Research Article

A Product of Theta-Functions Analogous to Ramanujan's Remarkable Product of Theta-Functions and Applications

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We define a product $l_{k,n}$ for any positive real numbers k and n involving Ramanujan's theta-functions $\phi(q)$ and $\psi(q)$ which is analogous to Ramanujan's remarkable product of theta-functions recorded by Ramanujan (1957) and study its several properties. We prove general theorems for the explicit evaluations of $l_{k,n}$ and find some explicit values. As application of the product $l_{k,n}$, we also offer explicit formulas for explicit values of Ramanujan's continued fraction $V(q)$ in terms of $l_{k,n}$ and give examples.

1. Introduction

Ramanujan's theta-functions $\phi(q)$, $\psi(q)$, and $f(-q)$ are defined as

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}}, \quad (1)$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}, \quad (2)$$

where $(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k)$.

On page 338 of his first notebook, Ramanujan [1] defined the remarkable product of theta-functions as

$$a_{m,n} = ne^{-(\pi/4)(n-1)\sqrt{m/n}} \frac{\psi^2(e^{-\pi\sqrt{mn}}) \phi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{m/n}}) \phi^2(-e^{-2\pi\sqrt{m/n}})}, \quad (3)$$

where m and n are positive real numbers. He then, on pages 338 and 339, offered a list of eighteen particular values of the product $a_{m,n}$. All these eighteen values are proved by Berndt et al. [2]. An account of these can be found in Berndt's book [3]. Naika and Dharmendra [4] also established some general

theorems for explicit evaluations of the product $a_{m,n}$ and found some new explicit values therefrom. Further results on $a_{m,n}$ can be found in [5, 6].

In [7], Mahadeva Naika et al. defined the product

$$b_{m,n} = ne^{-(\pi/4)(n-1)\sqrt{m/n}} \frac{\psi^2(-e^{-\pi\sqrt{mn}}) \phi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{m/n}}) \phi^2(-e^{-2\pi\sqrt{m/n}})}. \quad (4)$$

They established general theorems for explicit evaluations of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [8] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots, \quad |q| < 1. \quad (5)$$

and found some particular values of $V(q)$.

Motivated by the above work, in this paper, we define the product of theta-functions $l_{k,n}$ as

$$l_{k,n} = \frac{\psi(-q) \phi(q^k)}{q^{(k-1)/8} \psi(-q^k) \phi(q)}; \quad q := e^{-\pi\sqrt{n/k}}, \quad (6)$$

where n and k are positive real numbers. We establish several properties of the product $l_{k,n}$. We prove general formulas for explicit evaluations of $l_{k,n}$ and find its explicit values. As

application of the product $l_{k,n}$, we also offer explicit formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of $l_{k,n}$ and give examples.

In Section 2, we collect some preliminary results. In Section 3, we prove several properties of the product $l_{k,n}$. Section 4, is devoted to find explicit values of $l_{k,n}$. Finally in Section 5, we offer explicit formulas for explicit evaluations of continued fraction $V(q)$ in terms of $l_{k,n}$ with examples.

To end this introduction, we define Ramanujan's modular equation. Let K , K' , L , and L' denote the complete elliptic integrals of the first kind associated with the moduli k , k' , l , and l' , respectively. Suppose that the equality

$$n \frac{K'}{K} = \frac{L'}{L} \quad (7)$$

holds for some positive integer n . Then a modular equation of degree n is a relation between the moduli k and l which is implied by (7). Ramanujan recorded his modular equations in terms of α and β , where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α . By denoting $z_r = \phi^2(q^r)$, where $q = \exp(-\pi K'/K)$, $|q| < 1$, the multiplier m connecting α and β is defined by $m = z_1/z_n$.

2. Preliminary Results

Lemma 1 (see [9, p. 43, Entry 27(i)]). *If $\alpha\beta = \pi$ then*

$$\sqrt{\alpha}\phi(e^{-\alpha^2}) = \sqrt{\beta}\phi(e^{-\beta^2}). \quad (8)$$

Lemma 2 (see [10, p. 1049, (1.13)]). *If $\alpha\beta = \pi^2$ then*

$$e^{-\alpha/8} \sqrt[4]{\alpha} \psi(-e^{-\alpha}) = e^{-\beta/8} \sqrt[4]{\beta} \psi(-e^{-\beta}). \quad (9)$$

