Research Article

Differential Subordination for Certain Subclasses of p -Valent Functions Associated with Generalized Linear Operator

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Received 14 October 2012; Accepted 20 January 2013

Academic Editor: Liwei Zhang

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By making use of the differential subordination analytic functions, we investigate inclusion relationships among certain classes of analytic and p -valent functions defined by generalized linear operator.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1)$$

which are analytic and p -valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For functions $f \in \mathcal{A}(p)$, given by (1) and g given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \quad (2)$$

the Hadamard product (or convolution) of f and g is defined by

$$\begin{aligned} (f * g)(z) &= z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \\ &= (g * f)(z) \quad (z \in U; p \in \mathbb{N}). \end{aligned} \quad (3)$$

For $f, g \in \mathcal{A}(p)$, we say that the function f is subordinate to g denoted by $f(z) < g(z)$, if there exists a Schwarz function w , that is, $w \in \mathcal{A}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. It is well known that, if the function g is univalent in U , then $f(z) < g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1, 2]).

Now, we introduce the linear multiplier operator $\mathcal{L}_{c,p,\lambda}^{m,\delta} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ by

$$\begin{aligned} \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) &= z^p + \sum_{k=1}^{\infty} \left(1 + \frac{\lambda k}{p}\right)^m \left(\frac{c+p}{c+k+p}\right)^\delta a_{k+p} z^{k+p} \\ & \quad (\delta \geq 0; c > -p; \lambda \geq 0; m \in \mathbb{N}_0). \end{aligned} \quad (4)$$

It is easily verified from (4) that

$$z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)' = (c+p) \mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z) - c \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z), \quad (5)$$

$$\lambda z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)' = p \mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z) - p(1-\lambda) \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z). \quad (6)$$

By specializing the parameters c, λ, δ, p , and m , we obtain the following operators:

- (i) $\mathcal{L}_{c,p,0}^{1,\delta} f(z) = K_{c,p}^\delta f(z)$ (see [3]);
- (ii) $\mathcal{L}_{a-1,1,\lambda}^{0,\delta} f(z) = L_a^\delta f(z)$ (see [4, 5]);
- (iii) $\mathcal{L}_{1,p,0}^{1,\delta} f(z) = \mathcal{J}_p^\delta f(z)$ (see [6, 7]);
- (iv) $\mathcal{L}_{c,p,\lambda}^{m,0} f(z) = D_{\lambda,p}^m f(z)$ (see [8]);

- (v) $\mathcal{L}_{c,1,0}^{1,\delta} f(z) = \mathcal{P}_c^\delta f(z)$ (see [9, 10]);
- (vi) $\mathcal{L}_{c,1,0}^{1,1} f(z) = \mathcal{L}_c f(z)$ (see [11]);
- (vii) $\mathcal{L}_{1,1,0}^{1,\delta} f(z) = \mathcal{J}^\delta f(z)$ (see [12]);
- (viii) $\mathcal{L}_{c,1,\lambda}^{m,0} f(z) = \mathcal{D}_\lambda^m f(z)$ (see [13]);
- (ix) $\mathcal{L}_{c,1,1}^{m,0} f(z) = \mathcal{D}^m f(z)$ (see [14]).

By using the multiplier operator $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)$, we define the following classes of functions.

Definition 1. For fixed parameters A and B , with $-1 \leq B < A \leq 1$ and $p > \alpha$, we say that the function $f \in \mathcal{A}(p)$ is in the class $\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) < \frac{1 + Az}{1 + Bz}. \quad (7)$$

We note that

- (i) $\mathcal{S}_{1,p,0}^{1,0}(\alpha; A, B) = \mathcal{S}_p^*(\alpha; A, B)$ was studied by Cho et al. [15];
- (ii) $\mathcal{S}_{1,p,0}^{1,\delta}(\alpha; A, B) = \mathcal{S}_p^\delta(\alpha; A, B)$ was studied by Aouf et al. [16];
- (iii) $\mathcal{S}_{c,1,\lambda}^{m,0}(0; A, B) = \mathcal{S}_\lambda^m(A, B)$ was studied by Patel [17].

