \in \omega : \{ k \in \mathbb{N} : |Z^p x|^q_k \geq \epsilon \} \in I \};

\[ \mathcal{X}_{\infty}^I(q) = \{ x = (x_k) \in \omega : \sup_k |Z^p x|^q_k < \infty \} \}

(12)

We also denote by

\[ m^I_x(q) = \mathcal{X}_{\infty}^I(q) \cap \mathcal{X}^I(q), \]

\[ m^0_x(q) = \mathcal{X}_{\infty}^I(q) \cap \mathcal{X}^0(q), \]

(13)

where \( q = (q_k) \) is a sequence of positive real numbers.

Throughout the paper, for the sake of convenience now we will denote by \( Z^p x = x', Z^p y = y', Z^p z = z' \) for all \( x, y, z \in \omega \).

**2. Main Results**

**Theorem 12.** The classes of sequences \( \mathcal{X}^I(q), \mathcal{X}^0(q), m^I_x(q) \) and \( m^0_x(q) \) are linear spaces.

**Proof.** We shall prove the result for the space \( \mathcal{X}^I(q) \).

The proof for the other spaces will follow similarly.

Let \( (x_k), (y_k) \in \mathcal{X}^I(q) \), and let \( \alpha, \beta \) be scalars. Then for a given \( \epsilon > 0 \), we have:

\[ \{ k \in \mathbb{N} : |x_k - L|^q_k \geq \epsilon \}, \quad \text{for some } L \in \mathbb{C} \} \in I \]

(14)

\[ \{ k \in \mathbb{N} : |y_k - L|^q_k \geq \epsilon \}, \quad \text{for some } L \in \mathbb{C} \} \in I, \]

where

\[ M_1 = D \cdot \max \left\{ 1, \sup \alpha^q_k \right\}, \]

\[ M_2 = D \cdot \max \left\{ 1, \sup \beta^q_k \right\}, \]

(15)

\[ D = \max \{ 1, 2^{M-1} \} \quad \text{where } H = \sup k q_k \geq 0. \]

Let

\[ A_1 = \{ k \in \mathbb{N} : |x_k - L|^q_k < \frac{\epsilon}{2M_1}, \text{ for some } L \in \mathbb{C} \} \in I \]

\[ A_2 = \{ k \in \mathbb{N} : |y_k - L|^q_k < \frac{\epsilon}{2M_2}, \text{ for some } L \in \mathbb{C} \} \in I \]

(16)

be such that \( A_1 \cap A_2 \subseteq I \). Hence \( (\alpha x_k + \beta y_k) \in \mathcal{X}^I(q) \). Therefore \( \mathcal{X}^I(q) \) is a linear space. The rest of the result follows similarly.

**Theorem 13.** Let \((q_k) \in l_\infty\). Then \( m^I_x(q) \) and \( m^0_x(q) \) are paranormed spaces, paranormed by \( g(x) = \sup_k |x_k|^q_k/M \) where \( M = \max \{ 1, \sup_k q_k \} \).

**Proof.** Let \( x = (x_k), y = (y_k) \in m^I_x(q) \).

(1) Clearly, \( g(x) = 0 \) if and only if \( x = 0 \).

(2) \( g(x) = g(-x) \) is obvious.

(3) Since \( q_k/M \leq 1 \) and \( M > 1 \), using Minkowski’s inequality, we have

\[ \sup_k |x_k + y_k|^q_k/M \leq \sup_k |x_k|^q_k/M + \sup_k |y_k|^q_k/M \]

(18)

(4) Now for any complex \( \lambda \), we have \( \lambda \) such that \( \lambda \to \lambda \) \( \lambda \to \infty \).

Let \( x_k \in m^I_x(q) \) such that \( |x_k - L|^q_k \geq \epsilon \).

Therefore, \( g(x - L) = \sup_k |x_k - L|^q_k \leq \sup_k |x_k|^q_k/M + \sup_k |L|^q_k/M \), where \( e = (1, 1, 1, ...) \).

Hence \( g((\lambda x_k - \lambda L)) \leq g((\lambda x_k)) + g(\lambda L) = \lambda_n g(x) + \lambda g(L) \) as \( k \to \infty \).

Hence \( m^I_x(q) \) is a paranormed space.

The rest of the result follows similarly.
Theorem 14. $m_I^I(q)$ is a closed subspace of $I_\infty(q)$.

Proof. Let $(x^{(n)}_k)$ be a Cauchy sequence in $m_I^I(q)$ such that $x^{(n)}_k \to x$.
We show that $x \in m_I^I(q)$.

Since $(x^{(n)}_k) \in m_I^I(q)$, then there exists $a_n$ such that
$$\{k \in \mathbb{N} : |x^{(n)}_k - a_n| \geq \epsilon \} \in I.$$  \hspace{1cm} (19)

We need to show that

(1) $(a_n)$ converges to $a$,

(2) if $U = \{k \in \mathbb{N} : |x_k - a| < \epsilon \}$, then $U^c \in I$.

