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Research Article

Dynamical Analysis of a Modified Lorenz System

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This paper presents another new modified Lorenz system which is chaotic in a certain range of parameters. Besides that, this paper also presents explanations to solve the new modified Lorenz system. Furthermore, some of the dynamical properties of the system are shown and stated. Basically, this paper shows the finding that led to the discovery of fixed points for the system, dynamical analysis using complementary-cluster energy-barrier criterion (CCEBC), finding the Jacobian matrix, finding eigenvalues for stability, finding the Lyapunov functions, and finding the Lyapunov exponents to investigate some of the dynamical behaviours of the system. Pictures and diagrams will be shown for the chaotic systems using the aide of MAPLE in 2D and 3D views. Nevertheless, this paper is to introduce the new modified Lorenz system.

1. Introduction

In real life, the dynamical system is well known for its various uses such as in a population growth model. We can study the changes of population through time and can make a long-term prediction about the growth in population. Besides that, Devaney describes dynamical systems as "the branch of mathematics that attempts to describe processes in motion" [1]. In addition, Lorenz introduced the chaotic dynamical system in the early 1960s when he was doing a weather forecast. He realized that small changes or differences in the initial points of the weather system could alter the outcome with surprising results. He then named it the "butterfly effect" as the system is dependent on its initial conditions [2].

In simple words, chaos is a general term used to represent the chaotic dynamical system. A lot of research has been done and the system has been widely investigated in all kinds of characteristics. For example, Edward Lorenz has published reports of a "strange attractor" where he discovered this attractor as a result of using computers to find the approximate numerical solutions to a system of differential equations in the weather model [3–5].

A modern view of chaos theory by [6] in their book called *Chaos Theory* mentions that "A chaotic system is defined as one that shows sensitivity to initial condition. That is, any uncertainty in the initial state of the given system, no matter how small, will lead to rapidly growing errors in any effort to predict the future behaviour. In other words, the system is chaotic. Its behaviour can be predicted only if the initial conditions are known to an infinite degree of accuracy, which is impossible."

Li et al. [7] describes that chaos may well be considered together with relativity and quantum mechanics as one of the three monumental discoveries of the twentieth century. Over the past four decades chaos has matured as a science (i.e., still evolving) that has given us deep insights into previously intractable and inherently nonlinear natural phenomena. The term chaos associated with an interval map was first formally introduced into mathematics by Li and Yorke themselves in 1975, where they established a simple criterion for chaos in one-dimensional difference equations, the well-known "period three implies chaos."

Some of the researchers who have modified the Lorenz system have stumbled upon or sought to discover the applications in real life. Some of the previous work by Zhou et al. [8], Qi et al. [9], and Yan [10] have presented the modified Lorenz system and has discussed the system in terms of stability and dynamical behavior. Tigan [11] shows a promising modified Lorenz system that has the potential application in secure communications.

Besides this, there are other chaotic models such as the Rösler system, the four-wing hyperchaotic attractor, and transient chaos generated from a new 4-D quadratic autonomous system by [12]. Lü et al. [13, 14] have managed to find one of the modified chaotic dynamical systems from the Lorenz system itself. The new chaotic attractor is a simple three-dimensional autonomous system, which connects the Lorenz attractor and Chen's attractor and represents the transition from one to the other.

Further research by [15] did state that the unified chaotic system contains the Lorenz and the Chen systems as two dual systems at the two extremes of its parameter spectrum. Plus, the new system represents the continued transition from the Lorenz to the Chen system and is chaotic over the entire spectrum of the key system parameter.

One of the latest researches was solving Zhou's chaotic system using Euler's method [16, 17]. It stated that it is one of the simplest approaches to obtain the numerical solution of a differential equation. Even though the method is not efficient compared to the Runge-Kutta method, it does provide a simpler way to analyse the system.

Sanjaya et al. have managed to establish on the basis of Hindermarsh-Rose neuron model and the relation of bidirectional coupling strength of the gap junction, of which the synchronization is discussed in detail, see [18]. The sufficient condition of complete synchronization is obtained from rigorous mathematical derivation. The synchronizations of periodic neurons and chaotic bursting neurons are studied also.

There are a lot of applications that chaos can be applied to. Furthermore, a chaotic dynamical system is not only important and useful but it may also introduce a new method or system that can help people understand mathematics.

