Research Article

Generalized $H(\cdot, \cdot, \cdot)$ - η -Cocoercive Operators and Generalized Set-Valued Variational-Like Inclusions

Shamshad Husain,¹ Sanjeev Gupta,¹ and Vishnu Narayan Mishra²

¹ Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh 202002, India

² Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Road, Surat 395 007, India

Correspondence should be addressed to Sanjeev Gupta; guptasanmp@gmail.com

Received 18 March 2013; Revised 4 May 2013; Accepted 8 May 2013

Academic Editor: Kaleem R. Kazmi

Copyright © 2013 Shamshad Husain et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate a new class of cocoercive operators named generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operators in Hilbert spaces. We prove that generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator is single-valued and Lipschitz continuous and extends the concept of resolvent operators associated with $H(\cdot, \cdot)$ -cocoercive operators to the generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operators. Some examples are given to justify the definition of generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operators. Further, we consider a generalized set-valued variational-like inclusion problem involving generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator. In terms of the new resolvent operator technique, we give the approximate solution and suggest an iterative algorithm for the generalized set-valued variational-like inclusions. Furthermore, we discuss the convergence criteria of iterative algorithm under some suitable conditions. Our results can be viewed as a generalization of some known results in the literature.

1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. One of the most interesting and important problems in the theory of variational inclusions include variational, quasi-variational, and variational-like inequalities as special cases. For applications of variational inclusions, see, for example, [1]. Various kinds of iterative methods have been studied to solve the variational inclusions. Among these methods, the resolvent operator technique for the study of variational inclusions has been widely used by many authors. For details, we refer to [2–16].

Recently Fang and Huang, Kazmi and Khan, and Lan et al. investigated several resolvent operators for generalized operators such as H -monotone [5], H -accretive [6], (P, η) -proximal point [11], (P, η) -accretive [12], (H, η) -monotone [7], (A, η) -accretive [13], and mappings. Very recently, Zou and Huang [16] introduced and studied $H(\cdot, \cdot)$ -accretive operators, Kazmi et al. [8–10] introduced and studied generalized

$H(\cdot, \cdot)$ -accretive operators, $H(\cdot, \cdot)$ - η -proximal point mapping, Xu and Wang [15] introduced and studied $(H(\cdot, \cdot), \eta)$ -monotone operators, and Ahmad et al. [2] introduced and studied $H(\cdot, \cdot)$ -cocoercive operators.

Motivated by the recent work going in this direction, we consider a class of cocoercive operators called generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive, a natural generalization of monotone (accretive) operators in Hilbert (Banach) spaces. For details, we refer to [2, 5–7, 13–16]. We prove that generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator is single-valued and Lipschitz continuous and extends the concept of resolvent operators associated with $H(\cdot, \cdot)$ -cocoercive operators to the generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operators. Further, we consider the generalized set-valued variational-like inclusion problem involving generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator in Hilbert spaces. Using new a resolvent operator technique, we prove the existence of solutions and suggest an iterative algorithm for the generalized set-valued variational-like inclusions. Furthermore, we discuss the convergence criteria of the iterative algorithm under some suitable conditions. Our results can

be viewed as an extension and generalization of some known results [2, 15, 16]. For illustration of Definitions 2 and 7 and examples 2.3 and 3.2 are given, respectively.

2. Preliminaries

Throughout this paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$, 2^X (resp., $\text{CB}(X)$) is the family of all the nonempty (resp., closed and bounded) subsets of X , and $\mathcal{D}(\cdot, \cdot)$ is the Hausdorff metric on $\text{CB}(X)$ defined by

$$\mathcal{D}(P, Q) = \max \left\{ \sup_{x \in P} d(x, Q), \sup_{y \in Q} d(P, y) \right\}, \quad (1)$$

$$\forall P, Q \in \text{CB}(X),$$

where $d(x, Q) = \inf_{y \in Q} \|x - y\|$ and $d(P, y) = \inf_{x \in P} \|x - y\|$. In the sequel, let us recall some concepts.

Definition 1 (see [4, 17]). Let $P : X \rightarrow X$ and $\eta : X \times X \rightarrow X$ be two mappings. Then, P is said to be

(i) η -monotone if

$$\langle P(x) - P(y), \eta(x, y) \rangle \geq 0, \quad \forall x, y \in X, \quad (2)$$

(ii) δ_1 - η -strongly monotone if there exists a constant $\delta_1 > 0$ such that

$$\langle P(x) - P(y), \eta(x, y) \rangle \geq \delta_1 \|x - y\|^2, \quad \forall x, y \in X, \quad (3)$$

(iii) μ_1 - η -cocoercive if there exists a constant $\mu_1 > 0$ such that

$$\langle P(x) - P(y), \eta(x, y) \rangle \geq \mu_1 \|P(x) - P(y)\|^2, \quad \forall x, y \in X, \quad (4)$$

(iv) γ_1 - η -relaxed cocoercive if there exists a constant $\gamma_1 > 0$ such that

$$\langle P(x) - P(y), \eta(x, y) \rangle \geq (-\gamma_1) \|P(x) - P(y)\|^2, \quad \forall x, y \in X, \quad (5)$$

(v) λ_P -Lipschitz continuous, if there exists a constant $\lambda_P > 0$ such that

$$\|P(x) - P(y)\| \leq \lambda_P \|x - y\|, \quad \forall x, y \in X, \quad (6)$$

(vi) α -expansive, if there exists a constant $\alpha > 0$ such that

$$\|P(x) - P(y)\| \geq \alpha \|x - y\|, \quad \forall x, y \in X, \quad (7)$$

if $\alpha = 1$, then it is expansive;

(vii) η is said to be τ -Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X. \quad (8)$$

If $\eta(x, y) = x - y$, for all $x, y \in X$, then definitions (i) to (iv) reduce to the Definitions of monotonicity, strong monotonicity [18], cocoercivity [19], and relaxed cocoercive, respectively.