Lemma 3 (see [9, p. 122, Entry 10(i)]). *One has*

$$\phi(q) = \sqrt{z_1}. \quad (10)$$

Lemma 4 (see [9, p. 123, Entry 11(ii)]). *One has*

$$\psi(-q) = \sqrt{\frac{z_1}{2}} \{\alpha(1-\alpha)\}^{1/8} q^{-1/8}. \quad (11)$$

Lemma 5 (see [9, p. 124, Entry 12(v), (vi)]). *One has*

$$\begin{aligned} \chi(q) &= 2^{1/6} \left\{ \frac{\alpha(1-\alpha)}{q} \right\}^{-1/24}, \\ \chi(-q) &= 2^{1/6} (1-\alpha)^{1/12} \left(\frac{\alpha}{q} \right)^{-1/24}. \end{aligned} \quad (12)$$

One also notes that if we replace q by q^n in the Lemmas 3 and 4, then z_1 and α will be replaced by z_n and β , respectively, where β has degree n over α .

Lemma 6 (see [9, p. 345, Entry 1(i), (ii)]). *If $V(q)$ is as defined in (5), then*

$$1 + \frac{1}{V(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \quad (13)$$

$$1 + \frac{1}{V^3(q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}, \quad (14)$$

$$2V(q) = 1 - \frac{\phi(-q^{1/3})}{\phi(-q^3)}. \quad (15)$$

Lemma 7 (see [9, p. 347]). *If $V(q)$ is as defined in (5), then*

$$1 - 8V^3(q) = \frac{\phi^4(-q)}{\phi^4(-q^3)}. \quad (16)$$

Lemma 8 (see [9, p. 231, Entry 5(xii)]). *Let $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q = (\beta(1-\beta)/\alpha(1-\alpha))^{1/4}$, then*

$$Q + Q^{-1} + 2\sqrt{2}(P - P^{-1}) = 0, \quad (17)$$

where β has degree 3 over α .

Lemma 9 (see [9, p. 282, Entry 13(xiv)]). *Let $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ and $Q = (\beta(1-\beta)/\alpha(1-\alpha))^{1/8}$, then*

$$Q + Q^{-1} + 2(P - P^{-1}) = 0, \quad (18)$$

where β has degree 5 over α .

3. Some Properties of $l_{k,n}$

In this section, we study some properties of the product $l_{k,n}$.

Theorem 10. *For all positive real numbers k and n , one has*

- (i) $l_{k,1} = 1$,
- (ii) $l_{k,1/n} = 1/l_{k,n}$,
- (iii) $l_{k,n} = l_{n,k}$.

Proof. Using the definition of $l_{k,n}$ and Lemmas 1 and 2, we easily arrive at (i). Replacing n by $1/n$ in $l_{k,n}$ and using Lemmas 1 and 2, we find that $l_{k,n} l_{k,1/n} = 1$ which completes the proof of (ii). To prove (iii), we interchange n and k in $l_{k,n}$. $\square$

Remark 11. By using the definitions of $\phi(q)$, $\psi(q)$, and $l_{k,n}$, it can be seen that $l_{k,n}$ has positive real value and that the values of $l_{k,n}$ increases as n increases when $k > 1$. Thus, by Theorem 10(i), $l_{k,n} > 1$ for all $n > 1$ if $k > 1$.

Theorem 12. *For all positive real numbers k , m , and n , one has*

$$l_{k,n/m} = \frac{l_{mk,n}}{l_{nk,m}}. \quad (19)$$

Proof. Using the definition of $l_{k,n}$, we find that

$$\frac{l_{mk,n}}{l_{nk,m}} = e^{-\pi(\sqrt{n/mk} - \sqrt{m/nk})/8} \frac{\psi\left(-e^{-\pi\sqrt{n/mk}}\right)\phi\left(e^{-\pi\sqrt{m/nk}}\right)}{\psi\left(-e^{-\pi\sqrt{m/nk}}\right)\phi\left(e^{-\pi\sqrt{n/mk}}\right)}. \quad (20)$$

Applying Lemmas 1 and 2 in the denominator of right hand side of (20) and simplifying using Theorem 10(ii) and (iii), we complete the proof. $\square$