This research will be mostly based on the Lorenz system [4, 5, 19, 20] where the system has been modified from the Lorenz system. The equations of Lorenz system is described as follows:

$$\begin{aligned} \dot{x} &= a\left(y - x\right), \\ \dot{y} &= cx - xz - y, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

where a, b, and c are real constant parameters and the system is chaotic when $a = 10$, $b = 8/3$, and $c = 28$.

2. The Modified Lorenz System

This paper presents another new modified Lorenz system which has chaotic behaviour. The new modified Lorenz system is as follows:

$$\begin{aligned} \dot{x} &= a\left(by - x\right), \\ \dot{y} &= cx - xz, \\ \dot{z} &= xy - bz, \end{aligned} \tag{2}$$

where x, y, and z are variables and a, b, and c are real parameters.

2.1. Fixed Points of the System. In order to obtain the fixed points of the system, we let $\dot{x} = \dot{y} = \dot{z} = 0$. Thus, we obtain

$$\begin{aligned} P_1 &= (0, 0, 0), \\ P_2 &= \left(b\sqrt{c}, \sqrt{c}, c\right), \\ P_3 &= \left(-b\sqrt{c}, -\sqrt{c}, c\right). \end{aligned} \tag{3}$$

Furthermore, we obtain a Jacobian matrix from the system (2) to find the eigenvalues.

$$\mathrm{DF} = \begin{bmatrix} -a & ab & 0 \\ c - z & 0 & -x \\ y & x & -b \end{bmatrix}. \tag{4}$$

When the fixed point is $P_1(0, 0, 0)$,

$$(\lambda + b)\left(\lambda^2 + a\lambda - abc\right) = 0. \tag{5}$$

As for the fixed points $P_2(b\sqrt{c}, \sqrt{c}, c)$ or $P_3(-b\sqrt{c}, -\sqrt{c}, c)$, we obtain

$$\lambda^3 + (a + b)\lambda^2 + \left(ab + b^2c\right)\lambda + 2ab^2c = 0. \tag{6}$$

Thus, by solving (5), we will also obtain the three eigenvalues that are

$$\lambda_1 = -b < 0, \tag{7}$$

$$\lambda_2 = \frac{-a - \sqrt{a^2 + 4abc}}{2} < 0, \tag{8}$$

$$\lambda_3 = \frac{-a + \sqrt{a^2 + 4abc}}{2} > 0. \tag{9}$$

It is obvious that the λ_1 (7) is always negative since $b > 0$ which indicates that the system is stable. As for λ_2 (8), its value is always negative if $a > 0$, $b > 0$, and $c > 0$. Thus, the fixed point is always stable. Last but not least, the value of λ_3 (9) is always positive. This means that the system is always unstable with the existence of λ_3.

2.2. Some Mathematical Properties. We utilise the Lyapunov function proposed in [19] to show that the fixed point is an attractor:

$$L\left(x, y, z\right) = \frac{x^2}{2a} + \frac{y^2}{2} + \frac{z^2}{2}. \tag{10}$$

So, we will differentiate the above Lyapunov function (10) with respect to t to obtain

$$\frac{dL}{dt} = \frac{x\dot{x}}{a} + y\dot{y} + z\dot{z}. \tag{11}$$

Next, we substitute (2) into (11) to obtain

$$\frac{dL}{dt} = -\left(x^2 + bz^2\right) + (b + c)\,xy. \tag{12}$$

It is easy to verify that the system (2) is stable if $(x^2 + bz^2) > (b+c)xy$. This is possible given that the z value is large enough compared to the value of x and y.