For complex numbers a, b , and c , the Gaussian hypergeometric function is defined by

$$\begin{aligned} {}_2F_1(a, b, d; z) &= 1 + \frac{a \cdot b}{d} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{d(d+1)} \frac{z^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(d)_k} \frac{z^k}{k!}, \quad a, b \in \mathbb{C}, \\ &\quad d \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \end{aligned} \quad (8)$$

where $(\lambda)_k = \lambda(\lambda+1) \cdots (\lambda+k-1)$ and $(\lambda)_0 = 1$. The series (8) converges absolutely for $z \in U$, hence it represents an analytic function in U (see [18, Chapter 14]).

If $B \neq -1$, from the fact that $h(\bar{z}) = \overline{h(z)}$, $z \in U$, we deduce that the image $h(U)$ is symmetric with respect to the real axis, and that h maps the unit disc U onto the disc $|w - (1-AB)/(1-B^2)| < (A-B)/(1-B^2)$. If $B = -1$, the function h maps the unit disc U onto the half plan $\operatorname{Re} w > (1-A)/2$, hence we obtain the following.

Remark 2. The function $f \in \mathcal{A}(p)$ is in the class $\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$ if and only if

$$\left| \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}, \quad (9)$$

$z \in U$, for $B \neq -1$,

$$\operatorname{Re} \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) > \frac{1-A}{2}, \quad (10)$$

$z \in U$, for $B = -1$.

Definition 3. The function $f \in \mathcal{A}(p)$ is in the class $\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho)$ if it satisfies the following the inequality:

$$\operatorname{Re} \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) > \rho, \quad z \in U, \quad (11)$$

where $\rho < 1$; from (9) and (10) it follows, respectively, that

$$\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta} \left(\alpha; \frac{1-A}{1-B} \right), \quad (12)$$

$$\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B) = \mathcal{S}_{c,p,\lambda}^{m,\delta} \left(\alpha; \frac{1-A}{1-B} \right) \iff B = -1.$$

We note that when $c = 1$ and $m = 0$, the class $\mathcal{S}_{1,p,0}^{1,\delta}(\alpha; \rho) = \mathcal{S}_p^\delta(\alpha; \rho)$ was studied by Aouf et al. [16].

Let us consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z)) < h(z). \quad (13)$$

A univalent function q is called its dominant, if $\varphi(z) < q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\bar{q}$ is called the best dominant, if $\bar{q}(z) < q(z)$ for all dominants q . For the general theory of the first-order differential subordination and its applications, we refer the reader to [1, 2].

The object of the present paper is to obtain several inclusion relationships and other interesting properties of functions belonging to the subclasses $\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$ and $\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho)$ by using the theory of differential subordination.

To establish our main results, we will require the following lemmas.

Lemma 4 (see [19]). Let $\beta, \gamma \in \mathbb{C}$, and let h be a convex function with

$$\operatorname{Re} [\beta h(z) + \gamma] > 0, \quad z \in U. \quad (14)$$

If p is analytic in U , with $p(0) = h(0)$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z) \implies p(z) < h(z). \quad (15)$$

Lemma 5 (see [20]). Let $\beta > 0$, $\beta + \gamma > 0$, and consider the integral operator $J_{\beta,\gamma}$ defined by

$$J_{\beta,\gamma}(f)(z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma-1} dt \right]^{1/\beta}, \quad (16)$$

where the powers are the principal ones.