Corollary 13. For all positive real numbers k and n , one has

$$l_{k^2,n} = l_{nk,n} l_{k,n/k}. \quad (21)$$

Proof. Setting $k = n$ in Theorem 12 and simplifying using Theorem 10(ii), we obtain

$$l_{k^2,m} = l_{mk,k} l_{k,m/k}. \quad (22)$$

Replacing m by n , we complete the proof. $\square$

Theorem 14. Let k, a, b, c , and d be positive real numbers such that $ab = cd$. Then

$$l_{a,b} l_{kc,kd} = l_{ka,kb} l_{c,d}. \quad (23)$$

Proof. From the definition of $l_{k,n}$ and using $ab = cd$ for positive real numbers k, a, b, c , and d , we deduce that

$$l_{ka,kb} l_{a,b}^{-1} = l_{kc,kd} l_{c,d}^{-1}. \quad (24)$$

Rearranging the terms in (24) we arrive at the desired result. $\square$

Corollary 15. For any positive real numbers n and p , one has

$$l_{np,np} = l_{np^2,n} l_{p,p}. \quad (25)$$

Proof. The result follows immediately from Theorem 14 with $a = p^2, b = 1, c = d = p$ and $k = n$. $\square$

Theorem 16. For all positive real numbers a, b, c , and d , one has

$$l_{a/b,c/d} = \frac{l_{ad,bc}}{l_{ac,bd}}. \quad (26)$$

Proof. Applying Theorem 10(iii) in Theorem 12, we deduce that, for all positive real numbers a, b , and n

$$l_{a/b,n} = l_{a,bn} l_{b,an}^{-1}. \quad (27)$$

Now setting $n = c/d$ and again employing Theorems 12 and 10(iii) in (27), we complete the proof. $\square$

Theorem 17. For all positive real numbers a and b , one has

- (i) $l_{a/b,a/b} = l_{b,b} l_{a,a/b^2}$,
- (ii) $l_{a,a} l_{a,b^2/a} = l_{b,b} l_{b,a^2/b}$,
- (iii) $l_{a,a} l_{b,a^2b} = l_{b,b} l_{a,ab^2}$.

Proof. By using Theorems 10(ii) and 16, we find that

$$l_{a/b,a/b} = l_{b,b} l_{a,a/b^2}. \quad (28)$$

So we complete the proof of (i). Setting $n = a/b$ in (27), we find that

$$l_{a/b,a/b} = l_{a,a} l_{b,a^2/b}^{-1}. \quad (29)$$

Now (ii) follows from (28) and (29) and Theorem 10(ii).

By using Theorems 10(ii) and 12, we find that

$$l_{a/b,a/b} = l_{b,b} l_{ab^2,a} l_{a^2,b^2}^{-1}. \quad (30)$$

Similarly, we find that

$$l_{b/a,b/a} = l_{a,a} l_{a^2b,b} l_{b^2,a^2}^{-1}. \quad (31)$$

From (30), (31), and Theorem 10(ii) and (iii), we complete the proof of (iii). $\square$

4. Explicit Values of $l_{k,n}$

In this section, we prove general theorems for explicit evaluations of $l_{k,n}$ and find some explicit values. First we define Ramanujan's class invariants. Ramanujan's two class invariants G_n and g_n which are defined by

$$G_n = 2^{-1/4} q^{-1/24} \chi(q), \quad (32)$$

$$g_n = 2^{-1/4} q^{-1/24} \chi(-q), \quad q := e^{-\pi\sqrt{n}},$$

where $\chi(q) = (-q; q^2)_\infty$ and n is a positive rational number. Employing Lemma 5 in (32) it follows that

$$G_n = \{4\alpha(1-\alpha)\}^{-1/24}, \quad (33)$$

$$g_n = 2^{-1/12} (1-\alpha)^{1/12} \alpha^{-1/24}.$$

Also, if β has degree r over α , then

$$G_{r^2n} = \{4\beta(1-\beta)\}^{-1/24}, \quad (34)$$

$$g_{r^2n} = 2^{-1/12} (1-\beta)^{1/12} \beta^{-1/24}.$$

In his notebooks [1] and paper [11], Ramanujan recorded a total of 116 class invariants or monic polynomials satisfied by them. An account of these can also be found in Berndt's book [3]. For further references, see [2, 12–17].

Theorem 18. One has

$$l_{k,n} = \left(\frac{G_{nk}}{G_{n/k}} \right)^3. \quad (35)$$

Proof. Employing Lemmas 3 and 4 in the definition of $l_{k,n}$, we find that

$$l_{k,n} = \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/8}. \quad (36)$$

Again from (32) and (33) with $q := e^{-\pi\sqrt{n/k}}$, we find that

$$\frac{G_{nk}}{G_{n/k}} = \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/24}. \quad (37)$$

Combining (36) and (37), we complete the proof. $\square$

From Theorem 18, it is obvious that if we know $(G_{nk}/G_{n/k})$ or $(G_{nk} \text{ and } G_{n/k})$ then explicit values of $l_{k,n}$ can easily be determined. For example, we find some values of $l_{k,n}$ in next theorem by employing the values of $(G_{nk}/G_{n/k})$ or $(G_{nk} \text{ and } G_{n/k})$ from literature.