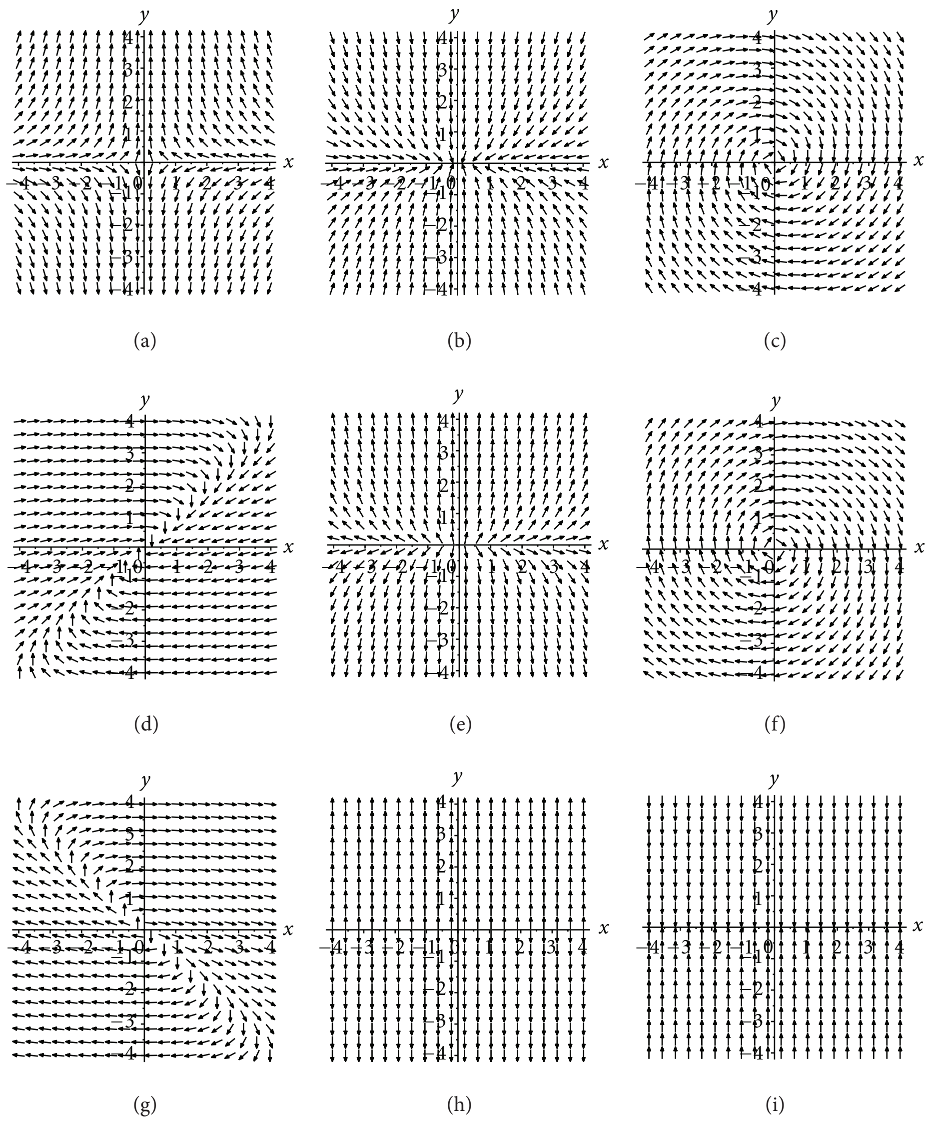

Figure 1: The possible phase portraits for system (16): (a) saddle point, (b) stable node point, (c) stable focus, (d) degenerate stable node, (e) unstable node point, (f) unstable focus, (g) degenerate unstable node, (h) $\lambda_2 > 0$, and (i) $\lambda_2 < 0$.

It is obvious that the origin point, x^*, is an attractor given that $L(x) > L(x^*)$ is satisfied for all x in U, where U is in the neighbourhood of x^*. Furthermore, it is also true for $\dot{L}(0) = 0$ for all x in U.

As we can see, the new modified Lorenz system has a symmetrical property. The equation in the system (2) does not change or is invariant when the following happens:

$$(x, y, z) \longrightarrow (-x, -y, z). \tag{13}$$

System (2) is affected by this symmetry. Hence, from the diagram (Figures 6 and 7), we can observe that the variable x and y are symmetrical with respect to the origin $(0, 0, 0)$. Furthermore, this hints that a pitchfork bifurcation for the fixed point and periodic solutions is possible [21].

Now, we will show that system (2) is dissipative. From the equations in (2), we will obtain

$$\operatorname{div} V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(a + b), \tag{14}$$

where V is volume. Since $\operatorname{div} V < 0$, it is obvious that system (2) is dissipative with an exponent contraction rate of

$$\frac{dV}{dt} = e^{-(a+b)t}. \tag{15}$$

That is, a volume element of V_0 is contracted by the flow into a volume element $V_0 e^{-(a+b)t}$ in time t. This also means that each volume which has the system trajectory will shrink to 0 when $t \rightarrow \infty$ at an exponent rate of $-(a + b)$. Hence, the orbits of the systems will be confined to a specific subset with zero volume and the asymptotic motion will settle onto the attractor [21].