Definition 2. Let $P, Q, R : X \rightarrow X$, $\eta : X \times X \rightarrow X$, and $H : X \times X \times X \rightarrow X$ be the single-valued mappings. Then,

(i) $H(P, \cdot, \cdot)$ is said to be μ - η -cocoercive with respect to P if there exists a constant $\mu > 0$ such that

$$\langle H(Px, u, u) - H(Py, u, u), \eta(x, y) \rangle \geq \mu \|Px - Py\|^2, \quad \forall x, y, u \in X, \quad (9)$$

(ii) $H(\cdot, Q, \cdot)$ is said to be γ - η -relaxed cocoercive with respect to Q if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Qx, u) - H(u, Qy, u), \eta(x, y) \rangle \geq (-\gamma) \|Qx - Qy\|^2, \quad \forall x, y, u \in X, \quad (10)$$

(iii) $H(\cdot, \cdot, R)$ is said to be δ - η -strongly monotone with respect to R if there exists a constant $\delta > 0$ such that

$$\langle H(u, u, Rx) - H(u, u, Ry), \eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall x, y, u \in X, \quad (11)$$

(iv) $H(P, \cdot, \cdot)$ is said to be r_1 -Lipschitz continuous with respect to P if there exists a constant $r_1 > 0$ such that

$$\|H(Px, \cdot, \cdot) - H(Py, \cdot, \cdot)\| \leq r_1 \|x - y\|, \quad \forall x, y \in X, \quad (12)$$

(v) $H(\cdot, Q, \cdot)$ is said to be r_2 -Lipschitz continuous with respect to Q if there exists a constant $r_2 > 0$ such that

$$\|H(\cdot, Qx, \cdot) - H(\cdot, Qy, \cdot)\| \leq r_2 \|x - y\|, \quad \forall x, y \in X, \quad (13)$$

(vi) $H(\cdot, \cdot, R)$ is said to be r_3 -Lipschitz continuous with respect to R if there exists a constant $r_3 > 0$ such that

$$\|H(\cdot, \cdot, Rx) - H(\cdot, \cdot, Ry)\| \leq r_3 \|x - y\|, \quad \forall x, y \in X. \quad (14)$$

If $\eta(x, y) = x - y$, for all $x, y \in H$ and $H(\cdot, \cdot, \cdot) = H(\cdot, \cdot, \cdot)$, then Definitions (i)-(ii) are reduced to the definition of cocoercivity and relaxed cocoercive [2], respectively, and (iii) reduces to strong accretivity [16].

Example 3. Let $X = \mathbb{R}^2$ with usual inner product. Let $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $P, Q, R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$\eta(x, y) = \begin{pmatrix} mx_1 + nx_2 \\ -my_1 - ny_2 \end{pmatrix}, \quad Px = \begin{pmatrix} mx_1 - mx_2 \\ -mx_1 + nx_2 \end{pmatrix}, \quad (15)$$

$$Qy = \begin{pmatrix} -my_1 + my_2 \\ -ny_2 \end{pmatrix}, \quad Rz = \begin{pmatrix} nz_1 \\ mz_2 \end{pmatrix},$$

for all scalars $n, m \in \mathbb{R}$ with $n > m$ and for all $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$.

Suppose that $H(Px, Qy, Rz) = Px + Qy + Rz$. Then, $H(P, Q, R)$ is $(1/2)$ - η -cocoercive with respect to P , $(1/2)$ - η -relaxed cocoercive with respect to Q , m^2 - η -strongly monotone with respect to R , and $\eta(x, y)$ is n -Lipschitz continuous.

Indeed, let for any $u \in X$,

$$\begin{aligned}
 & \langle H(Px, u, u) - H(Py, u, u), \eta(x, y) \rangle \\
 &= \langle Px - Py, \eta(x, y) \rangle \\
 &= \langle (mx_1 - mx_2, -mx_1 + nx_2) - (my_1 - my_2, -my_1 + ny_2), \\
 & \quad (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\
 &= \langle (m(x_1 - y_1) - m(x_2 - y_2), -m(x_1 - y_1) + n(x_2 - y_2)), \\
 & \quad (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\
 &= m^2(x_1 - y_1)^2 - m(m + n)(x_1 - y_1)(x_2 - y_2) \\
 & \quad + n^2(x_2 - y_2)^2, \\
 & \|Px - Py\|^2 \\
 &= \langle Px - Py, Px - Py \rangle \\
 &= \langle (mx_1 - mx_2, -mx_1 + nx_2) - (my_1 - y_2, -my_1 + ny_2), \\
 & \quad (mx_1 - mx_2, -mx_1 + nx_2) - (my_1 - y_2, -my_1 + ny_2) \rangle \\
 &= 2m^2(x_1 - y_1)^2 - 2m(m + n)(x_1 - y_1)(x_2 - y_2) \\
 & \quad + (m^2 + n^2)(x_2 - y_2)^2 \\
 &\leq 2m^2(x_1 - y_1)^2 - 2m(m + n)(x_1 - y_1)(x_2 - y_2) \\
 & \quad + 2n^2(x_2 - y_2)^2 \\
 &= 2 \langle H(Px, u, u) - H(Py, u, u), \eta(x, y) \rangle, \\
 & \langle H(Px, u, u) - H(Py, u, u), \eta(x, y) \rangle \geq \frac{1}{2} \|Px - Py\|^2,
 \end{aligned} \tag{16}$$

that is, $H(P, Q, R)$ is $(1/2)$ - η -cocoercive with respect to P . Consider

$$\begin{aligned}
 & \langle H(u, Qx, u) - H(u, Qy, u), \eta(x, y) \rangle \\
 &= \langle Qx - Qy, \eta(x, y) \rangle \\
 &= \langle (-mx_1 + mx_2, -nx_2) - (-my_1 + my_2, -ny_2), \\
 & \quad (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\
 &= \langle (-m(x_1 - y_1) + m(x_2 - y_2), -n(x_2 - y_2)), \\
 & \quad (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\
 &= -m^2(x_1 - y_1)^2 + m^2(x_1 - y_1)(x_2 - y_2) - n^2(x_2 - y_2)^2,
 \end{aligned}$$

$$\begin{aligned}
 & \|Qx - Qy\|^2 \\
 &= \langle Qx - Qy, Qx - Qy \rangle \\
 &= \langle (-mx_1 + mx_2, -nx_2) - (-my_1 + my_2, -ny_2), \\
 & \quad (-mx_1 + mx_2, -nx_2) - (-my_1 + my_2, -ny_2) \rangle \\
 &= m^2(x_1 - y_1)^2 - 2m^2(x_1 - y_1)(x_2 - y_2) \\
 & \quad + (m^2 + n^2)(x_2 - y_2)^2 \\
 &\leq 2m^2(x_1 - y_1)^2 - 2m^2(x_1 - y_1)(x_2 - y_2) + 2n^2(x_2 - y_2)^2 \\
 &= (-2) \langle H(u, Qx, u) - H(u, Qy, u), \eta(x, y) \rangle, \\
 & \langle H(u, Qx, u) - H(u, Qy, u), \eta(x, y) \rangle \geq -\frac{1}{2} \|Qx - Qy\|^2,
 \end{aligned} \tag{17}$$

that is, $H(P, Q, R)$ is $(1/2)$ - η -relaxed cocoercive with respect to Q . In addition,