If $\sigma \in [-\gamma/\beta, 1)$ then the order of starlikeness of the class $J_{\beta,\gamma}(S^*(\sigma))$, that is, the largest number $\delta(\sigma; \beta, \gamma)$ such that $J_{\beta,\gamma}(S^*(\sigma)) \subset S^*(\delta)$, is given by the number $\delta(\sigma; \beta, \gamma) = \inf \{ \operatorname{Re} q(z) : z \in U \}$, where

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}, \quad (17)$$

$$Q(z) = \int_0^1 \left(\frac{1-z}{1-tz} \right)^{2\beta(1-\sigma)} t^{\beta+\gamma-1} dt.$$

Moreover, if $\sigma \in [\sigma_0, 1)$, where $\sigma_0 = \max\{(\beta - \gamma - 1)/2\beta, -\gamma/\beta\}$ and $g = J_{\beta,\gamma}(f)(z)$ with $f \in S^*(\sigma)$, then

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > \delta(\sigma; \beta, \gamma) \quad (z \in U), \quad (18)$$

where

$$\delta(\sigma; \beta, \gamma) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{{}_2F_1(1, 2\beta(1-\sigma), \beta + \gamma + 1; 1/2)} - \gamma \right]. \quad (19)$$

Lemma 6 (see [21]). Let ϕ be analytic in U with $\phi(0) = 1$ and $\phi(z) \neq 0$ for $0 < |z| < 1$, and let $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$.

(i) Let $B \neq 0$ and $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ satisfy either $|(\gamma(A - B)/B) - 1| \leq 1$ or $|(\gamma(A - B)/B) + 1| \leq 1$. If ϕ satisfies

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} < \frac{1 + Az}{1 + Bz} \quad (20)$$

then

$$\phi(z) < (1 + Bz)^{\gamma(A-B)/B}, \quad (21)$$

and this is the best dominant.

(ii) Let $\gamma \in \mathbb{C}^*$ be such that $|\gamma A| < \pi$, and if ϕ satisfies

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} < 1 + Az \quad (22)$$

then

$$\phi(z) < e^{\gamma Az}, \quad (23)$$

and this is the best dominant.

2. Inclusion Relationships

Unless otherwise mentioned, we assume throughout this paper that

$$\begin{aligned} -1 \leq B < A \leq 1, \quad p > \alpha, \quad \delta \geq 0, \\ c > -p, \quad \lambda \geq 0, \quad p \in \mathbb{N}, \quad m \in \mathbb{Z} \end{aligned} \quad (24)$$

and the power is the principal one.

Theorem 7. Let

$$(p - \alpha)(1 - A) + (\alpha + c)(1 - B) \geq 0. \quad (25)$$

(1) Supposing that $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \neq 0$ for all $z \in \dot{U} = U \setminus \{0\}$, then $\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$.

(2) Moreover, if we suppose in addition that

$$A \leq 1 + \left(\frac{1-B}{p-\alpha} \right) \left(\alpha + \frac{c-p}{2} + \frac{1}{2} \right), \quad (26)$$

then

$$\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho(p, \alpha, c, A, B)), \quad (27)$$

where the bound

$$\begin{aligned} \rho(p, \alpha, c, A, B) &= \frac{1}{p - \alpha} \\ &\times \left[\frac{p + c}{{}_2F_1(1, 2(p - \alpha)(A - B)/(1 - B), p + c + 1; 1/2)} \right. \\ &\quad \left. - (\alpha + c) \right], \end{aligned} \quad (28)$$

is the best possible.

Proof. Let $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B)$, and put

$$g(z) = z \left(\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right)^{1/(p-\alpha)} \quad (z \in U), \quad (29)$$

the function g is analytic in U , with $g(0) = 0$ and $g'(0) = 1$. Differentiating (29) logarithmically with respect to z , we have

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p - \alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) \quad (30)$$

$$(z \in U),$$

then, using (5) in (30), we obtain

$$\begin{aligned} (c + p) \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} &= (p - \alpha) \phi(z) + (\alpha + c). \end{aligned} \quad (31)$$

By differentiating both sides of (31) logarithmically with respect to z and multiplying by z , we have

$$\begin{aligned} \frac{1}{p - \alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)} - \alpha \right) &= \phi(z) + \frac{z\phi'(z)}{(p - \alpha)\phi(z) + (\alpha + c)}. \end{aligned} \quad (32)$$