Now, we have used Runge-Kutta fourth-order method to solve the new system (2) using MAPLE. In order to determine that the system (2) is chaotic, we have used Lyapunov exponent using MATLAB command [16]. As for plotting the system, we have used the aide of MAPLE instead.

2.3. Dynamical Analysis Using Complementary-Cluster Energy-Barrier Criterion (CCEBC). Now, we are going to determine the dynamical behaviours of the dynamical system (2) based on the complementary-cluster energy-barrier criterion (CCEBC) [22]. Let the first two differential equations from system (2) be

$$\begin{aligned} \dot{x} &= a(by - x), \\ \dot{y} &= -xz + cx, \end{aligned} \tag{16}$$

where $z = z(t)$ is a known function of the time variable, t. As for a, b, and c, they are constant parameters with the case of $t = t_0$. Hence, system (16) is a two-dimensional linear system with constant coefficients such that system (16) is simple and global.

By computing the Jacobian matrix, we obtained the following:

$$J = \begin{bmatrix} -a & ab \\ c - z & 0 \end{bmatrix}. \tag{17}$$

By linearising system (16) with the fixed point origin (0,0), it gives the following characteristic equation as below:

$$\lambda^2 + a\lambda + ab(z - c). \tag{18}$$

Hence, we analysed the above equation under certain circumstances.

(1) When $a > 0$, $b > 0$, and $z < c$, system (16) has two real eigenvalues such that $\lambda_1 < 0 < \lambda_2$. This implies that the fixed point $(0, 0)$ is a saddle point in the two-dimensional plane (Figure 1(a)).

(2) When $a > 0$, $b > 0$, and $c < z < c + a/4b$, the system (16) has two negative real eigenvalues such that $\lambda_1 < 0$, $\lambda_2 < 0$. This implies that the fixed point $(0, 0)$ is a stable node point in the two-dimensional plane (Figure 1(b)).

(3) When $a > 0$, $b > 0$, and $c + a/4b < z$, system (16) has two complex eigenvalues such that $\lambda_{1,2} = \alpha \pm \beta i$ with $\mathrm{Re}(\lambda)$ being negative. This implies that the fixed point $(0, 0)$ is a stable focus in the two-dimensional plane (Figure 1(c)).

(4) When $a > 0$, $b > 0$, and $z = c + a/4b$, system (16) has two repeated real eigenvalues such that $\lambda_{1,2} = \alpha$ (negative). This implies that the fixed point $(0, 0)$ is a degenerate stable node in the two-dimensional plane (Figure 1(d)).

(5) When $a < 0$, $b > 0$, and $c + a/4b < z < c$, system (16) has two positive real eigenvalues such that $\lambda_1 > 0$, $\lambda_2 > 0$. This implies that the fixed point $(0, 0)$ is an unstable node point in the two-dimensional plane (Figure 1(e)).

(6) When $a < 0$, $b > 0$, and $z < c + a/4b$, system (16) has two complex eigenvalues such that $\lambda_{1,2} = \alpha \pm \beta i$ with $\mathrm{Re}(\lambda)$ being positive. This implies that the fixed point $(0, 0)$ is an unstable focus in the two-dimensional plane (Figure 1(f)).

(7) When $a < 0$, $b > 0$, and $z = c + a/4b$, system (16) has two repeated real eigenvalues such that $\lambda_{1,2} = \alpha$ (positive). This implies that the fixed point $(0, 0)$ is a degenerate unstable node in the two-dimensional plane (Figure 1(g)).

(8) When $a < 0$, and $z = c$, system (16) has the first eigenvalue equal to zero and the second eigenvalue is positive such that $\lambda_1 = 0$, $\lambda_2 > 0$. (Figure 1(h))

(9) When $a = 0$ and $z = c$, system (16) has the first eigenvalue equal to zero and the second eigenvalue is equal to zero also such that $\lambda_{1,2} = 0$.

