Research Article

An Analytical Approach on Thermally Induced Vibrations of Nonhomogeneous Tapered Plate

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Received 10 January 2013; Accepted 21 April 2013

Academic Editor: Harold Benson

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A mathematical model to control the vibrations of a rectangular plate is constructed with an aim to assist engineers in designing and fabrication of various structures used in the field of science and technology, mostly used in satellite and aeronautical engineering. The present study is related to the analysis of free vibrations of nonhomogeneous rectangular plate clamped at all the four edges. Authors studied the bilinear effect of thickness as well as temperature variations in both x and y directions. Variation in Poisson's ratio is also considered linearly in x -direction due to nonhomogeneity. Rayleigh-Ritz method is used to analyze the frequencies for the first two modes of vibrations for different values of thermal gradient, nonhomogeneity constant, taper constants and aspect ratio. All the numerical computations have been performed for an alloy of aluminum, that is, duralumin. All the results are presented in the form of graphs.

1. Introduction

Vibration effects on complex systems have always been a major concern for structural engineers. In the field of science and technology, it is desired to design large machines for smooth operations with controlled vibrations. Study of the phenomenon of vibration is not just restricted to science but has become part of our daily life. Recent studies in the field of vibrational behavior of different materials have drawn attention of scientists and engineers involved in the design and construction of complex systems such as ships, submarines, aircrafts, launch vehicles, missiles, satellites. These studies are also useful for architectural engineers to construct earthquake-resistant buildings. Plates of variable thickness are commonly used in many engineering applications. So what is required at the moment is an in-depth knowledge of the plates' behavior under vibrations which would ultimately help to see their potential in several areas. The prevalent interest in the effects of high temperatures on nonhomogeneous rectangular plates of variable thickness is due to their applications in various engineering branches such as nuclear power plants, aeronautical engineering, chemical plants and

so forth, where metals and their alloys exhibit viscoelastic behavior. The reason for this is that during heating-up periods, structures are exposed to high-intensity heat fluxes and material properties undergo significant changes; in particular thermal effect can not be negligible. A lot of literature is available in one dimensional variation in temperature with thickness variation of plates, but negligible work is found on two-dimensional temperature variations.

Laura et al. [1] had studied the transverse vibrations of rectangular plates with linear variation of the thickness in the x and y directions. Leissa [2] had evaluated the effect of thermal gradient on the vibration of parallelogram plate with linearly varying thickness in both directions and thermal effect in linear form only. Gupta et al. [3] analyzed the effect of nonhomogeneity on thermally induced vibration of orthotropic viscoelastic rectangular plate of linearly varying thickness. Gupta et al. [4] had evaluated time period and deflection for the first two modes of vibration of viscoelastic rectangular plate and for various bilinear thickness variation. Jain and Soni [5] had studied the free vibrations of rectangular plates with thickness varying parabolically. Kumar [6] analyzed the vibration of viscoelastic isotropic

rectangular plate with varying thickness in two directions that is, linearly in one and parabolically in the other direction. Lal and Dhanapati [7] had worked on the vibrations of nonhomogeneous orthotropic rectangular plates of varying thickness with two opposite simply supported edges and resting on two-parameter foundation. Singh and Saxena [8] had studied the transverse vibrations of a rectangular plate of variable thickness with different combinations of boundary conditions at the four edges. Tomar and Gupta [9] had evaluated the thermal gradient effect on the vibration of a rectangular plate having bidirectional variation in thickness. Tomar and Gupta [10] had studied the effect of thermal gradient on frequencies of an orthotropic rectangular plate whose thickness varies in two directions. Li [11] gave an analysis on modal characteristics of vibrations of a rectangular plate with general elastic supports along its edges. Khanna et al. [12] analyzed the effect of varying Poisson's ratio on thermally induced vibrations of nonhomogeneous rectangular plate.