Combining (32) together with $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B)$, we obtain that the function ϕ satisfies the Briot-Bouquet differential subordination as follows:

$$\begin{aligned} \phi(z) + \frac{z\phi'(z)}{(p-\alpha)\phi(z) + (\alpha+c)} \\ < \frac{1+Az}{1+Bz} \equiv h(z). \end{aligned} \quad (33)$$

Now we will use Lemma 4 for the special case $\beta = p - \alpha$ and $\gamma = \alpha + c$. Since h is a convex function in U , a simple computation shows that

$$\operatorname{Re}\{(p-\alpha)h(z) + (\alpha+c)\} > 0, \quad z \in U, \quad (34)$$

whenever (25) holds, we have $\phi(z) < h(z)$; that is, $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$. If in addition, we suppose that the inequality (26) holds, then all the assumptions of Lemma 5 are verified for the above values of β , γ , and $\sigma = (1-A)/(1-B)$. Then it follows the inclusion $\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho(p, \alpha, c, A, B))$, where the bound $\rho(p, \alpha, c, A, B)$ given by (28) is the best possible.

From Theorem 7, according to the definitions (7) and (11), we deduce the next inclusions. $\square$

Corollary 8. Let $p > \alpha$, such that (25) holds.

(1) Supposing that $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \neq 0$ for all $z \in \dot{U}$, then

$$\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}\left(\alpha; \frac{1-A}{1-B}\right). \quad (35)$$

(2) If we suppose in addition that (26) holds, then

$$\begin{aligned} \mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; A, B) &\subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B) \\ &\subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho(p, \alpha, c, A, B)), \end{aligned} \quad (36)$$

where $\rho(p, \alpha, c, A, B)$ is given by (28). As a consequence of the last inclusion, one has $\rho(p, \alpha, c, A, B) \geq (1-A)/(1-B)$.

For the special case $B = -1$, Theorem 7 reduces to the following.

Corollary 9. Let $a \geq -(c+\alpha)/(p-\alpha)$.

(1) Supposing that $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \neq 0$ for all $z \in \dot{U}$, then

$$\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; a) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; a). \quad (37)$$

(2) If we suppose in addition that

$$a \geq \max \left\{ -\frac{2\alpha+c-p+1}{2(p-\alpha)}; -\frac{\alpha+c}{(p-\alpha)} \right\}, \quad (38)$$

then

$$\mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; a) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho(p, \alpha, c, a)), \quad (39)$$

where the bound

$$\begin{aligned} \rho(p, \alpha, c, a) &= \frac{1}{p-\alpha} \\ &\times \left[\frac{p+c}{{}_2F_1(1, 2(p-\alpha)(1-a), p+c+1; 1/2)} \right. \\ &\quad \left. - (\alpha+c) \right], \end{aligned} \quad (40)$$

is the best possible.

Theorem 10. Let $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho)$, where $\rho < 1$ and $\rho > \alpha$, then $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; \rho)$ for $|z| < R(p, \alpha, c, \rho)$, where

$$\begin{aligned} R(p, \alpha, c, \rho) &= \min \{r > 0 : \theta(p, \alpha, c, \rho, r) = 0\}, \\ \theta(p, \alpha, c, \rho, r) &= 2r((1-r)(p-\alpha) \\ &\quad \times |(1-\rho)(1-r) \\ &\quad - |\rho + (\alpha+c)/(p-\alpha)|(1+r)|)^{-1}. \end{aligned} \quad (41)$$

Proof. Since $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; \rho)$, the function u given by

$$u(z) = \frac{1}{p-\alpha} \left(\frac{z(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z))'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) \quad (z \in U), \quad (42)$$

is analytic in U with $u(0) = 1$ and $\operatorname{Re} u(z) > \rho$. Using (5) in (42) and taking the logarithmic differentiation in the resulting equation, we obtain

$$\begin{aligned} \frac{1}{p-\alpha} \left(\frac{z(\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z))'}{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)} - \alpha \right) \\ = u(z) + \frac{zu'(z)}{(p-\alpha)u(z) + (\alpha+c)}. \end{aligned} \quad (43)$$