(10) When $a > 0$ and $z = c$, system (16) has the first eigenvalue equal to zero and the second eigenvalue is negative such that $\lambda_1 = 0$, $\lambda_2 < 0$ (Figure 1(i)).

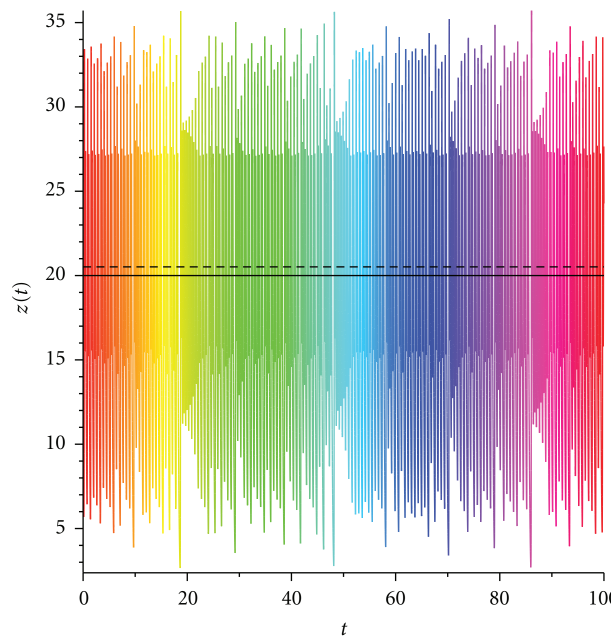

Figure 2: The chaotic time series of $z(t)$ when $a = 10$, $b = 4.845$, and $c = 20$ with two straight lines $z = 20$ (solid) and $z = 20.515$ (long dash).

Figure 2 shows the graph of chaotic time series of the function $z(t)$ and two straight lines with $z(t) = c$ (solid) and $z(t) = c + a/4b$ (long dash). It is obvious that as time, t, goes to infinity, the function $z(t)$ passes through the straight line alternatively. The $z(t)$ is divided into 3 partitioned disjointed domains: $(-\infty, c)$, $(c, c + a/4b)$, and $(c + a/4b, \infty)$.

Thus, the system (16) has different dynamical behaviours in the stated different domains. When time, t, goes to infinity,

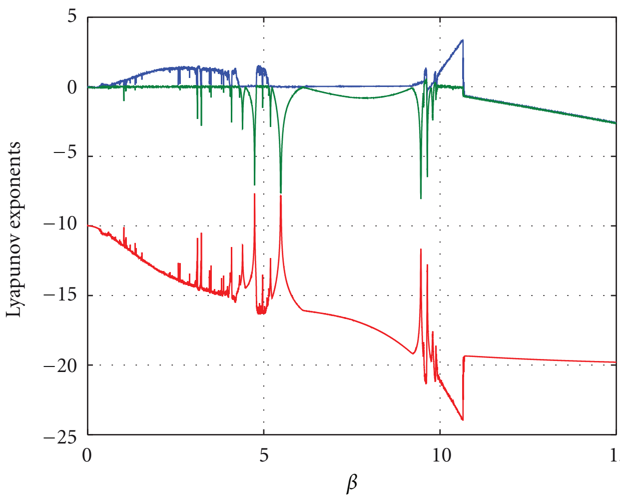

Figure 3: The Lyapunov exponents versus the parameter b of the new system (2).

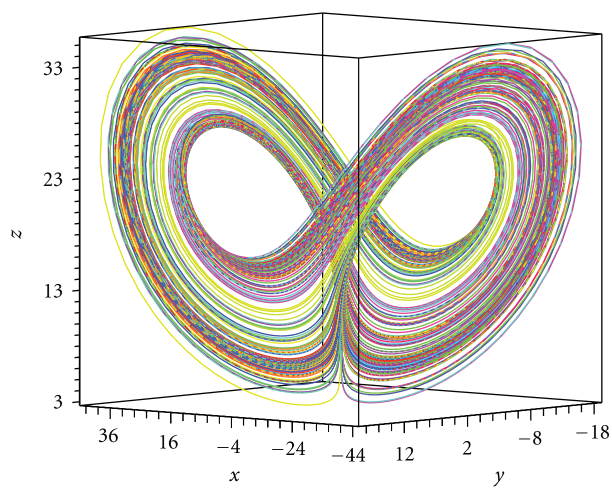

Figure 4: 3D view of the new system (2) in the x-y-z space with the parameters $a = 10$, $b = 4.845$, and $c = 20$.

the dynamical behaviour changes in system (6) and passes through these domains repeatedly, leading to a complicated-dynamical behaviour.