Vibrations are of the most important mode of failure in plates. Hence, engineers often try to know about some modes of vibration before finalizing the design of a structure or machine. In this paper, our endeavor is to provide a mathematical model for analyzing the vibrational behavior of viscoelastic nonhomogeneous isotropic rectangular plate with bilinear varying thickness and temperature variations. The rectangular plate is clamped at all four edges. Due to nonhomogeneity in the material of plate, it is considered that Poisson's ratio varies linearly along x -direction. The Rayleigh Ritz method has been used to calculate first two modes of frequencies for different values of aspect ratio, taper constants, thermal gradient, and nonhomogeneity constant.

2. Analysis of Motion

The differential equation of the motion of a viscoelastic isotropic rectangular plate may be written as [13]

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = \rho h \frac{\partial^2 w}{\partial t^2}. \quad (1)$$

Here, M_x and M_y are bending moments, M_{xy} is twisting moment per unit length of plate, ρ is mass per unit volume, h is thickness of plate, and w is displacement at time t .

The expressions for M_x , M_y , and M_{xy} are given by

$$\begin{aligned} M_x &= -\bar{D} D_1 \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right], \\ M_y &= -\bar{D} D_1 \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right], \\ M_{xy} &= -\bar{D} D_1 (1 - \nu) \frac{\partial^2 w}{\partial x \partial y}, \end{aligned} \quad (2)$$

where $D_1 = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate's material and $\bar{D}$ is viscoelastic operator.

On putting the values of M_x , M_y , and M_{xy} from (2) in (1), one gets

$$\begin{aligned} \bar{D} \left[D_1 \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + 2 \frac{\partial D_1}{\partial x} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \right. \\ + 2 \frac{\partial D_1}{\partial y} \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y \partial x^2} \right) + \frac{\partial^2 D_1}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ + \frac{\partial^2 D_1}{\partial y^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial^2 D_1}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \left. \right] \\ + \rho h \frac{\partial^2 w}{\partial t^2} = 0. \end{aligned} \quad (3)$$

Take deflection w as a product of two functions as follows:

$$w(x, y, t) = W(x, y) T(t), \quad (4)$$

where $W(x, y)$ is deflection function in x and y and $T(t)$ is a time function.

Substituting (4) into (3), one obtains

$$\begin{aligned} \left[D_1 \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + 2 \frac{\partial D_1}{\partial x} \left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} \right) \right. \\ + 2 \frac{\partial D_1}{\partial y} \left(\frac{\partial^3 W}{\partial y^3} + \frac{\partial^3 W}{\partial y \partial x^2} \right) + \frac{\partial^2 D_1}{\partial x^2} \left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right) \\ + \frac{\partial^2 D_1}{\partial y^2} \left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial^2 D_1}{\partial x \partial y} \frac{\partial^2 W}{\partial x \partial y} \left. \right] \\ \times (\rho h W)^{-1} = - \frac{(\partial^2 T / \partial t^2)}{\bar{D} T}. \end{aligned} \quad (5)$$

Taking that both sides of (5) are equal to a constant p^2 , we have

$$\begin{aligned} \left[D_1 \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + 2 \frac{\partial D_1}{\partial x} \left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} \right) \right. \\ + 2 \frac{\partial D_1}{\partial y} \left(\frac{\partial^3 W}{\partial y^3} + \frac{\partial^3 W}{\partial y \partial x^2} \right) + \frac{\partial^2 D_1}{\partial x^2} \left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right) \\ + \frac{\partial^2 D_1}{\partial y^2} \left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right) + 2(1 - \nu) \frac{\partial^2 D_1}{\partial x \partial y} \frac{\partial^2 W}{\partial x \partial y} \left. \right] \\ - \rho p^2 h W = 0, \end{aligned} \quad (6)$$

$$\frac{\partial^2 T}{\partial t^2} + p^2 \bar{D} T = 0. \quad (7)$$

These are the differential equations of motion (6) and time function (7) for viscoelastic isotropic rectangular plate of variable thickness in Cartesian coordinate, respectively.