If we denote $k(z) = (u(z) - \rho)/(1 - \rho)$, then $k(0) = 1$ and $\operatorname{Re} k(z) > 0$, $z \in U$ and substituting in (43) we obtain

$$\begin{aligned} \frac{1}{p-\alpha} \left(\frac{z(\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z))'}{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)} - \alpha \right) - \rho \\ = (1-\rho) \\ \times \left[k(z) \right. \\ \left. + \frac{zk'(z)}{(p-\alpha)(1-\rho)k(z) + \rho(p-\alpha) + (\alpha+c)} \right], \end{aligned} \quad (44)$$

hence

$$\begin{aligned} & \operatorname{Re} \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)} - \alpha \right) - \rho \\ & \geq (1-\rho) \\ & \quad \times \left[\operatorname{Re} k(z) - \left(|zk'(z)| \right) ((p-\alpha) \right. \\ & \quad \left. \times |(1-\rho)k(z) - |\rho + ((\alpha+c)/(p-\alpha))||)^{-1} \right]. \end{aligned} \quad (45)$$

By using the well-known results [22]

$$\begin{aligned} |zk'(z)| & \leq \frac{2r}{1-r^2} \operatorname{Re} k(z), \quad \operatorname{Re} k(z) \geq \frac{1-r}{1+r}, \\ |z| & = r < 1, \end{aligned} \quad (46)$$

together with the inequality (45), we get

$$\begin{aligned} & \operatorname{Re} \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)} - \alpha \right) - \rho \\ & \geq (1-\rho) [1 - \theta(p, \alpha, c, \rho, r)] \operatorname{Re} k(z), \quad |z| = r. \end{aligned} \quad (47)$$

Since the right hand side term of the inequality (47) is non-negative whenever $|z| \leq R(p, \alpha, c, \rho)$ is given by (41), using the fact that the real part of an analytic function is harmonic, we deduce that $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta-1}(\alpha; \rho)$ for $|z| < R(p, \alpha, c, \rho)$.

For a function $f \in \mathcal{A}(p)$, let the integral operator $F_{\mu,p} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ defined by Saitoh [23] and Saitoh et al. [24]

$$\begin{aligned} F_{\mu,p}(f(z)) & = \frac{\mu+p}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \\ & = z^p + \sum_{k=p+1}^{\infty} \frac{\mu+p}{k+\mu} a_k z^k \\ & = \left(z^p + \sum_{k=p+1}^{\infty} \frac{\mu+p}{k+\mu} z^k \right) * f(z) \\ & = z^p {}_2F_1(1, \mu+p; \mu+p+1; z) \\ & \quad * f(z) \quad (z \in U; \mu > -p). \end{aligned} \quad (48)$$

From (4) and (48), we have

$$\begin{aligned} & z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z)) \right)' \\ & = (\mu+p) \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) - \mu \mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z)), \end{aligned} \quad (49)$$

$$\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z)) = F_{\mu,p}(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)), \quad f \in \mathcal{A}(p).$$

□

We now prove the next result.

Theorem 11. Let $\alpha + \mu > 0$ and

$$(\alpha + \mu)(1-B) + (p-\alpha)(1-A) \geq 0. \quad (50)$$

(i) Supposing that $F_{\mu,p}(f(z)) \neq 0$ for all $z \in \dot{U} = U \setminus \{0\}$, then

$$F_{\mu,p}(\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B). \quad (51)$$

(ii) Moreover, if we suppose in addition that

$$A \leq 1 + \left(\frac{1-B}{p-\alpha} \right) \min \left\{ \frac{\mu+2\alpha-p+1}{2}; \mu+\alpha \right\}, \quad (52)$$

then

$$F_{\mu,p}(\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; r(p, \alpha, \mu, A, B)), \quad (53)$$

where the bound

$$\begin{aligned} r(p, \alpha, \mu, A, B) & = \frac{1}{p-\alpha} \\ & \quad \times \left[\frac{p+\mu}{{}_2F_1(1, 2(p-\alpha)(A-B)/(1-B), \mu+p+1; 1/2)} \right. \\ & \quad \left. - (\mu+\alpha) \right], \end{aligned} \quad (54)$$

is the best possible.