(1) Since $(x^{(n)}_k)$ is a Cauchy sequence in $m_I^I(q)$ then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that
$$\sup_k |x^{(n)}_k - x^{(j)}_k| < \frac{\epsilon}{3}, \ \forall n, i \geq k_0.$$  \hspace{1cm} (20)

For a given $\epsilon > 0$, we have
$$B_{n\epsilon} = \{k \in \mathbb{N} : |x^{(n)}_k - x^{(j)}_k| < \frac{\epsilon}{3}\},$$
$$B_{\epsilon} = \{k \in \mathbb{N} : |x^{(j)}_k - a| < \frac{\epsilon}{3}\},$$
$$B_{n\epsilon} = \{k \in \mathbb{N} : |x^{(n)}_k - a| < \frac{\epsilon}{3}\}.$$  \hspace{1cm} (21)

Then $B_{n\epsilon} \cap B_{\epsilon} \cap B_{n\epsilon} \in I$.

Let $B = B_{n\epsilon} \cap B_{\epsilon} \cap B_{n\epsilon}$, where $B = \{k \in \mathbb{N} : |a_n - a| < \epsilon\}$. Then $B \in I$.

We choose $k_0 \in B$, then for each $n, i \geq k_0$, we have
$$\{k \in \mathbb{N} : |a_i - a_n| < \epsilon \} \supseteq \{k \in \mathbb{N} : |x^{(j)}_k - a| < \frac{\epsilon}{3}\}$$
$$\cap \{k \in \mathbb{N} : |x^{(n)}_k - x^{(j)}_k| < \frac{\epsilon}{3}\}$$
$$\cap \{k \in \mathbb{N} : |x^{(n)}_k - a| < \frac{\epsilon}{3}\}.$$  \hspace{1cm} (22)

Then $(a_n)$ is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $a_n \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then we show that if $U = \{k \in \mathbb{N} : |x_k - a|^k \geq \delta\}$, then $U^c \in I$.

Since $x^{(n)}_k \to x$, then there exists $q_0 \in \mathbb{N}$ such that
$$P = \{k \in \mathbb{N} : |x^{(n)}_k - x| < \left(\frac{\delta}{3D}\right)^M\}.$$  \hspace{1cm} (23)

which implies that $P^c \in I$.

The number $q_0$ can be so chosen that together with (23), we have
$$Q = \{k \in \mathbb{N} : |a_k - a|^k < \left(\frac{\delta}{3D}\right)^M\}.$$  \hspace{1cm} (24)

such that $Q^c \in I$.

Since $\{k \in \mathbb{N} : |x^{(n)}_k - a_k|^k \geq \delta\} \in I$. Then we have a subset $S$ of $\mathbb{N}$ such that $S^c \in I$, where
$$S = \left\{k \in \mathbb{N} : |x^{(n)}_k - a_k|^k < \left(\frac{\delta}{3D}\right)^M\right\}.$$  \hspace{1cm} (25)

Let $U^c = P^c \cap Q^c \cap S^c$, where $U = \{k \in \mathbb{N} : |x_k - a|^k < \delta\}$. Therefore for each $k \in U^c$, we have
$$\{k \in \mathbb{N} : |x_k - a|^k < \delta\},$$
$$\supseteq \left\{k \in \mathbb{N} : |x^{(n)}_k - x_k|^k < \left(\frac{\delta}{3D}\right)^M\right\}$$
$$\cap \left\{k \in \mathbb{N} : |a_k - a|^k < \left(\frac{\delta}{3D}\right)^M\right\}.$$  \hspace{1cm} (26)

Then the result follows.

Since the inclusions $m_I^I(q) \subset I_\infty(q)$ and $m_I^I(q) \subset I_\infty(q)$ are strict, so in view of Theorem 14 we have the following result.

Theorem 15. The spaces $m_I^I(q)$ and $m_I^I(q)$ are nowhere dense subsets of $I_\infty(q)$.

Theorem 16. The spaces $m_I^I(q)$ and $m_I^I(q)$ are not separable.

Proof. We shall prove the result for the space $m_I^I(q)$.

The proof for the other spaces will follow similarly.

Let $M$ be an infinite subset of $\mathbb{N}$ of increasing natural numbers such that $M \in I$.

Let
$$q_k = \begin{cases} 1, & \text{if } k \in M, \\ 2, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (27)

Let $P_0 = \{(x_k) : x_k = 0 \text{ or } 1, \text{for } k \in M \text{ and } x_k = 0, \text{ otherwise}\}$. Clearly $P_0$ is uncountable.

Consider the class of open balls $B_1 = \{B(z, 1/2) : z \in P_0\}$. Let $C_1$ be an open cover of $m_I^I(q)$ containing $B_1$.

Since $B_1$ is uncountable, so $C_1$ cannot be reduced to a countable subcover for $m_I^I(q)$.

Thus $m_I^I(q)$ is not separable.

Theorem 17. Let $G = \sup_k q_k < \infty$ and $I$ an admissible ideal. Then the following is equivalent.

(a) $(x_k) \in \mathcal{L}^I(q)$;
(b) there exists $(y_k) \in \mathcal{L}(q)$ such that $x_k = y_k$, for $a.a.k.r.I$;
(c) there exists $(y_k) \in \mathcal{L}(q)$ and $(x_k) \in \mathcal{L}_0(q)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L| \geq \epsilon\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $K \in \mathcal{L}(I)$ and $\lim_{n \to \infty} |x_{k_n} - L|^k = 0$. 