3. Results and Discussions

By using the Lyapunov exponent, we observed that system (2) has chaotic behaviours with the value of $a = 10$, $b = 4.845$, and $c = 20$. To obtain these values, the parameters a and c are fixed parameters while the parameter b will be used for dynamic variation. We have fixed the iterations for the value b to 0.001 per iteration and we obtain $b = 4.845$ after one thousand iterations. With the aid of MATLAB, we have succeeded in obtaining the Lyapunov exponent value and to plot the Lyapunov exponents versus the parameter b of the new system (2). Figure 3 is the graph obtained from the Lyapunov exponent.

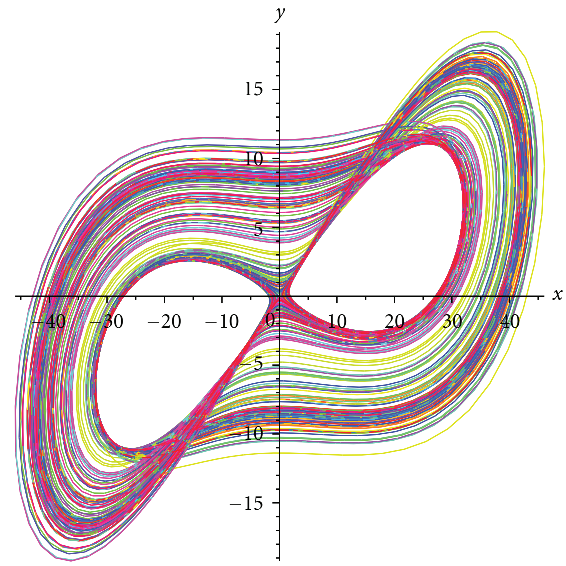

Figure 5: Projection of the new system (2) on the x-y plane.

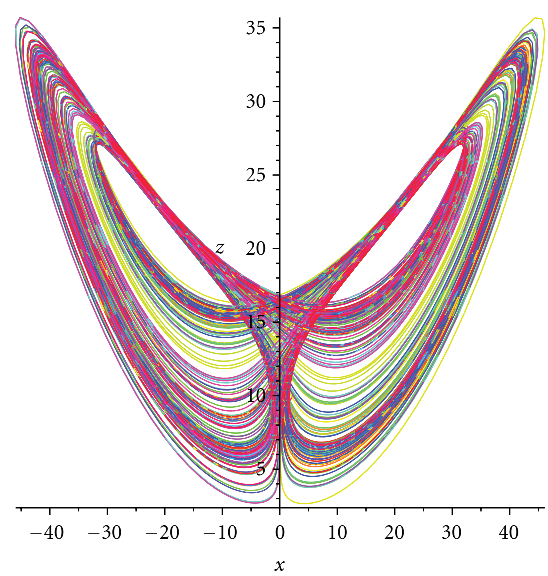

Figure 6: Projection of the new system (2) on the x-z plane.

The Lyapunov exponents show some of the positive values which mean the system itself is chaotic at a certain range of values. Thus, in the graph, the highest value for the Lyapunov exponent with two positive values occurs when the b value is 4.845. The corresponding Lyapunov exponents values are $l_1 = 1.534557$, $l_2 = 0.013385$, and $l_3 = -16.365683$.

The Figures 4, 5, 6, and 7 are the phase portraits of the system with the parameters $a = 10$, $b = 4.845$, and $c = 20$ with the chosen initial points being $(2.3, -1.3, 10)$. As for Figures 8, 9, and 10 are the portraits of the time series plot for the corresponding variables x, y, and z with the parameters $a = 10$, $b = 4.845$, and $c = 20$ with the chosen initial points being $(2.3, -1.3, 10)$.