It is assumed that the thickness of the viscoelastic isotropic rectangular plate varies bilinearly in both directions, that is,

$$h = h_0 \left(1 + \beta_1 \frac{x}{a}\right) \left(1 + \beta_2 \frac{y}{b}\right), \quad (8)$$

where a and b are the length and breadth of the rectangle plate respectively and β_1 and β_2 are taper parameters. Also, $h = h_0$ at $x = y = 0$.

It is considered that the rectangular plate of engineering material has a steady bilinear temperature variations that is,

$$\tau = \tau_0 \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right), \quad (9)$$

where τ denotes the temperature excess above the reference temperature at any point on the plate and τ_0 denotes the temperature at any point on the boundary of plate.

The temperature dependence of the modulus of elasticity for most of engineering materials can be expressed as follows:

$$E = E_0 (1 - \gamma\tau), \quad (10)$$

where E_0 is the value of Young's modulus at reference temperature, that is, $\tau = 0$ and γ is the slope of variation of E and τ . The modulus variation (10) becomes

$$E = E_0 \left[1 - \alpha \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right)\right], \quad (11a)$$

where $\alpha = \gamma\tau_0$ ($0 \leq \alpha < 1$) is thermal gradient.

Again, it is assumed that the Poisson ratio of the plate's material varies linearly in x -direction that is,

$$v = v_0 \left(1 - \alpha_1 \frac{x}{a}\right), \quad (11b)$$

where α_1 is the nonhomogeneity constant.

Putting the values of h , E , and v from (8), (11a) and (11b) in the expression D_1 , one obtains

$$D_1 = \frac{E_0 [1 - \alpha(1 - x/a)(1 - y/b)] h_0^3 (1 + \beta_1(x/a))^3 (1 + \beta_2(y/b))^3}{12(1 - v_0^2(1 - \alpha_1(x/a))^2)}. \quad (12)$$

3. Solution and Frequency Equations

Rayleigh-Ritz technique is applied to solve the frequency equation. In this method, one requirement is that maximum strain energy (P_E) must be equal to the maximum kinetic energy (K_E). So it is necessary for the problem under consideration that

$$\delta(P_E - K_E) = 0 \quad (13)$$

for arbitrary variations of W satisfying relevant geometrical boundary conditions.

Since the plate is assumed as clamped at all the four edges, the boundary conditions are

$$\begin{aligned} W = \frac{\partial W}{\partial x} = 0 \quad \text{at } x = 0, a, \\ W = \frac{\partial W}{\partial y} = 0 \quad \text{at } y = 0, b. \end{aligned} \quad (14)$$

To satisfy the above boundary conditions, two-term deflection function is taken as [4]

$$\begin{aligned} W = & \left[\left(\frac{x}{a}\right) \left(\frac{y}{b}\right) \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \right]^2 \\ & \times \left[A_1 + A_2 \left(\frac{x}{a}\right) \left(\frac{y}{b}\right) \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \right]. \end{aligned} \quad (15)$$

Now assuming the nondimensional variables as

$$X = \frac{x}{a}, \quad Y = \frac{y}{b}, \quad \bar{W} = \frac{W}{a}, \quad \bar{h} = \frac{h}{a}. \quad (16)$$

The expressions for kinetic energy K_E and strain energy V_E are [1]