$$\begin{aligned}
 & \langle H(u, u, Rx) - H(u, u, Ry), \eta(x, y) \rangle \\
 &= \langle Rx - Ry, \eta(x, y) \rangle \\
 &= \langle (nx_1, mx_2) - (ny_2, my_2), (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\
 &= mn(x_1 - y_1)^2 + mn(x_2 - y_2)^2 \\
 &\geq m^2 \|x - y\|^2, \\
 & \langle H(u, u, Rx) - H(u, u, Ry), \eta(x, y) \rangle \geq m^2 \|x - y\|^2,
 \end{aligned} \tag{18}$$

that is, $H(P, Q, R)$ is m^2 - η -strongly monotone with respect to R . Moreover,

$$\begin{aligned}
 & \|\eta(x, y)\| = \|m(x_1 - y_1), n(x_2 - y_2)\| \\
 &= \sqrt{m^2(x_1 - y_1)^2 + n^2(x_2 - y_2)^2} \\
 &\leq \sqrt{n^2(x_1 - y_1)^2 + n^2(x_2 - y_2)^2} \\
 &\leq n \|x - y\|,
 \end{aligned} \tag{19}$$

that is, $\eta(x, y)$ is n -Lipschitz continuous.

Definition 4. A set-valued mapping $M : X \rightarrow 2^X$ is said to be m - η -relaxed monotone if there exists a constant $m > 0$ such that

$$\begin{aligned}
 & \langle u - v, \eta(x, y) \rangle \geq (-m) \|x - y\|^2, \quad \forall x, y \in X, \\
 & u \in M(x), \quad v \in M(y).
 \end{aligned} \tag{20}$$

Definition 5. A set-valued mapping $T : X \rightarrow CB(X)$ is said to be $\mathcal{D}$ -Lipschitz continuous, if there exists a constant $l > 0$ such that

$$\mathcal{D}(Tx, Ty) \leq l \|x - y\|, \quad \forall x, y \in X. \tag{21}$$

Definition 6. Let $S, T : X \rightarrow \text{CB}(X)$ be the set-valued mappings. A mapping $N : X \times X \rightarrow X$ is said to be

- (i) ϵ_1 -Lipschitz continuous in the first argument with respect to S , if there exists a constant $\epsilon_1 > 0$ such that

$$\begin{aligned} \|N(w_1, \cdot) - N(w_2, \cdot)\| &\leq \epsilon_1 \|w_1 - w_2\|, \quad \forall u_1, u_2 \in X, \\ w_1 &\in S(u_1), \quad w_2 \in S(u_2), \end{aligned} \quad (22)$$

- (ii) ϵ_2 -Lipschitz continuous in the second argument with respect to T , if there exists a constant $\epsilon_2 > 0$ such that

$$\begin{aligned} \|N(\cdot, v_1) - N(\cdot, v_2)\| &\leq \epsilon_2 \|v_1 - v_2\|, \quad \forall u_1, u_2 \in X, \\ v_1 &\in T(u_1), \quad v_2 \in T(u_2). \end{aligned} \quad (23)$$

3. Generalized $H(\cdot, \cdot, \cdot)$ - η -Cocoercive Operators

Definition 7. Let $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, and $\eta : X \times X \rightarrow X$ be the single-valued mappings. Let $H(P, Q, R)$ be μ - η -cocoercive with respect to P , γ - η -relaxed cocoercive with respect to Q , and δ - η -strongly monotone with respect to R . Then, the set-valued mapping $M : X \rightarrow 2^X$ is said to be a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive with respect to the mappings P, Q , and R , if

- (i) M is m - η -relaxed monotone;
(ii) $(H(P, Q, R) + \lambda M)(X) = X$, for all $\lambda > 0$.

Example 8. Let X, H, P, Q, R , and η be the same as in Example 3, and let $M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $M(x) = (-nx_1, -mx_2)$, for all $x = (x_1, x_2) \in \mathbb{R}^2$.

We claim that M is n^2 - η -relaxed monotone mapping. Indeed, for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} &\langle Mx - My, \eta(x, y) \rangle \\ &= \langle ((-nx_1 - mx_2) - (-ny_1 - my_2)), \\ &\quad (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\ &= \langle (-n(x_1 - y_1), -m(x_2 - y_2)), (m(x_1 - y_1), n(x_2 - y_2)) \rangle \\ &= -mn \{ (x_1 - y_1)^2 + (x_2 - y_2)^2 \} \\ &\geq -n^2 \{ (x_1 - y_1)^2 + (x_2 - y_2)^2 \} \\ &= -n^2 \|x - y\|^2, \\ &\langle Mx - My, \eta(x, y) \rangle \geq (-n^2) \|x - y\|^2. \end{aligned} \quad (24)$$

Furthermore, M is also a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator since $(H(P, Q, R) + \lambda M)(\mathbb{R}^2) = \mathbb{R}^2$ for any $\lambda > 0$.

Remark 9. If $H(P, Q, R) = H(P, Q)$, P is α -strongly monotone, and Q is β -relaxed monotone, then generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator reduces to $H(\cdot, \cdot)$ - η -monotone operator introduced and studied by Xu and Wang [15].

Proposition 10. Let set-valued mapping $M : X \rightarrow 2^X$ be a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator with respect to the mappings P, Q , and R . If P is α -expansive, Q is β -Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^2 - \gamma\beta^2 + \delta > m$, then the following inequality

$$\langle u - v, \eta(x, y) \rangle \geq 0 \quad (25)$$

holds for all $(y, v) \in \text{Graph}(M)$ and implies $u \in Mx$, where

$$\text{Graph}(M) = \{(a, b) \in X \times X : b \in M(a)\}. \quad (26)$$

Proof. Suppose on the contrary that there exists $(x_0, u_0) \notin \text{Graph } M$ such that

$$\langle u_0 - v, \eta(x_0, y) \rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(M). \quad (27)$$

Since M is a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive, we know that $(H(P, Q, R) + \lambda M)(X) = X$ holds for all $\lambda > 0$, and so there exists $(x_1, u_1) \in \text{Graph}(M)$ such that

$$H(Px_0, Qx_0, Rx_0) + \lambda u_0 = H(Px_1, Qx_1, Rx_1) + \lambda u_1 \in X. \quad (28)$$

Now, $\lambda u_0 - \lambda u_1 = H(Px_1, Qx_1, Rx_1) - H(Px_0, Qx_0, Rx_0) \in X$

$$\begin{aligned} &\langle \lambda u_0 - \lambda u_1, \eta(x_0, x_1) \rangle \\ &= -\langle H(Px_0, Qx_0, Rx_0) - H(Px_1, Qx_1, Rx_1), \eta(x_0, x_1) \rangle. \end{aligned} \quad (29)$$