Theorem 19. *One has*

$$\begin{aligned} \text{(i)} \quad l_{3,5} &= (G_{15}G_{5/3})^3 = \left(\frac{3 + \sqrt{5}}{2} \right), \\ \text{(ii)} \quad l_{3,7} &= (G_{21}/G_{7/3})^3 = (3\sqrt{3} + 2\sqrt{7})^{1/2}, \\ \text{(iii)} \quad l_{3,11} &= (G_{33}/G_{11/3})^3 = (3 + \sqrt{11}/\sqrt{2}), \\ \text{(iv)} \quad l_{3,13} &= (G_{39}/G_{13/3})^3 = 1/2^{3/2}((3 + \sqrt{13})/2 + \sqrt{(3 + 3\sqrt{13})/2})^{3/2}, \\ \text{(v)} \quad l_{5,11} &= (G_{55}/G_{11/5})^3 = 1/2((9 + 5\sqrt{5})/2 + \sqrt{(95 + 45\sqrt{5})/2}), \\ \text{(vi)} \quad l_{5,13} &= (G_{65}/G_{13/5})^3 = (\sqrt{(7 + \sqrt{65})/8} + \sqrt{(\sqrt{65} - 1)/8})^3, \\ \text{(vii)} \quad l_{3,23} &= (G_{69}/G_{23/3})^3 = (24 + 5\sqrt{23})^{1/4}(25 + 3\sqrt{69}/2)^{3/8}, \\ \text{(viii)} \quad l_{3,47} &= (G_{141}/G_{47/3})^3 = (48 + 7\sqrt{47})^{3/12}(95 + 8\sqrt{141})^{3/8}, \\ \text{(ix)} \quad l_{3,71} &= (G_{213}/G_{71/3})^3 = (3480 + 413\sqrt{71})^{1/4}((73 + 5\sqrt{213})/2)^{3/8}. \end{aligned}$$

Proof. For (i)–(vi), the corresponding values of $(G_{nk}/G_{n/k})$ can be found in equations (4.10), (4.20), (4.29), (4.37), (4.51), and (4.60), respectively of [12]. For (vii) to (ix), the corresponding values of can be found in equations [p. 227, (5.8)], [p. 229, (5.15)], and [p. 233, (5.25)], respectively of [3]. $\square$

Corollary 20. *One has*

$$l_{n,n} = (G_{n^2})^3. \quad (38)$$

Proof. We set $k = n$ in Theorem 18 and use the result $G_1 = 1$, to complete the proof. $\square$

From Corollary 20, it is obvious explicit values of $l_{n,n}$ can easily be determined if the corresponding values of the class invariants G_{n^2} are known. We give some examples in the next theorem.

Theorem 21. *One has*

$$\begin{aligned} \text{(i)} \quad l_{2,2} &= G_4^3 = (1 + \sqrt{2})^{3/4}/2^{9/16}, \\ \text{(ii)} \quad l_{4,4} &= G_{16}^3 = 2^{-3/4}(16 + 15 \cdot 2^{1/4} + 12\sqrt{2} + 9 \cdot 2^{3/4})^3, \\ \text{(iii)} \quad l_{3,3} &= G_9^3 = (1 + \sqrt{3})/\sqrt{2}, \\ \text{(iv)} \quad l_{5,5} &= G_{25}^3 = ((7^{1/4} + \sqrt{4 + \sqrt{7}})/(2))^{3^3}, \\ \text{(v)} \quad l_{7,7} &= G_{49}^3 = ((1 + \sqrt{5})/(2))^{3^3}, \\ \text{(vi)} \quad l_{9,9} &= G_{81}^3 = (\sqrt[3]{2(\sqrt{3} + 1)} + 1)/(\sqrt[3]{2(\sqrt{3} - 1)} + 1). \end{aligned}$$

Proof. For (i) and (ii), we use the values of G_4 and G_{16} from [18, p. 114–115, Theorem 6.2.2(ii) (vi)]. For (iii)–(vi), we use the corresponding values of G_n from [3, p. 189–193]. $\square$

Ramanujan's Schaffli type modular equations of prime degree can also be used to explicit values of the product $l_{k,n}$. We offer two theorems as examples.

