Research Article

Coefficient Inequalities of Analytic Functions Related to Robertson Functions

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We introduce and study a subclass of analytic functions related to Robertson functions. Here we discuss the coefficient estimate for function in this class.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Also let S^* and C denote the well-known classes of starlike and convex functions, respectively.

For any two analytic functions f given by (1) and g with

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \text{for } z \in E, \quad (2)$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad \text{for } z \in E. \quad (3)$$

Using the concept of convolution, Ruscheweyh [1] introduced a differential operator D^δ given by

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\delta) a_n z^n, \quad (4)$$

$(\delta > -1)$

with

$$\varphi_n(\delta) = \frac{(\delta+1)_{n-1}}{(n-1)!}, \quad (5)$$

where $(x)_n$ is a Pochhammer symbol given as

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n \in \mathbb{N}. \end{cases} \quad (6)$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, and

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad \forall \delta = n \in N_0 = \{0, 1, 2, \dots\}. \quad (7)$$

The following identity can easily be established:

$$(\delta+1)D^{\delta+1}f(z) = \delta D^\delta f(z) + z(D^\delta f(z))'. \quad (8)$$

Now with the help of Ruscheweyh derivative, we define a class $VD_\lambda(\alpha, \beta, b, \delta)$ of analytic functions as follows.

Definition 1. Let $f(z) \in A$. Then, $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$, if and only if

$$\begin{aligned} & \operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1}f(z)}{D^\delta f(z)} \right) \right\} \\ & > \alpha \left| \frac{2}{b} \left(\frac{D^{\delta+1}f(z)}{D^\delta f(z)} - 1 \right) \right| + \beta \cos \lambda, \end{aligned} \quad (9)$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta > -1$, λ is real with $|\lambda| < \pi/2$, and $b \in \mathbb{C} \setminus \{0\}$.

By giving specific values to α , β , λ , b , and δ in $VD_\lambda(\alpha, \beta, b, \delta)$, we obtain many important subclasses studied by various authors in earlier papers, see for details [2–5], and list some of them as follows:

- (i) $VD_\lambda(0, 0, 2, 0) \equiv S_\lambda^*$ and $VD_\lambda(0, 0, 1, 1) \equiv K_\lambda$, studied by Spacek [6] and Robertson [7], respectively; for the advancement work, see [8, 9].
- (ii) $VD_0(\alpha, \beta, 2, 0) \equiv SD(\alpha, \beta)$ and $VD_0(\alpha, \beta, 1, 1) \equiv KD(\alpha, \beta)$, studied by both Owa et al. and Shams et al. [10, 11].
- (iii) $VD_\lambda(1, 0, 2, 0) \equiv USP(\lambda)$, $VD_\lambda(1, 0, 1, 1) \equiv UCSP(\lambda)$, introduced by Ravichandran et al. [12].
- (iv) $VD_0(\alpha, \beta, b, \delta) \equiv VD(\alpha, \beta, b, \delta)$, considered by Latha [13].
- (v) $VD_0(0, \beta, 2, 0) \equiv S^*(\beta)$, $VD_0(0, \beta, 1, 1) \equiv C(\beta)$, the well-known classes of starlike and convex functions of order β .

From the above special cases, we note that this class provides a continuous passage from the class of starlike functions to the class of convex functions.

We will assume throughout our discussion, unless otherwise stated, that $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta > -1$, λ is real with $|\lambda| < \pi/2$, and $b \in \mathbb{C} \setminus \{0\}$.

2. Some Properties of the Class $VD_\lambda(\alpha, \beta, b, \delta)$

Theorem 2. If $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$ with $0 \leq \alpha \leq \beta$, then

$$f(z) \in VD_\lambda\left(0, \left(\frac{\beta - \alpha}{1 - \alpha}\right), b, \delta\right). \quad (10)$$

Proof. Since $\operatorname{Re} w \leq |w|$ for any complex number w , $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$ implies that

$$\begin{aligned} & \operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} \right) \right\} \\ & > \alpha \left| \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} - \frac{2}{b} \right| + \beta \cos \lambda \\ & \geq \alpha \operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} \right) \right\} \\ & \quad + (\beta - \alpha) \cos \lambda \end{aligned} \quad (11)$$

which implies that

$$\begin{aligned} & \operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} \right) \right\} \\ & > \frac{(\beta - \alpha) \cos \lambda}{(1 - \alpha)}, \quad (z \in E). \end{aligned} \quad (12)$$

And hence, we obtain the required result.