Proof. Let $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$, and suppose that $F_{\mu,p}(f(z)) \neq 0$ for all $z \in \dot{U}$. Let

$$\begin{aligned} g(z) & = z \left(\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z))}{z^p} \right)^{1/(p-\alpha)} \\ & \quad (z \in U), \end{aligned} \quad (55)$$

then g is analytic in U , with $g(0) = 0$ and $g'(0) = 1$. Taking the logarithmic differentiation in (55), we have

$$\begin{aligned} \phi(z) & = \frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z)) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z))} - \alpha \right) \\ & \quad (z \in U). \end{aligned} \quad (56)$$

Now, by using (49) in (56), we obtain

$$(p+\mu) \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z))} = (p-\alpha) \phi(z) + (\mu+\alpha). \quad (57)$$

By differentiating in both sides of (57) logarithmical with respect to z and multiplying by z , we have

$$\frac{1}{p-\alpha} \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \alpha \right) = \phi(z) + \frac{z\phi'(z)}{(p-\alpha)\phi(z) + (\mu+\alpha)}. \quad (58)$$

Since $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$, from (58), we obtain that the function ϕ satisfies the Briot-Bouquet differential subordination

$$\phi(z) + \frac{z\phi'(z)}{(p-\alpha)\phi(z) + (\mu+\alpha)} < \frac{1+Az}{1+Bz} \equiv h(z). \quad (59)$$

Now we will use Lemma 4 for the special case $\beta = p - \alpha$ and $\gamma = \mu + \alpha$; we have $\phi(z) < h(z)$, that is, $F_{\mu,p} \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$. If we suppose in addition that the inequality (52) holds, then all the assumptions of the Lemma 5 are satisfied for β , γ , and $\sigma = (1-A)/(1-B)$, hence it follows the inclusion $F_{\mu,p}(\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; r(p, \alpha, \mu, A, B))$, and the bound $r(p, \alpha, \mu, A, B)$ given by (54) is the best possible. $\square$

Taking $B = -1$ in Theorem 11, we obtain the next corollary.

Corollary 12. Let $\alpha + \mu > 0$ and $a \geq -(\mu + \alpha)/(p - \alpha)$.

- (1) Supposing that $\mathcal{L}_{c,p,\lambda}^{m,\delta} F_{\mu,p}(f(z)) \neq 0$ for all $z \in \dot{U}$, then

$$F_{\mu,p}(\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; a)) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; a). \quad (60)$$

- (2) If we suppose in addition that

$$a \geq \max \left\{ -\frac{\mu + 2\alpha - p + 1}{2(p - \alpha)}; -\frac{\mu + \alpha}{p - \alpha} \right\}, \quad (61)$$

then

$$F_{\mu,p}(\mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; a)) \subset \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; r(p, \alpha, \mu, a)), \quad (62)$$

where the bound

$$\begin{aligned} r(p, \alpha, \mu, a) &= \frac{1}{p - \alpha} \\ &\times \left[\frac{\mu + p}{{}_2F_1(1, 2(p - \alpha)(1 - a), \mu + p + 1; 1/2)} - (\mu + \alpha) \right], \end{aligned} \quad (63)$$

is the best possible.

Theorem 13. Let $v \in \mathbb{C}^*$, and let $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that

$$\begin{aligned} \left| \frac{v(c+p)(A-B)}{B} - 1 \right| &\leq 1 \quad \text{or} \\ \left| \frac{v(c+p)(A-B)}{B} + 1 \right| &\leq 1, \quad \text{if } B \neq 0, \\ |vA| &\leq \frac{\pi}{c+p}, \quad \text{if } B = 0. \end{aligned} \quad (64)$$

If $f \in \mathcal{A}(p)$ with $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \neq 0$ for all $z \in \dot{U}$, then

$$\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} < \frac{1+Az}{1+Bz} \quad (65)$$

implies

$$\left(\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right)^v < q_1(z), \quad (66)$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{v(c+p)(A-B)/B} & \text{if } B \neq 0, \\ e^{v(c+p)Az} & \text{if } B = 0, \end{cases} \quad (67)$$

is the best dominant.