Setting $(y, v) = (x_1, u_1)$ in (27), and then from the resultant, (28) and m - η -relaxed monotonicity of M , we obtain

$$\begin{aligned} &-m \|x_0 - x_1\|^2 \\ &\leq \lambda \langle u_0 - u_1, \eta(x_0, x_1) \rangle \\ &= -\langle H(Px_0, Qx_0, Rx_0) - H(Px_1, Qx_1, Rx_1), \eta(x_0, x_1) \rangle \\ &= -\langle H(Px_0, Qx_0, Rx_0) - H(Px_1, Qx_0, Rx_0), \eta(x_0, x_1) \rangle \\ &\quad - \langle H(Px_1, Qx_0, Rx_0) - H(Px_1, Qx_1, Rx_0), \eta(x_0, x_1) \rangle \\ &\quad - \langle H(Px_1, Qx_1, Rx_0) - H(Px_1, Qx_1, Rx_1), \eta(x_0, x_1) \rangle. \end{aligned} \quad (30)$$

Since $H(P, Q, R)$ is μ - η -cocoercive with respect to P , γ - η -relaxed cocoercive with respect to Q , δ - η -strongly monotone

with respect to R , and P is α -expansive, Q is β -Lipschitz continuous, thus (30) becomes

$$\begin{aligned}
 & -m\|x_0 - x_1\|^2 \\
 & \leq -\mu\|Px_0 - Px_1\|^2 + \gamma\|Qx_0 - Qx_1\|^2 - \delta\|x_0 - x_1\|^2 \\
 & \leq -(\mu\alpha^2 - \gamma\beta^2 + \delta)\|x_0 - x_1\|^2 \\
 & = -r\|x_0 - x_1\|^2 \leq 0, \\
 & \leq -(r - m)\|x_0 - x_1\|^2 \leq 0, \quad \text{where } r = \mu\alpha^2 - \gamma\beta^2 + \delta,
 \end{aligned} \tag{31}$$

which gives $x_0 = x_1$ since $r > m$. By (27), we have $u_0 = u_1$, a contradiction. This completes the proof. $\square$

Theorem 11. Let set-valued mapping $M : X \rightarrow 2^X$ be a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator with respect to the mappings P, Q , and R . If P is α -expansive, Q is β -Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^2 - \gamma\beta^2 + \delta > \lambda m$, then $(H(P, Q, R) + \lambda M)^{-1}$ is single-valued.

Proof. For any given $x \in X$, let $u, v \in (H(P, Q, R) + \lambda M)^{-1}(x)$. It follows that

$$\begin{aligned}
 & -H(Pu, Qu, Ru) + x \in \lambda Mu, \\
 & -H(Pv, Qv, Rv) + x \in \lambda Mv.
 \end{aligned} \tag{32}$$

Since M is m - η -relaxed monotone, we have

$$\begin{aligned}
 & -m\|u - v\|^2 \\
 & \leq \frac{1}{\lambda} \langle -H(Pu, Qu, Ru) \\
 & \quad + x - (-H(Pv, Qv, Rv) + x), \eta(u, v) \rangle, \\
 & -\lambda m\|u - v\|^2 \\
 & \leq \langle -H(Pu, Qu, Ru) + x - (-H(Pv, Qv, Rv) + x), \eta(u, v) \rangle \\
 & = -\langle H(Pu, Qu, Ru) - H(Pv, Qv, Rv), \eta(u, v) \rangle \\
 & = -\langle H(Pu, Qu, Ru) - H(Pv, Qu, Ru), \eta(u, v) \rangle \\
 & \quad - \langle H(Pv, Qu, Ru) - H(Pv, Qv, Ru), \eta(u, v) \rangle \\
 & \quad - \langle H(Pv, Qv, Ru) - H(Pv, Qv, Rv), \eta(u, v) \rangle.
 \end{aligned} \tag{33}$$

Since $H(P, Q, R)$ is μ - η -cocoercive with respect to P , γ - η -relaxed cocoercive with respect to Q , δ - η -strongly monotone

with respect to R , and P is α -expansive, Q is β -Lipschitz continuous, thus (33) becomes

$$\begin{aligned}
 & -\lambda m\|u - v\|^2 \\
 & \leq -\mu\|Pu - Pv\|^2 + \gamma\|Qu - Qv\|^2 - \delta\|u - v\|^2 \\
 & \leq -(\mu\alpha^2 - \gamma\beta^2 + \delta)\|u - v\|^2 \\
 & = -r\|u - v\|^2 \leq 0, \\
 & \leq -(r - \lambda m)\|u - v\|^2 \leq 0, \quad \text{where } r = \mu\alpha^2 - \gamma\beta^2 + \delta,
 \end{aligned} \tag{34}$$

since $r > \lambda m$. Hence, it follows that $\|u - v\| \leq 0$. This implies that $u = v$, and so $(H(P, Q, R) + \lambda M)^{-1}$ is single-valued. $\square$

Definition 12. Let set-valued mapping $M : X \rightarrow 2^X$ be a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator with respect to the mappings P, Q , and R . If P is α -expansive, Q is β -Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ and $r = \mu\alpha^2 - \gamma\beta^2 + \delta > \lambda m$, then the resolvent operator $R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} : X \rightarrow X$ is defined by

$$R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u) = (H(P, Q, R) + \lambda M)^{-1}(u), \quad \forall u \in X. \tag{35}$$

Now, we prove that the resolvent operator defined by (35) is Lipschitz continuous.

Theorem 13. Let set-valued mapping $M : X \rightarrow 2^X$ be a generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator with respect to the mappings P, Q , and R . If P is α -expansive, Q is β -Lipschitz continuous, η is τ -Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^2 - \gamma\beta^2 + \delta > \lambda m$, then the resolvent operator $R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} : X \rightarrow X$ is $(\tau/(r - \lambda m))$ -Lipschitz continuous, that is,

$$\|R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u) - R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)\| \leq \frac{\tau}{r - \lambda m} \|u - v\|, \quad \forall u, v \in X. \tag{36}$$

Proof. Let $u, v \in X$ be any given points. It follows from (35) that

$$\begin{aligned}
 & R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u) = (H(P, Q, R) + \lambda M)^{-1}(u), \\
 & R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v) = (H(P, Q, R) + \lambda M)^{-1}(v). \\
 & \frac{1}{\lambda} (u - H(P(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u)), Q(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u)), \\
 & \quad R(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u)))) \in M(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u)), \\
 & \frac{1}{\lambda} (v - H(P(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)), Q(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)), \\
 & \quad R(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)))) \in M(R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)).
 \end{aligned} \tag{37}$$

Let $z_1 = R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u)$ and $z_2 = R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)$.