Put $\lambda = 0$, $b = 2$, and $\delta = 0$ in Theorem 2; we obtain the following result. $\square$

Corollary 3 (see [10]). If $f(z) \in SD(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$, then

$$f(z) \in S^*\left(\frac{\beta - \alpha}{1 - \alpha}\right). \quad (13)$$

Set $\lambda = 0$, $b = 1$, and $\delta = 1$ in Theorem 2; one has the following result.

Corollary 4 (see [10]). If $f(z) \in KD(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$, then

$$f(z) \in K\left(\frac{\beta - \alpha}{1 - \alpha}\right). \quad (14)$$

Theorem 5. If $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$, then

$$|a_2| \leq \frac{|b| |\eta|}{|1 - \alpha|}, \quad (15)$$

$$|a_n| \leq \frac{(\delta + 1) |b| |\eta|}{(n - 1) |1 - \alpha| \varphi_n(\delta)} \prod_{j=1}^{n-2} \left(1 + \frac{(\delta + 1) |b| |\eta|}{j |1 - \alpha|} \right), \quad (16)$$

$$n \geq 3,$$

where

$$|\eta| = \sqrt{(1 - \beta)^2 \cos^2 \lambda + (1 - \alpha)^2 \sin^2 \lambda}. \quad (17)$$

Proof. We note that for $f(z) \in VD_\lambda(\alpha, \beta, b, \delta)$,

$$\operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} \right) \right\} > \frac{(\beta - \alpha)}{(1 - \alpha)} \cos \lambda, \quad (18)$$

$z \in E$.

Let us define the function $p(z)$ by

$$\begin{aligned} p(z) &= ((1 - \alpha) \\ & \quad \times [e^{i\lambda} (1 - 2/b + (2/b) (D^{\delta+1} f(z) / D^\delta f(z)))] \\ & \quad - (\beta - \alpha) \cos \lambda) \times ((1 - \beta) \cos \lambda + i(1 - \alpha) \sin \lambda)^{-1}. \end{aligned} \quad (19)$$

Then, $p(z)$ is analytic in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \quad (20)$$

Then, (19) can be written as

$$1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+1} f(z)}{D^\delta f(z)} \quad (21)$$

$$= 1 + \frac{(1 - \beta) \cos \lambda + i(1 - \alpha) \sin \lambda}{e^{i\lambda} (1 - \alpha)} \sum_{n=1}^{\infty} p_n z^n,$$

and using (8), we have

$$\begin{aligned} & 2e^{i\lambda} \frac{(1 - \alpha)}{(\delta + 1)} (z(D^\delta f(z))' - (D^\delta f(z))) \\ & = b\eta (D^\delta f(z)) \left(\sum_{n=1}^{\infty} p_n z^n \right) \end{aligned} \quad (22)$$

which implies that

$$e^{i\lambda} a_n = \frac{b\eta(\delta+1)}{2(1-\alpha)(n-1)\varphi_n(\delta)} \times \{p_{n-1} + \varphi_2(\delta)a_2p_{n-2} + \cdots + \varphi_{n-1}(\delta)a_{n-1}p_1\}, \quad (23)$$

where we have used (4) and (5). Now applying the coefficient estimates $|p_n| \leq 2$ for Caratheodory function [14], we obtain

$$|a_n| \leq \frac{(\delta+1)|b||\eta|}{(n-1)|1-\alpha|\varphi_n(\delta)} \times [1 + \varphi_2(\delta)|a_2| + \cdots + \varphi_{n-1}(\delta)|a_{n-1}|]. \quad (24)$$

For $n = 2$,

$$|a_2| \leq \frac{|b||\eta|}{|1-\alpha|} \quad (25)$$

which proves (15).

For $n = 3$,

$$|a_3| \leq \frac{(\delta+1)|b||\eta|}{2|1-\alpha|\varphi_3(\delta)} \left[1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|} \right]. \quad (26)$$

Therefore, (16) holds for $n = 3$. Assume that (16) is true for all $n = 3, 4, \dots, k$ and consider

$$\begin{aligned} |a_{k+1}| &\leq \frac{(\delta+1)|b||\eta|}{(k+1-1)|1-\alpha|\varphi_{k+1}(\delta)} \\ &\times \left\{ \left(1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|} \right) \right. \\ &\quad + \frac{(\delta+1)|b||\eta|}{2|1-\alpha|} \left(1 + \frac{(\delta+1)|b||\eta|}{|1-\alpha|} \right) \\ &\quad \left. + \cdots + \frac{(\delta+1)|b||\eta|}{(k-1)|1-\alpha|} \prod_{j=1}^{k-2} 1 + \frac{(\delta+1)|b||\eta|}{j|1-\alpha|} \right\} \\ &= \frac{(\delta+1)|b||\eta|}{k|1-\alpha|\varphi_{k+1}(\delta)} \prod_{j=1}^{k-1} \left(1 + \frac{(\delta+1)|b||\eta|}{j|1-\alpha|} \right). \end{aligned} \quad (27)$$

Thus, the result is true for $n = k + 1$, and hence by induction, (16) holds for all $n \geq 3$.