Proof. Let us put

$$\phi(z) = \left(\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right)^v \quad (z \in U), \quad (68)$$

then ϕ is analytic in U , with $\phi(0) = 1$ and $\phi(z) \neq 0$ for all $z \in \dot{U}$. By differentiating both sides of (68) logarithmical with respect to z and using (5), we have

$$\begin{aligned} 1 + \frac{z\phi'(z)}{v(c+p)\phi(z)} &= \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} \\ &< \frac{1+Az}{1+Bz}. \end{aligned} \quad (69)$$

Now the assertions of Theorem 13 follows by using Lemma 6 for the special case $\gamma = v(c+p)$. $\square$

Putting $B = -1$ and $A = 1 - 2\rho$, $0 \leq \rho < 1$, in Theorem 13, we obtain the following corollary.

Corollary 14. Assume that $v \in \mathbb{C}^*$ satisfies either

$$\begin{aligned} |2v(c+p)(1-\rho) - 1| &\leq 1 \quad \text{or} \\ |2v(c+p)(1-\rho) + 1| &\leq 1, \quad \text{if } B \neq 0, \end{aligned} \quad (70)$$

If $f \in \mathcal{A}(p)$ with $\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \neq 0$ for all $z \in \dot{U}$, then

$$\operatorname{Re} \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z)}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} > \rho \quad (71)$$

implies

$$\left(\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right)^v < q_2(z) = (1-z)^{-2v(c+p)(1-\rho)}, \quad (72)$$

and q_2 is the best dominant.

3. Properties Involving the Multiplier Operator $\mathcal{L}_{c,p,\lambda}^{m,\delta}$

Theorem 15. If $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$, then for all $s, t \in \mathbb{C}$ with $|s| \leq 1$, $|t| \leq 1$ and $s \neq t$, the next subordination holds

$$\frac{t^p \mathcal{L}_{c,p,\lambda}^{m,\delta} f(zs)}{s^p \mathcal{L}_{c,p,\lambda}^{m,\delta} f(zt)} < \begin{cases} \left(\frac{1+Bzs}{1+Bzt} \right)^{(p-\alpha)(A-B)/B}, & \text{for } B \neq 0 \\ \exp[(p-\alpha)Az(s-t)], & \text{for } B = 0. \end{cases} \quad (73)$$

Proof. If $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$, from (7) it follows that

$$\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} < \frac{p + [pB + (p-\alpha)(A-B)]z}{1+Bz} \equiv k(z). \quad (74)$$

Moreover, the function k defined by (74) and the function h given by

$$h(z) \equiv h(z; s, t) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu} \right) du \quad (75)$$

are convex in U . By combining a general subordination theorem [25, Theorem 4] with (74), we get

$$\begin{aligned} & \left(\frac{z \left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - p \right) * h(z) \\ & < \frac{(p-\alpha)(A-B)z}{1+Bz} * h(z). \end{aligned} \quad (76)$$

For every analytic function ϕ in U with $\phi(0) = 0$, we have

$$\phi(z) * h(z) = \int_{tz}^{sz} \frac{\phi(u)}{u} du, \quad (77)$$

and thus, from (76) and (77), we deduce

$$\begin{aligned} & \int_{tz}^{sz} \left(\frac{\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(u) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(u)} - \frac{p}{u} \right) du \\ & < (p-\alpha)(A-B) \int_{tz}^{sz} \frac{du}{1+Bu}. \end{aligned} \quad (78)$$