Theorem 22. *One has*

$$l_{3,n}^{-2} + l_{3,n}^2 + 2\sqrt{2} \left\{ (G_{3n}G_{n/3})^{-3} - (G_{3n}G_{n/3})^3 \right\} = 0. \quad (39)$$

Proof. We use the definitions of class invariants $l_{k,n}$ and G_n in Lemma 8, to complete the proof. $\square$

Theorem 23. *One has*

$$l_{5,n} + l_{5,n}^{-1} + 2 \left\{ (G_{5n}G_{n/5})^{-2} - (G_{5n}G_{n/5})^2 \right\} = 0. \quad (40)$$

Proof. We use the definitions of class invariants $l_{k,n}$ and G_n in Lemma 9, to complete the proof. $\square$

5. Evaluations of Ramanujan's Cubic Continued Fraction

In this section, we prove two formulas of explicit evaluations Ramanujan's cubic continued fraction $V(q)$ in terms of the product $l_{k,n}$. We also give examples.

Theorem 24. *One has*

$$V^3 \left(-e^{-\pi\sqrt{n/3}} \right) = \frac{l_{3,n}^4 + 1 - \sqrt{l_{3,n}^8 + 34 l_{3,n}^4 + 1}}{16 l_{3,n}^4}. \quad (41)$$

Proof. Replacing q by $-q$ in (14) and Lemma 7, we obtain

$$-\left(1 + \frac{1}{V^3(q)} \right) = \frac{\psi^4(-q)}{q\psi^4(-q^3)}, \quad (42)$$

$$1 - 8V^3(-q) = \frac{\phi^4(q)}{\phi^4(q^3)}. \quad (43)$$

Dividing (42) by (43), setting $q := e^{-\pi\sqrt{n/3}}$ and simplifying using the definition of $l_{k,n}$, we obtain

$$8l_{3,4}^4 V^6 \left(-e^{-\pi\sqrt{n/3}} \right) - (l_{3,4}^4 + 1) V^3 \left(-e^{-\pi\sqrt{n/3}} \right) - 1 = 0. \quad (44)$$

Solving (2) for $V^3(-e^{-\pi/\sqrt{n/3}})$ and noting that $V(-e^{-\pi/\sqrt{n/3}}) < 0$ and the fact in Remark 11, we arrive at the desired result. $\square$

From Theorem 24, it is clear that to find explicit values of $V^3(-e^{-\pi/\sqrt{n/3}})$ it is enough to know the explicit values of $l_{3,n}$. For example, noting $l_{3,1} = 1$ from Theorem 10(i) we evaluate

$$V^3(-e^{-\pi/\sqrt{3}}) = -1/4. \quad (45)$$

Theorem 25. *One has*

$$V(-e^{-\pi/\sqrt{n}}) = \frac{l_{9,n} + 1 - \sqrt{l_{9,n}^2 + 10 l_{9,n} + 1}}{4 l_{9,n}}. \quad (46)$$

Proof. Replacing q by q^3 and then replacing q by $-q$ in (13) and (15), we obtain

$$-\left(1 + \frac{1}{V(-q^3)}\right) = \frac{\psi(-q)}{q\psi(-q^9)}, \quad (47)$$

$$1 - 2V(-q^3) = \frac{\phi(q)}{\phi(q^9)}. \quad (48)$$

Dividing (47) by (48), setting $q := e^{-\pi/\sqrt{n/9}}$ and simplifying by using the definition of $l_{k,n}$, we arrive at

$$2 l_{9,n} V^2(-e^{-\pi/\sqrt{n}}) - (l_{9,n} + 1) V(-e^{-\pi/\sqrt{n}}) - 1 = 0. \quad (49)$$

Solving (49) for $V(-e^{-\pi/\sqrt{n}})$ and noting $V(-e^{-\pi/\sqrt{n}}) < 0$ and the fact in Remark 11, we complete the proof. $\square$

From Theorem 25, it is obvious that if we know the values of $l_{9,n}$ the corresponding values of $V(-e^{-\pi/\sqrt{n}})$ can easily be evaluated. For example, by noting $l_{9,1} = 1$ from Theorem 10(i) we calculate

$$V(-e^{-\pi}) = \frac{1 - \sqrt{3}}{2}. \quad (50)$$

Similarly, one can find explicit values of $V(-e^{-3\pi})$ by using the value of $l_{9,9}$ from Theorem 21(vi).

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