$$\begin{aligned} K_E = & \frac{1}{2} \rho p^2 \bar{h}_0 a^5 \int_0^1 \int_0^{b/a} \left[\left(1 + \beta_1 X\right) \left(1 + \beta_2 \frac{aY}{b}\right) \bar{W}^2 \right] dY dX, \\ P_E = & \frac{E_0 h_0^3 a^3}{24} \int_0^1 \int_0^{b/a} \frac{1}{\{1 - v_0^2(1 - \alpha_1 X)^2\}} \\ & \times (1 + \beta_1 X)^3 \times \left(1 + \beta_2 \frac{aY}{b}\right)^3 \\ & \times \left[\left(\frac{\partial^2 \bar{W}}{\partial X^2}\right)^2 + \left(\frac{\partial^2 \bar{W}}{\partial Y^2}\right)^2 \right. \\ & + 2\{v_0(1 - \alpha_1 X)\} \frac{\partial^2 \bar{W}}{\partial X^2} \frac{\partial^2 \bar{W}}{\partial Y^2} \\ & \left. + 2\{1 - v_0(1 - \alpha_1 X)\} \left(\frac{\partial^2 \bar{W}}{\partial X \partial Y}\right)^2 \right] dY dX. \end{aligned} \quad (17)$$

Using (17) in (13), one gets

$$P_E^* - \lambda^2 K_E^* = 0, \quad (18)$$

FIGURE 1: Frequency versus nonhomogeneity constant.

where

$$\begin{aligned}
 P_E^* &= \int_0^1 \int_0^{b/a} \frac{1}{\{1 - v_0^2(1 - \alpha_1 X)^2\}} \\
 &\quad \times (1 + \beta_1 X)^3 \times \left(1 + \beta_2 \frac{aY}{b}\right)^3 \\
 &\quad \times \left[(\overline{W}_{XX})^2 + (\overline{W}_{YY})^2\right. \\
 &\quad \left.+ 2\{v_0(1 - \alpha_1 X)\} \overline{W}_{XX} \overline{W}_{YY}\right. \\
 &\quad \left.+ 2\{1 - v_0(1 - \alpha_1 X)\} (\overline{W}_{XY})^2\right] dY dX, \\
 K_E^* &= \int_0^1 \int_0^{b/a} \left[(1 + \beta_1 X) \left(1 + \beta_2 \frac{aY}{b}\right) \overline{W}^2\right] dY dX.
 \end{aligned} \quad (19)$$

Here, $\lambda^2 = 12\rho p^2/E_0 h_0^2$ is a frequency parameter.

Equation (18) consists of two unknown constants that is, A_1 and A_2 arising due to the substitution of W .

These two constants are to be determined as follows:

$$\frac{\partial}{\partial A_n} (P_E^* - \lambda^2 K_E^*) = 0, \quad n = 1, 2. \quad (20)$$

On simplifying (20), one gets

$$b_{n1} A_1 + b_{n2} A_2 = 0, \quad n = 1, 2, \quad (21)$$

where b_{n1} and b_{n2} involve parametric constant and the frequency parameter.

For a nontrivial solution, the determinant of the coefficient must be zero. So, one gets the frequency equation as

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0. \quad (22)$$

With the help of (22), one can obtain a quadratic equation in λ^2 from which the two values of frequency parameters for both modes of vibration can be found easily.

4. Results and Discussion

For calculating the values of frequency (λ) for a rectangular plate with different values of aspect ratio (a/b), thermal gradient (α), nonhomogeneity constant (α_1), and taper constants (β_1 and β_2) for the first two modes of vibrations, the following material parameters are used which are for duralumin reported at (2007): $E = 7.08 \times 10^{10} \text{ N/M}^2$, $\rho = 2.80 \times 10^3 \text{ Kg/M}^3$, and $\nu = 0.345$. The thickness of the plate at the center is taken as $h_0 = 0.01 \text{ m}$.

Computations have been made for calculating frequencies for different values of taper constants (β_1 and β_2) and aspect ratio (a/b) for the first two modes of vibration. Various cases of time period against nonhomogeneity constant, taper constant, aspect ratio, and thermal gradient which are stated as below were considered.

4.1. Frequency versus Nonhomogeneity Constant. From Figures 1(a) and 1(b), it is clear that frequency is decreasing continuously as nonhomogeneity constant (α_1) increases from 0.0 to 1.0 frequency increases continuously for increasing value of taper constant and thermal gradient for the first two modes of vibration.