Since M is m - η -relaxed monotone, we have

$$\begin{aligned} & \left\langle \frac{1}{\lambda} (u - H(P(z_1), Q(z_1), R(z_1))) \right. \\ & \quad \left. - (v - H(P(z_2), Q(z_2), R(z_2))), \eta(z_1, z_2) \right\rangle \\ & \geq -m \|z_1 - z_2\|^2, \\ & \left\langle \frac{1}{\lambda} (u - v - H(P(z_1), Q(z_1), R(z_1))) \right. \\ & \quad \left. + H(P(z_2), Q(z_2), R(z_2)), \eta(z_1, z_2) \right\rangle \end{aligned} \quad (38)$$

$$\geq -m \|z_1 - z_2\|^2,$$

which implies

$$\begin{aligned} & \langle u - v, \eta(z_1, z_2) \rangle \\ & \geq \langle H(P(z_1), Q(z_1), R(z_1)) - H(P(z_2), Q(z_2), R(z_2)), \\ & \quad \eta(z_1, z_2) \rangle - \lambda m \|z_1 - z_2\|^2. \end{aligned} \quad (39)$$

Further, we have

$$\begin{aligned} & \tau \|u - v\| \|z_1 - z_2\| \\ & \geq \|u - v\| \|\eta(z_1, z_2)\| \\ & \geq \langle u - v, \eta(z_1, z_2) \rangle \\ & \geq \langle H(P(z_1), Q(z_1), R(z_1)) \\ & \quad - H(P(z_2), Q(z_2), R(z_2)), \\ & \quad \eta(z_1, z_2) \rangle - \lambda m \|z_1 - z_2\|^2 \\ & = \langle H(P(z_1), Q(z_1), R(z_1)) \\ & \quad - H(P(z_2), Q(z_1), R(z_1)), \eta(z_1, z_2) \rangle \\ & \quad + \langle H(P(z_2), Q(z_1), R(z_1)) \\ & \quad - H(P(z_2), Q(z_2), R(z_1)), \eta(z_1, z_2) \rangle \\ & \quad + \langle H(P(z_2), Q(z_2), R(z_1)) \\ & \quad - H(P(z_2), Q(z_2), R(z_2)), \\ & \quad \eta(z_1, z_2) \rangle - \lambda m \|z_1 - z_2\|^2 \\ & \geq \mu \|P(z_1) - P(z_2)\|^2 - \gamma \|Q(z_1) - Q(z_2)\|^2 \\ & \quad + \delta \|R(z_1) - R(z_2)\|^2 - \lambda m \|z_1 - z_2\|^2 \\ & = (\mu\alpha^2 - \gamma\beta^2 + \delta) \|z_1 - z_2\|^2 - \lambda m \|z_1 - z_2\|^2 \\ & = (r - \lambda m) \|z_1 - z_2\|^2, \quad \text{where } r = \mu\alpha^2 - \gamma\beta^2 + \delta, \end{aligned} \quad (40)$$

and hence,

$$\tau \|u - v\| \|z_1 - z_2\| \geq (r - \lambda m) \|z_1 - z_2\|^2, \quad (41)$$

that is,

$$\|R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u) - R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(v)\| \leq \frac{\tau}{r - \lambda m} \|u - v\|, \quad \forall u, v \in X. \quad (42)$$

This completes the proof. $\square$

4. An Application of $H(\cdot, \cdot, \cdot)$ - η -Cocoercive Operators for Solving Generalized Variational Inclusions

In this section, we will show that under suitable assumptions, the generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive operator can also play important roles for solving the variational inclusion problem in Hilbert spaces.

Let $N : X \times X \rightarrow X$, $\eta : X \times X \rightarrow X$, $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, and $p : X \rightarrow X$ be the single-valued mappings, and $S, T : X \rightarrow \text{CB}(X)$, $M : X \rightarrow 2^X$ be the set-valued mappings such that M is generalized $H(\cdot, \cdot, \cdot)$ - η -cocoercive with respect to P, Q , and R and $\text{range}(p) \cap \text{dom } M \neq \emptyset$. Then we consider the problem to find $u \in X$, $w \in S(u)$, $v \in T(u)$ such that

$$0 \in N(w, v) + M(p(u)). \quad (43)$$

The problem (43) is called generalized set-valued variational-like inclusion problem. The problem of type (43) was introduced and studied by Chidume et al. [3] by applying η -proximal mapping. If $T = 0$ and $\eta(u, v) = u - v$, for all $u, v \in X$, and $N(\cdot, \cdot) = S(\cdot)$, where $S : X \rightarrow \text{CB}(X)$ is a set-valued mapping. Then, problem (43) reduces to the problem of finding $u \in X$, $w \in S(u)$ such that

$$0 \in w + M(p(u)). \quad (44)$$

The problem of type (44) was studied by Ahmad et al. [2] by applying $H(\cdot, \cdot)$ -cocoercive operators.

If $S, T = 0$, $N(\cdot, \cdot) = 0$, and $\eta(u, v) = u - v$, for all $u, v \in X$, then problem (43) reduces to the problem of finding $u \in X$ such that

$$0 \in M(p(u)). \quad (45)$$

The problem of type (45) was studied by Verma [14] in the setting of Banach spaces when M is A -maximal-relaxed accretive.

Lemma 14. *The (u, w, v) , where $u \in X$, $w \in S(u)$, $v \in T(u)$ is a solution of the problem (43), if and only if (u, w, v) satisfies the following relation:*

$$\begin{aligned} p(u) = & R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} [H(P(p(u)), Q(p(u)), R(p(u))) \\ & - \lambda N(w, v)], \end{aligned} \quad (46)$$

where $R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}(u) = (H(P, Q, R) + \lambda M)^{-1}(u)$ and $\lambda > 0$ is a constant.

Proof. By using the definitions of resolvent operators $R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}$, the conclusion follows directly. $\square$

Using Lemma 14 and using the technique of Chidume et al. [3] and Nadler [20], we develop an iterative algorithm for finding the approximate solution of problem (43) as follows.