This last subordination implies

$$\begin{aligned} & \exp \left(\int_{tz}^{sz} \left(\frac{\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(u) \right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(u)} - \frac{p}{u} \right) du \right) \\ & < \exp \left((p-\alpha)(A-B) \int_{tz}^{sz} \frac{du}{1+Bu} \right), \end{aligned} \quad (79)$$

and by simplification, we get the assertion of Theorem 15. $\square$

Corollary 16. If $f \in \mathcal{S}_{c,p,\lambda}^{m,\delta}(\alpha; A, B)$, then for $|z| = r < 1$, the next inequalities hold

$$\left| \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right| \leq \begin{cases} r^p (1+Br)^{(p-\alpha)(A-B)/B}, & \text{for } B \neq 0, \\ r^p \exp[(p-\alpha)Ar], & \text{for } B = 0, \end{cases} \quad (80)$$

$$\left| \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \right| \geq \begin{cases} r^p (1-Br)^{(p-\alpha)(A-B)/B}, & \text{for } B \neq 0, \\ r^p \exp[-(p-\alpha)Ar], & \text{for } B = 0, \end{cases} \quad (81)$$

$$\begin{aligned} & \left| \arg \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right| \\ & \leq \begin{cases} \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|r), & \text{for } B \neq 0, \\ (p-\alpha)Ar, & \text{for } B = 0. \end{cases} \end{aligned} \quad (82)$$

All of the estimates asserted here are sharp.

Proof. Taking $s = 1$ and $t = 0$ in (73) and using the definition of subordination, we obtain

$$\frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} = \begin{cases} (1+Bw(z))^{(p-\alpha)(A-B)/B}, & \text{for } B \neq 0, \\ \exp[(p-\alpha)Aw(z)], & \text{for } B = 0, \end{cases} \quad (83)$$

where w is analytic function in U with $w(0) = 0$ and $|w(z)| \leq 1$ for $z \in U$. According to the well-known Schwarz's theorem, we have $|w(z)| \leq |z|$ for all $z \in U$.

(i) If $B > 0$, then we find from (83) that

$$\begin{aligned} \left| \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right| &= \exp \left[\frac{(p-\alpha)(A-B)}{B} \log |1+Bw(z)| \right] \\ &= |1+Bw(z)|^{(p-\alpha)(A-B)/B} \\ &\leq (1+Br)^{(p-\alpha)(A-B)/B}. \end{aligned} \quad (84)$$

(ii) If $B < 0$, we can easily obtain

$$\begin{aligned} \left| \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right| &= |1+Bw(z)|^{(p-\alpha)(A-B)/-B} \\ &\leq [(1+Br)^{-1}]^{(p-\alpha)(A-B)/-B} \\ &= (1+Br)^{(p-\alpha)(A-B)/B}. \end{aligned} \quad (85)$$

This proves the inequality (80) for $B \neq 0$. Similarly, we can prove the other inequalities in (80) and (81). Now, for $|z| = r$ and $B \neq 0$, we observe from (83) that

$$\begin{aligned} \left| \arg \frac{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)}{z^p} \right| &= \frac{(p-\alpha)(A-B)}{|B|} |\arg(1+Bw(z))| \\ &\leq \frac{(p-\alpha)(A-B)}{|B|} \sin^{-1}(|B|r), \end{aligned} \quad (86)$$

and for $B = 0$, (82) is a direct consequence of (83).

It is easy to see that all of the estimates in Corollary 16 are sharp, being attained by the function f_0 defined by

$$\mathcal{L}_{c,p,\lambda}^{m,\delta} f_0(z) = \begin{cases} z^p(1+Bz)^{(p-\alpha)(A-B)/B}, & \text{for } B \neq 0, \\ z^p \exp[(p-\alpha)Az], & \text{for } B = 0. \end{cases} \quad (87)$$

□

Remark 17. (i) Putting $c = 1$ and $m = 0$ in our results, we obtain the results obtained by Aouf et al. [16].

(ii) By specializing the parameters c , λ , δ , p , and m , we obtain various results for different operators defined in Section 1.

Acknowledgment

The authors thank the referees for their valuable suggestions which led to improvement of this paper.

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