4.2. Frequency versus Thermal Gradient. Figures 2(a) and 2(b) show the numerical results of frequency with thermal gradient for different combinations of taper constant and nonhomogeneity constant, that is, (i) $\beta_1 = 0.0, \beta_2 = 0.0, \alpha = 0.0$, (ii) $\beta_1 = 0.0, \beta_2 = 0.0, \alpha = 0.2$, (iii) $\beta_1 = 0.0, \beta_2 = 0.0, \alpha = 0.6$, and (iv) $\beta_1 = 0.0, \beta_2 = 0.0, \alpha = 0.8$.

FIGURE 2: Frequency versus thermal gradient.

Authors can easily conclude that frequency continuously decreases as thermal gradient α increase from 0.0 to 1.0 for the first two modes of vibrations.

4.3. Frequency versus Taper Constant (β_2). From Figure 3, authors conclude that frequency increases continuously as taper constants β_1 increases from 0.0 to 1.0. For the fixed values of thermal gradient and nonhomogeneity constant ($\alpha = 0.4$ and $\alpha_1 = 0.4$), frequency continuously increases as β_2 increases from 0.0 to 1.0 for both modes of vibrations for the following cases: (i) $\beta_1 = 0.2$ and (ii) $\beta_1 = 0.6$ and for the fixed values of thermal gradient and nonhomogeneity constant ($\alpha = 0.2$ and $\alpha_1 = 0.2$).

4.4. Frequency versus Taper Constant (β_1). From Figure 4, authors conclude that frequency increases continuously as taper constants β_2 increases from 0.0 to 1.0. For the fixed values of thermal gradient and nonhomogeneity constant ($\alpha = 0.2$ and $\alpha_1 = 0.2$), frequency continuously increases as β_1 increases from 0.0 to 1.0 for both modes of vibrations for the following cases: (i) $\beta_2 = 0.4$ and (ii) $\beta_2 = 0.6$. On comparing these cases, one can easily see that as nonhomogeneity constant increases, frequency increases for both modes of vibrations.

4.5. Frequency versus Aspect Ratio. From Figure 5, one can clearly observe that frequency increases continuously as aspect ratio increases from 0.5 to 2.5 for different values of thermal gradient, nonhomogeneity constant, and taper constant (β_1 and β_2) for the first two modes of vibrations for the following cases:

- (i) $\alpha = 0.2, \alpha_1 = 0.2, \beta_1 = 0.0, \beta_2 = 0.0$,
- (ii) $\alpha = 0.2, \alpha_1 = 0.2, \beta_1 = 0.6, \beta_2 = 0.6$,
- (iii) $\alpha = 0.2, \alpha_1 = 0.2, \beta_1 = 0.4, \beta_2 = 0.2$, and
- (iv) $\alpha = 0.8, \alpha_1 = 0.8, \beta_1 = 0.2, \beta_2 = 0.4$.

FIGURE 3: Frequency versus taper constant (β_2). FIGURE 4: Frequency versus taper constant (β_1).

FIGURE 5: Frequency versus aspect ratio (a/b).

On comparing above cases, one can easily find that as nonhomogeneity constant increases, frequency also increases for both the modes of vibrations.

Also, it can be observed that as the combined values of β_1 and β_2 increase from 0.0 to 0.6 for fixed values of $\alpha = \alpha_1 = 0.2$, both modes of frequency increase rapidly.

5. Conclusion

On comparing the results of the present paper with those of [12], authors conclude that values of frequency for both modes of vibration in the present paper are slightly greater than those of [12] for the corresponding parameters. Engineers or practitioners are advised to analyze the numerical finding of the present paper to get the required values of frequency by appropriate tapering of plates. The paper aims at providing a kind of mathematical design so that scientists can perceive their potential in mechanical engineering field increase strength, durability, and efficiency of mechanical design and fabrication with a practical approach with higher levels of safety and economy.

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