Algorithm 15. Let $N : X \times X \rightarrow X$, $\eta : X \times X \rightarrow X$, $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, $p : X \rightarrow X$, and $S, T : X \rightarrow CB(X)$ be such that, for each $u \in X$, $F(u) \subseteq p(X)$, where $F : X \rightarrow 2^X$ is the set-valued mappings defined by

$$F(u) = \bigcup_{w \in S(u)} \bigcup_{v \in T(u)} \left(R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} (H(P(p(u)), Q(p(u)), R(p(u))) - \lambda N(w, v)) \right), \quad (47)$$

where $M : X \rightarrow 2^X$ is a set-valued mapping such that, M is generalized $H(\cdot, \cdot, \cdot) - \eta$ -cocoercive with respect to the mappings P, Q , and R .

For given $u_0 \in X$, take $w_0 \in S(u_0)$ and $v_0 \in T(u_0)$. Let

$$x_0 = R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} (H(P(p(u_0)), Q(p(u_0)), R(p(u_0))) - \lambda N(w_0, v_0)) \in F(u_0) \subseteq p(X). \quad (48)$$

Hence, there exists $u_1 \in X$ such that $x_0 = p(u_1)$. Since $w_0 \in S(u_0) \in CB(X)$ and $v_0 \in T(u_0) \in CB(X)$, then, by Nadler's result [20], there exist $w_1 \in S(u_1)$ and $v_1 \in T(u_1)$ such that

$$\begin{aligned} \|w_0 - w_1\| &\leq (1 + 1^{-1}) \mathcal{D}(S(u_0), S(u_1)), \\ \|v_0 - v_1\| &\leq (1 + 1^{-1}) \mathcal{D}(T(u_0), T(u_1)). \end{aligned} \quad (49)$$

Let

$$x_1 = R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} (H(P(p(u_1)), Q(p(u_1)), R(p(u_1))) - \lambda N(w_1, v_1)) \in F(u_1) \subseteq p(X). \quad (50)$$

Hence, there exists $u_2 \in X$ such that $x_1 = p(u_2)$. By induction, we can define iterative sequences $\{u_n\}$, $\{p(u_n)\}$, $\{w_n\}$, and $\{v_n\}$ as follows:

$$p(u_{n+1}) = R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} [H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) - \lambda N(w_n, v_n)], \quad (51)$$

$$\begin{aligned} w_n \in S(u_n), \quad \|w_n - w_{n+1}\| \\ \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_n), S(u_{n+1})), \end{aligned} \quad (52)$$

$$\begin{aligned} v_n \in T(u_n), \quad \|v_n - v_{n+1}\| \\ \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(u_n), T(u_{n+1})), \end{aligned} \quad (53)$$

for all $n = 0, 1, 2, \dots$, and $\lambda > 0$ is a constant.

If $N(w, v) = w$, for all $w, v \in X$, then Algorithm 15 reduces to the following algorithm for solving the problem (44).

Algorithm 16. For any $u_0 \in X$, and $w_0 \in S(u_0)$, compute the sequence $\{u_n\}$ and $\{w_n\}$ by the following:

$$\begin{aligned} p(u_n) \\ = R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} [H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) - \lambda w_n], \\ w_n \in S(u_n), \quad \|w_n - w_{n+1}\| \\ \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_n), S(u_{n+1})), \end{aligned} \quad (54)$$

for all $n = 0, 1, 2, \dots$, and $\lambda > 0$ is a constant.

If $N(w, v) = 0$, for all $w, v \in X$, then Algorithm 15 reduces to the following algorithm for solving the problem (45).

Algorithm 17. For any $u_0 \in X$, compute the sequence $\{u_n\}$ by the following:

$$p(u_n) = R_{\lambda, M}^{H(\cdot, \cdot, \cdot) - \eta} [H(P(p(u_n)), Q(p(u_n)), R(p(u_n)))], \quad (55)$$

for all $n = 0, 1, 2, \dots$, and $\lambda > 0$ is a constant.

Now, we prove the existence of a solution of problem (43) and the convergence of Algorithm 15.

Theorem 18. Let X be a real Hilbert space. Let $\eta, N : X \times X \rightarrow X$, $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, and $p : X \rightarrow X$ be single-valued mappings, and $S, T : X \rightarrow CB(X)$ and $M : X \rightarrow 2^X$ be the set-valued mappings such that M is generalized $H(P, Q, R) - \eta$ -cocoercive operator with respect to the mappings P, Q , and R and range $(p) \cap \text{dom } M \neq \emptyset$, and, for each $u \in X$, let $F(u) \subseteq p(X)$, where F is defined by (47). Assume that

- (i) S, T are $\mathcal{D}$ -Lipschitz continuous with constants l_1 and l_2 , respectively;
- (ii) η is τ -Lipschitz continuous, P is α -expansive, and Q is β -Lipschitz continuous;
- (iii) p is λ_p -Lipschitz continuous and ξ -strongly monotone;
- (iv) $H(P, Q, R)$ is r_1 -Lipschitz continuous with respect to P , r_2 -Lipschitz continuous with respect to Q , and r_3 -Lipschitz continuous with respect to R ;
- (v) N is ϵ_1 -Lipschitz continuous in the first argument with respect to S and ϵ_2 -Lipschitz continuous in the second argument with respect to T .

In addition,

$$(r_1 + r_2 + r_3) \lambda_p \tau + \lambda (\epsilon_1 l_1 + \epsilon_2 l_2) < \xi (r - \lambda m), \quad (56)$$

where $r = \mu \alpha^2 - \gamma \beta^2 + \delta > \lambda m$ and $\mu > \gamma$, $\alpha > \beta$, $\delta > 0$.

Then, generalized set-valued variational inclusion problem (43) has a solution $(u, w, v) \in X$, and the iterative sequences $\{u_n\}$, $\{p(u_n)\}$, $\{w_n\}$, and $\{v_n\}$ generated by Algorithm 15 converge strongly to u , $p(u)$, w , and v , respectively.

Proof. Since S, T are $\mathcal{D}$ -Lipschitz continuous with constants l_1 and l_2 , respectively, it follows from (52) and (53) that

$$\begin{aligned} \|w_{n+1} - w_n\| &\leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(S(u_{n+1}), S(u_n)) \\ &\leq \left(1 + \frac{1}{n+1}\right) l_1 \|u_{n+1} - u_n\|, \\ \|v_{n+1} - v_n\| &\leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(u_{n+1}), T(u_n)) \\ &\leq \left(1 + \frac{1}{n+1}\right) l_2 \|u_{n+1} - u_n\|, \end{aligned} \quad (57)$$

for $n = 0, 1, 2, \dots$

It follows from (51) and Theorem 13 that

$$\begin{aligned} &\|p(u_{n+1}) - p(u_n)\| \\ &= \|R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} \{H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) \\ &\quad - \lambda N(w_n, v_n)\} \\ &\quad - [R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} \{H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1}))) \\ &\quad - \lambda N(w_{n-1}, v_{n-1})\}]\| \\ &+ \frac{\tau}{r - \lambda m} \|H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) \\ &\quad - H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1})))\| \\ &+ \frac{\tau \lambda}{r - \lambda m} \|N(w_n, v_n) - N(w_{n-1}, v_{n-1})\|. \end{aligned} \quad (58)$$