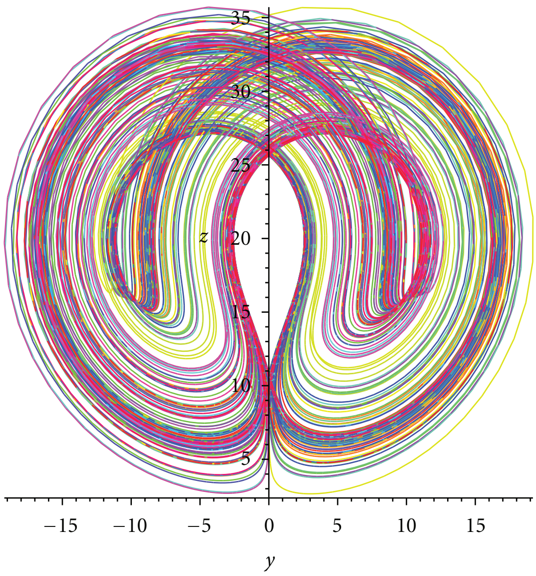

Figure 7: Projection of the new system (2) on the y-z plane.

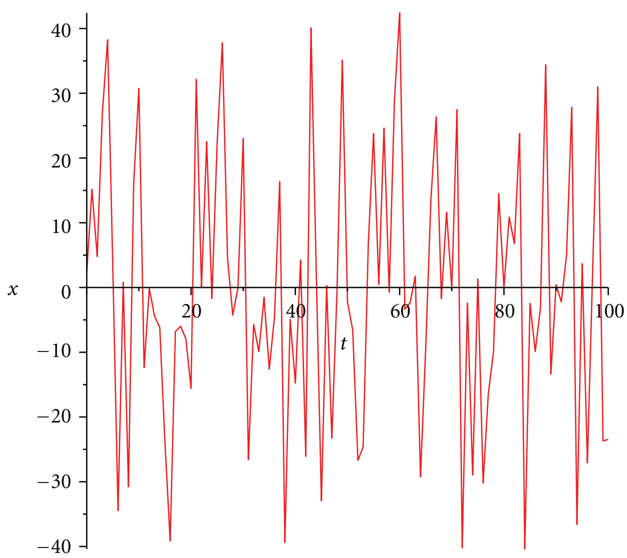

Figure 8: Time series for x in system (2) with $a = 10$, $b = 4.485$, and $c = 20$.

Next, we also found some chaotic behaviours in the system which portrays a different structure in the diagram. As we can see in the Lyapunov exponent, the highest peak in Figure 3 is at the point when b is equal to 10.629. Value b gives the highest value of the Lyaponuv exponent which is $l_1 = 3.354912$, $l_2 = -0.0864123$, and $l_3 = -23.879563$. Furthermore, the system gives another sudden change in the diagram when b is equal to 10.650. Figures 11 and 12 show the, respectively, phase portraits of the system.

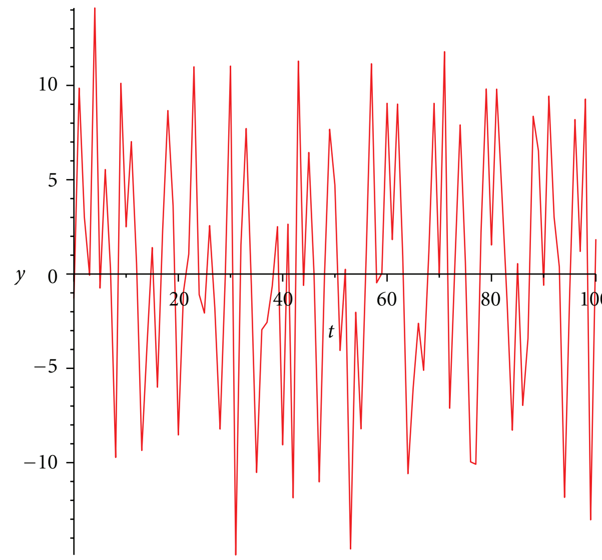

Figure 9: Time series for y in system (2) with $a = 10$, $b = 4.485$, and $c = 20$.