Proof. (a) implies (b).
Let $(x_k) \in \mathcal{I}(q)$. Then there exists $L \in \mathbb{C}$ such that
\[
\left\{ k \in \mathbb{N} : |x_k^j - L|^\rho_k \geq c \right\} \in I. \tag{28}
\]
Let $(m_k)$ be an increasing sequence with $m_k \in \mathbb{N}$ such that
\[
\left\{ k \leq m_k : |x_k^j - L|^\rho_k \geq t^{-1} \right\} \in I. \tag{29}
\]
Define a sequence $(y_k)$ as
\[
y_k = x_k, \quad \forall k \leq m_1, \tag{30}
\]
For $t < k \leq m_{t+1}, t \in \mathbb{N}$,
\[
y_k = \begin{cases} x_k, & \text{if } |x_k^j - L|^\rho_k < t^{-1}, \\ L, & \text{otherwise.} \end{cases} \tag{31}
\]
Then $(y_k) \in \mathcal{I}(q)$ and form the following inclusion:
\[
\left\{ k \leq m_1 : x_k \neq y_k \right\} \subseteq \left\{ k \leq m_1 : |x_k^j - L|^\rho_k \geq t^{-1} \right\} \subseteq I. \tag{32}
\]
we get $x_k = y_k$, for a.a.k.r.l.
(b) implies (c).
For $(x_k) \in \mathcal{I}(q)$, then $(y_k) \in \mathcal{I}(q)$ such that $x_k = y_k$, for a.a.k.r.l.
Let $K = \{ k \in \mathbb{N} : x_k \neq y_k \}$, then $k \in I$.
Define a sequence $(z_k)$ as
\[
z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases} \tag{33}
\]
Then $z_k \in \mathcal{I}(q)$ and $y_k \in \mathcal{I}(q)$.

(c) implies (d).
Suppose (c) holds.
Let $\epsilon > 0$ be given.
Let $P_1 = \{ k \in \mathbb{N} : |x_k^j|^\rho_k \geq \epsilon \} \in I$ and
\[
K = P_1^c = \{ k_1 < k_2 < k_3 < \cdots \} \in \mathcal{E}(I). \tag{34}
\]
Then we have $\lim_{n \to \infty} |x_{k_n}^j - L|^\rho_{k_n} = 0$.
(d) implies (a).
Let $K = \{ k_1 < k_2 < k_3 < \cdots \} \in \mathcal{E}(I)$ and $\lim_{n \to \infty} |x_{k_n}^j - L|^\rho_{k_n} = 0$.
Then for any $\epsilon > 0$, and Lemma 11, we have
\[
\left\{ k \in \mathbb{N} : |x_k^j - L|^\rho_k \geq \epsilon \right\} \subseteq K \cup \left\{ k \in K : |x_k^j - L|^\rho_k \geq \epsilon \right\}. \tag{35}
\]
Thus $(x_k) \in \mathcal{I}(q)$.

Theorem 18. Let $h = \inf_{k \in K} q_k$ and $G = \sup_{k \in K} q_k$. Then the following results are equivalent.

(a) $G < \infty$ and $h > 0$.
(b) $\mathcal{I}(q) = \mathcal{I}$.

Proof. Suppose that $G < \infty$ and $h > 0$, then the inequalities $\min\{1, s^h\} \leq s^h \leq \max\{1, s^h\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$.

Therefore the equivalence of (a) and (b) is obvious. □

Theorem 19. Let $(q_k)$ and $(r_k)$ be two sequences of positive real numbers. Then $m_{I_q}(q) \geq m_{I_r}(r)$ if and only if $\lim_{k \to K} \inf (q_k/r_k) > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. Let $\lim_{k \to K} \inf (q_k/r_k) > 0$ and $(x_k) \in m_{I_q}(q)$. Then there exists $\beta > 0$ such that $q_k > \beta r_k$, for all sufficiently large $k \in K$.

Since $(x_k) \in m_{I_q}(q)$ for a given $\epsilon > 0$, we have
\[
B_0 = \{ k \in \mathbb{N} : |x_k|^\beta_k \geq \epsilon \} \in I. \tag{36}
\]
Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$.

Then for all sufficiently large $k \in G_0$,
\[
\{ k \in \mathbb{N} : |x_k|^\beta_k \geq \epsilon \} \subseteq \{ k \in \mathbb{N} : |x_k|^\beta_k \geq \epsilon \} \in I. \tag{37}
\]
Therefore $(x_k) \in m_{I_r}(r)$.

The converse part of the result follows obviously. □

Theorem 20. Let $(q_k)$ and $(r_k)$ be two sequences of positive real numbers. Then $m_{I_q}(r) \geq m_{I_q}(q)$ if and only if $\lim_{k \in K} \inf (r_k/q_k) > 0$, where $K \subseteq \mathbb{N}$ such that $K \in I$.

Proof. By combining Theorems 19 and 20, we get the required result. □

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