Since $H(P, Q, R)$ is r_1 -Lipschitz continuous with respect to P , r_2 -Lipschitz continuous with respect to Q , and r_3 -Lipschitz continuous with respect to R , and p is λ_p -Lipschitz continuous, we have

$$\begin{aligned} &\|H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) \\ &\quad - H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1})))\| \\ &= \|H(P(p(u_n)), Q(p(u_n)), R(p(u_n))) \\ &\quad - H(P(p(u_{n-1})), Q(p(u_n)), R(p(u_n)))\| \\ &+ \|H(P(p(u_{n-1})), Q(p(u_n)), R(p(u_n))) \\ &\quad - H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1})))\| \\ &+ \|H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1}))) \\ &\quad - H(P(p(u_{n-1})), Q(p(u_{n-1})), R(p(u_{n-1})))\| \\ &\leq (r_1 + r_2 + r_3) \lambda_p \|u_n - u_{n-1}\|. \end{aligned} \quad (59)$$

Since N is ϵ_1 -Lipschitz continuous in the first argument with respect to S and ϵ_2 -Lipschitz continuous in the second argument with respect to T , and S, T are $\mathcal{D}$ -Lipschitz continuous with constants l_1 and l_2 , respectively, we have

$$\begin{aligned} &\|N(w_n, v_n) - N(w_{n-1}, v_{n-1})\| \\ &\leq \|N(w_n, v_n) - N(w_{n-1}, v_n)\| \\ &\quad + \|N(w_{n-1}, v_n) - N(w_{n-1}, v_{n-1})\| \\ &\leq \epsilon_1 \|w_n - w_{n-1}\| + \epsilon_2 \|v_n - v_{n-1}\| \\ &\leq \epsilon_1 \left(1 + \frac{1}{n}\right) \mathcal{D}(S(u_n), S(u_{n-1})) \\ &\quad + \epsilon_2 \left(1 + \frac{1}{n}\right) \mathcal{D}(T(u_n), T(u_{n-1})) \\ &\leq \epsilon_1 \left(1 + \frac{1}{n}\right) l_1 \|u_n - u_{n-1}\| \\ &\quad + \epsilon_2 \left(1 + \frac{1}{n}\right) l_2 \|u_n - u_{n-1}\| \\ &= (\epsilon_1 l_1 + \epsilon_2 l_2) \left(1 + \frac{1}{n}\right) \|u_n - u_{n-1}\|. \end{aligned} \quad (60)$$

Using (59), (60) in (58), we have

$$\begin{aligned} &\|p(u_{n+1}) - p(u_n)\| \\ &\leq \left\{ \frac{(r_1 + r_2 + r_3) \lambda_p \tau + \lambda (\epsilon_1 l_1 + \epsilon_2 l_2) (1 + (1/n))}{r - \lambda m} \right\} \\ &\quad \times \|u_n - u_{n-1}\|. \end{aligned} \quad (61)$$

Using the ξ -strong monotonicity of p , we have

$$\begin{aligned} &\|p(u_{n+1}) - p(u_n)\| \|u_{n+1} - u_n\| \\ &\geq \langle p(u_{n+1}) - p(u_n), u_{n+1} - u_n \rangle \\ &\geq \xi \|u_{n+1} - u_n\|, \end{aligned} \quad (62)$$

which implies that

$$\|u_{n+1} - u_n\| \leq \frac{1}{\xi} \|p(u_{n+1}) - p(u_n)\|. \quad (63)$$

Combining (61) and (63), we have

$$\|u_{n+1} - u_n\| \leq \theta_n \|u_n - u_{n-1}\|, \quad (64)$$

where

$$\theta_n = \frac{(r_1 + r_2 + r_3) \lambda_p \tau + \lambda (\epsilon_1 l_1 + \epsilon_2 l_2) (1 + (1/n))}{\xi (r - \lambda m)}. \quad (65)$$

Let

$$\theta = \frac{(r_1 + r_2 + r_3) \lambda_p \tau + \lambda (\epsilon_1 l_1 + \epsilon_2 l_2)}{\xi (r - \lambda m)}. \quad (66)$$

From (56), it is easy to see that $\theta < 1$. Therefore, (64) implies that $\{u_n\}$ is a Cauchy sequence in X . Since X is a Hilbert space, there exists $u \in X$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$. From (57), $\{w_n\}$ and $\{v_n\}$ are also Cauchy sequences in X ; thus, there exist $w, v \in X$ such that $w_n \rightarrow w$ and $v_n \rightarrow v$ as $n \rightarrow \infty$. By the continuity of $p, R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta}, H, \eta, N, P, Q, R$, and (51) of Algorithm 15, we have

$$\begin{aligned} p(u) &= R_{\lambda, M}^{H(\cdot, \cdot, \cdot)-\eta} [H(P(p(u)), Q(p(u)), R(p(u))) - \lambda N(w, v)]. \end{aligned} \quad (67)$$

Now, we prove that $w \in S(u)$. In fact, since $w_n \in S(u_n)$, we have

$$\begin{aligned} d(w, S(u)) &\leq \|w - w_n\| + d(w_n, S(u)) \\ &\leq \|w - w_n\| + \mathcal{D}(S(u_n), S(u)) \\ &\leq \|w - w_n\| + l_1 \|u_n - u\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (68)$$

which implies that $d(w, S(u)) = 0$. Since $S(u) \in \text{CB}(X)$, it follows that $w \in S(u)$. Similarly, it is easy to see that $v \in T(u)$. By Lemma 14, (u, w, v) is the solution of problem (43). This completes the proof. $\square$

Based on Lemma 14 and Algorithm 16, Theorem 18 reduced to the following result for solving problem (44).

Theorem 19. *Let X be a real Hilbert space. Let $\eta : X \times X \rightarrow X$, $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, and $p : X \rightarrow X$ be single-valued mappings, and let $S : X \rightarrow \text{CB}(X)$ and $M : X \rightarrow 2^X$ be the set-valued mappings such that M is generalized $H(P, Q, R)$ - η -cocoercive operator with respect to the mappings P, Q , and R and range $(p) \cap \text{dom } M \neq \emptyset$, and, for each $u \in X$, let $F(u) \subseteq p(X)$, where F is defined by (47). Assume that*

- (i) S is $\mathcal{D}$ -Lipschitz continuous with constants l ;
- (ii) η is τ -Lipschitz continuous, P is α -expansive, and Q is β -Lipschitz continuous;
- (iii) p is λ_p -Lipschitz continuous and ξ -strongly monotone;
- (iv) $H(P, Q, R)$ is r_1 -Lipschitz continuous with respect to P , r_2 -Lipschitz continuous with respect to Q , and r_3 -Lipschitz continuous with respect to R .