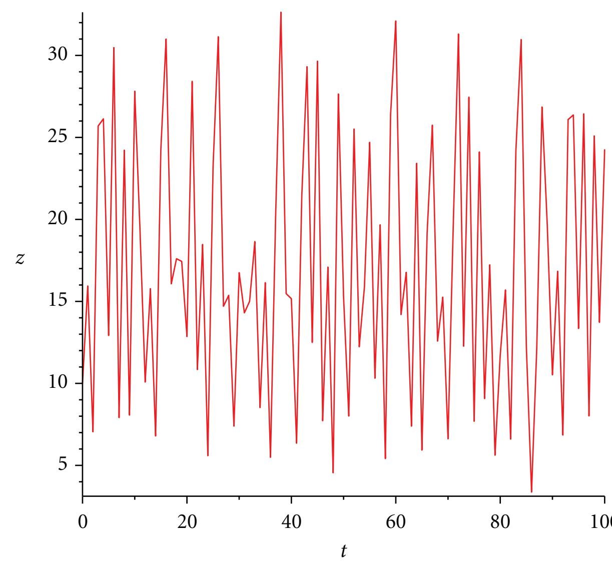

Figure 10: Time series for z in system (2) with $a = 10$, $b = 4.485$, and $c = 20$.

4. Conclusions and Suggestions

In this paper, we have found a new modified Lorenz system (2) with some interesting results. Besides that, we have studied the dynamic behaviour of the system and basic dynamic analysis. We also did complementary-cluster energy-barrier criterion (CCEBC) analysis and studied the chaotic behaviour by using Lyapunov exponent. The system behaved chaotically when $a = 10$, $c = 20$, and $b = 4.845$.

For the new system (2), there are still many other properties and dynamical behaviours that are still unknown. Many other analyses need to be conducted to investigate the new system. One of the dynamical analyses that is not shown here is the bifurcation of the system. Besides, the Poincaré

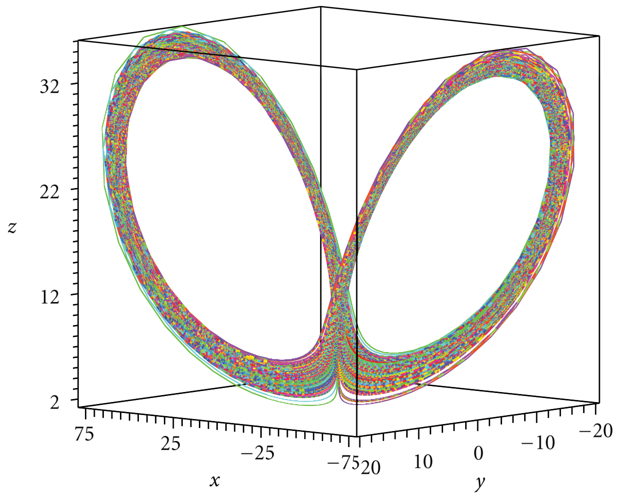

FIGURE 11: 3D view of the new system (2) in the x-y-z space with the parameters $a = 10$, $b = 10.629$, and $c = 20$.

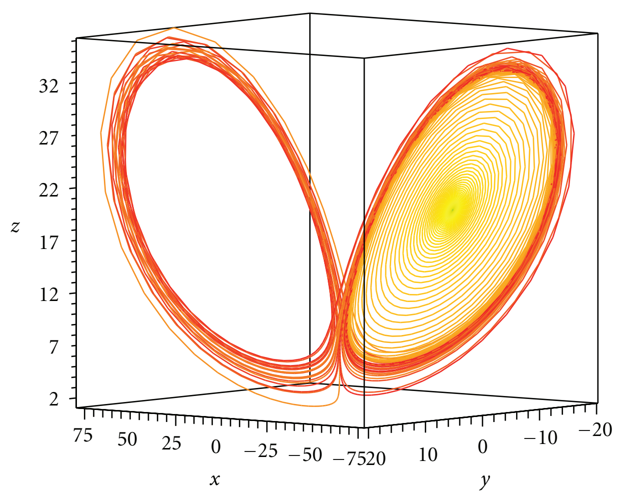

FIGURE 12: 3D view of the system (2) in the x-y-z space with the parameters $a = 10$, $b = 10.650$, and $c = 20$.

map of the system is not shown. Hence, in order to investigate the other properties of the system, further research need to be conducted. Furthermore, the step size used in the Lyapunov exponents may not be small enough to prove that the value of b obtained is accurate enough. Readers may choose a smaller step size to obtain a stronger value from the Lyapunov exponents.

In the next paper, there will be some basic dynamical property explanations especially in bifurcation on the new modified Lorenz system.Besides that, explanations on Poincaré map, manifold structure, and other dynamical behaviours will be shown and explained.

Acknowledgment

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