If we set $\lambda = 0$, $b = 2$, and $\delta = 0$ in Theorem 5, we get the result proved in [10]. $\square$

Corollary 6. If $f(z) \in SD(\alpha, \beta)$, then

$$|a_2| \leq \frac{2(1-\beta)}{|1-\alpha|}, \quad (28)$$

$$|a_n| \leq \frac{2(1-\beta)}{(n-1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right), \quad (n \geq 3).$$

Remark 7. If we take $\alpha = 0$ in Corollary 6, we have

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^n (j-2\beta), \quad (n \geq 2) \quad (29)$$

which was proved by Robertson [15].

By setting $\lambda = 0$, $b = 1$, and $\delta = 1$ in Theorem 5, we obtain the result in [10].

Corollary 8. If $f(z) \in KD(\alpha, \beta)$, then

$$|a_2| \leq \frac{(1-\beta)}{|1-\alpha|},$$

$$|a_n| \leq \frac{2(1-\beta)}{n(n-1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right), \quad (n \geq 3). \quad (30)$$

Remark 9. Letting $\alpha = 0$ in Corollary 8, we have

$$|a_n| \leq \frac{1}{n!} \prod_{j=2}^n (j-2\beta), \quad (n \geq 2) \quad (31)$$

given by Robertson [15].

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References

- [1] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.
- [2] E. Aqlan, J. M. Jahangiri, and S. R. Kulkarni, "New classes of k -uniformly convex and starlike functions," *Tamkang Journal of Mathematics*, vol. 35, no. 3, pp. 261–266, 2004.
- [3] S. Kanas and A. Wisniowska, "Conic regions and k -uniform convexity," *Journal of Computational and Applied Mathematics*, vol. 105, no. 1-2, pp. 327–336, 1999.
- [4] S. Kanas and A. Wiśniowska, "Conic domains and starlike functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 45, no. 4, pp. 647–657, 2000.
- [5] J. Nishiwaki and S. Owa, "Certain classes of analytic functions concerned with uniformly starlike and convex functions," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 350–355, 2007.
- [6] L. Spacek, "Prispevek k teorii funkei prostych," *Časopis pro Pěstování Matematiky a Fysik*, vol. 62, pp. 12–119, 1933.
- [7] M. S. Robertson, "Univalent functions $f(z)$ for which $zf'(z)$ is spirallike," *The Michigan Mathematical Journal*, vol. 16, pp. 97–101, 1969.
- [8] M. Arif, W. Haq, and M. Ismail, "Mapping properties of generalized Robertson functions under certain integral operators," *Applied Mathematics*, vol. 3, no. 1, pp. 52–55, 2012.

- [9] K. I. Noor, M. Arif, and A. Muhammad, "Mapping properties of some classes of analytic functions under an integral operator," *Journal of Mathematical Inequalities*, vol. 4, no. 4, pp. 593–600, 2010.
- [10] S. Owa, Y. Polatoğlu, and E. Yavuz, "Coefficient inequalities for classes of uniformly starlike and convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, article 160, 2006.
- [11] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, "Classes of uniformly starlike and convex functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 55, pp. 2959–2961, 2004.
- [12] V. Ravichandran, C. Selvaraj, and R. Rajagopal, "On uniformly convex spiral functions and uniformly spirallike functions," *Soochow Journal of Mathematics*, vol. 29, no. 4, pp. 393–405, 2003.
- [13] S. Latha, "Coefficient inequalities for certain classes of Ruscheweyh type analytic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 2, p. Article 52, 2008.
- [14] C. Caratheodory, "Über den variabilitätsbereich der Fourier'schen konstanten von positiven harmonischen funktionen," *Rendiconti del Circolo Matematico di Palermo*, vol. 32, pp. 193–217, 1911.
- [15] M. I. S. Robertson, "On the theory of univalent functions," *Annals of Mathematics*, vol. 37, no. 2, pp. 374–408, 1936.