In addition,

$$(r_1 + r_2 + r_3) \lambda_p \tau + \lambda l < \xi(r - \lambda m), \quad (69)$$

where $r = \mu\alpha^2 - \gamma\beta^2 + \delta > \lambda m$ and $\mu > \gamma, \alpha > \beta, \delta > 0$.

Then, generalized set-valued variational inclusion problem (44) has a solution $(u, w) \in X$, and the iterative sequences $\{u_n\}$, $\{p(u_n)\}$, and $\{w_n\}$ generated by Algorithm 16 converge strongly to u , $p(u)$ and w , respectively.

Based on Lemma 14 and Algorithm 17, Theorem 18 reduced to the following result for solving problem (45).

Theorem 20. *Let X be a real Hilbert space. Let $\eta : X \times X \rightarrow X$, $H : X \times X \times X \rightarrow X$, $P, Q, R : X \rightarrow X$, and $p : X \rightarrow X$ be single-valued mappings, and let $M : X \rightarrow 2^X$ be the set-valued mappings such that M is generalized $H(P, Q, R)$ - η -cocoercive operator with respect to the mappings P, Q , and R and range $(p) \cap \text{dom } M \neq \emptyset$, and, for each $u \in X$, let $F(u) \subseteq p(X)$, where F is defined by (47). Assume that*

- (i) η is τ -Lipschitz continuous, P is α -expansive, and Q is β -Lipschitz continuous;
- (ii) p is λ_p -Lipschitz continuous and ξ -strongly monotone;
- (iii) $H(P, Q, R)$ is r_1 -Lipschitz continuous with respect to P , r_2 -Lipschitz continuous with respect to Q , and r_3 -Lipschitz continuous with respect to R .

In addition,

$$(r_1 + r_2 + r_3) \lambda_p \tau < \xi(r - \lambda m), \quad (70)$$

where $r = \mu\alpha^2 - \gamma\beta^2 + \delta > \lambda m$ and $\mu > \gamma, \alpha > \beta, \delta > 0$.

Then, generalized set-valued variational inclusion problem (45) has a solution $u \in X$, and the iterative sequence $\{u_n\}$ and $\{p(u_n)\}$ generated by Algorithm 17 converge strongly to u and $p(u)$, respectively.

Acknowledgments

The authors are grateful to the editor and referees for valuable comments and suggestions.

References

- [1] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, Germany, 1984.
- [2] R. Ahmad, M. Dilshad, M.-M. Wong, and J.-C. Yao, " $H(\cdot, \cdot)$ -cocoercive operator and an application for solving generalized variational inclusions," *Abstract and Applied Analysis*, vol. 2011, Article ID 261534, 12 pages, 2011.
- [3] C. E. Chidume, K. R. Kazmi, and H. Zegeye, "Iterative approximation of a solution of a general variational-like inclusion in Banach spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 21–24, pp. 1159–1168, 2004.
- [4] X. P. Ding and J.-C. Yao, "Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces," *Computers & Mathematics with Applications*, vol. 49, no. 5–6, pp. 857–869, 2005.
- [5] Y.-P. Fang and N.-J. Huang, " H -monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2–3, pp. 795–803, 2003.
- [6] Y.-P. Fang and N.-J. Huang, " H -accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.
- [7] Y.-P. Fang, N.-J. Huang, and H. B. Thompson, "A new system of variational inclusions with (H, η) -monotone operators in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 49, no. 2–3, pp. 365–374, 2005.
- [8] K. R. Kazmi, N. Ahmad, and M. Shahzad, "Convergence and stability of an iterative algorithm for a system of generalized

- implicit variational-like inclusions in Banach spaces,” *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9208–9219, 2012.
- [9] K. R. Kazmi, M. I. Bhat, and N. Ahmad, “An iterative algorithm based on M -proximal mappings for a system of generalized implicit variational inclusions in Banach spaces,” *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 361–371, 2009.
- [10] K. R. Kazmi, F. A. Khan, and M. Shahzad, “A system of generalized variational inclusions involving generalized $H(\cdot, \cdot)$ -accretive mapping in real q -uniformly smooth Banach spaces,” *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9679–9688, 2011.
- [11] K. R. Kazmi and F. A. Khan, “Iterative approximation of a solution of multi-valued variational-like inclusion in Banach spaces: a P - η -proximal-point mapping approach,” *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 665–674, 2007.
- [12] K. R. Kazmi and F. A. Khan, “Sensitivity analysis for parametric generalized implicit quasi-variational-like inclusions involving P - η -accretive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1198–1210, 2008.
- [13] H.-Y. Lan, Y. J. Cho, and R. U. Verma, “Nonlinear relaxed cocoercive variational inclusions involving (A, η) -accretive mappings in Banach spaces,” *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1529–1538, 2006.
- [14] R. U. Verma, “The generalized relaxed proximal point algorithm involving A -maximal-relaxed accretive mappings with applications to Banach spaces,” *Mathematical and Computer Modelling*, vol. 50, no. 7-8, pp. 1026–1032, 2009.
- [15] Z. Xu and Z. Wang, “A generalized mixed variational inclusion involving $(H(\cdot, \cdot), \eta)$ -monotone operators in Banach spaces,” *Journal of Mathematics Research*, vol. 2, no. 3, pp. 47–56, 2010.
- [16] Y.-Z. Zou and N.-J. Huang, “ $H(\cdot, \cdot)$ -accretive operator with an application for solving variational inclusions in Banach spaces,” *Applied Mathematics and Computation*, vol. 204, no. 2, pp. 809–816, 2008.
- [17] Q. H. Ansari and J. C. Yao, “Iterative schemes for solving mixed variational-like inequalities,” *Journal of Optimization Theory and Applications*, vol. 108, no. 3, pp. 527–541, 2001.
- [18] S. Karamardian, “The nonlinear complementarity problem with applications. I, II,” *Journal of Optimization Theory and Applications*, vol. 4, pp. 167–181, 1969.
- [19] P. Tseng, “Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming,” *Mathematical Programming*, vol. 48, no. 2, pp. 249–263, 1990.
- [20] S. B. Nadler, “Multivalued contraction mapping,” *Pacific Journal of Mathematics*, vol. 30, no. 3, pp. 457–488, 1969.