Research Article

Convexity and Spirallikeness Conditions for Two New General Integral Operators

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We define two new general integral operators for certain analytic functions in the unit disc $\mathbb{U}$ and give some sufficient conditions for these integral operators on some subclasses of analytic functions.

1. Introduction

Let $\mathcal{A}_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, \dots\}) \quad (1)$$

which is analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In particular, we set

$$\mathcal{A}_p(1) := \mathcal{A}_p, \quad \mathcal{A}_1(1) = \mathcal{A}_1 := \mathcal{A}. \quad (2)$$

The fractional derivative of order α is defined, for a function f , by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta \quad (0 \leq \alpha < 1), \quad (3)$$

where the function f is analytic in a simply connected region of the complex z -plane containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

It readily follows from (3) that

$$D_z^\alpha z^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} z^{k-\alpha} \quad (0 \leq \alpha < 1, k \in \mathbb{N}). \quad (4)$$

Using the operator $\Omega_p^\alpha : \mathcal{A}_p(n) \rightarrow \mathcal{A}_p(n)$,

$$\begin{aligned} \Omega_p^\alpha f(z) &= \frac{\Gamma(p-\alpha+1)}{\Gamma(p+1)} z^\alpha D_z^\alpha f(z), \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} a_k z^k, \end{aligned} \quad (5)$$

$$\alpha \neq p+1, p+2, p+3, \dots$$

Bulut [1] defined the general differential operator $D_{\lambda,l,p}^{m,\alpha}$, namely, the generalization of the generalized Al-Oboudi differential operator as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_{\lambda,l,p}^{1,\alpha} f(z) &= \frac{p-\lambda p+l}{p+l} \Omega_p^\alpha f(z) + \frac{\lambda}{p+l} z \left(\Omega_p^\alpha f(z) \right)', \\ &= D_{\lambda,l,p}^\alpha f(z) \quad (\lambda, l \geq 0, 0 \leq \alpha < 1), \\ D_{\lambda,l,p}^{2,\alpha} f(z) &= D_{\lambda,l,p}^\alpha \left(D_{\lambda,l,p}^{1,\alpha} f(z) \right), \\ &\vdots \\ D_{\lambda,l,p}^{m,\alpha} f(z) &= D_{\lambda,l,p}^\alpha \left(D_{\lambda,l,p}^{m-1,\alpha} f(z) \right) \quad (m \in \mathbb{N}). \end{aligned} \quad (6)$$

If f is given by (1), then by (5) and (6), we see that

$$D_{\lambda,l,p}^{m,\alpha} f(z) = z^p + \sum_{k=p+n}^{\infty} \Psi_{k,m}(\alpha, \lambda, l, p) a_k z^k \quad (7)$$

$$(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where

$$\Psi_{k,m}(\alpha, \lambda, l, p) = \left[\frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} \frac{p+\lambda(k-p)+l}{p+l} \right]^m \quad (8)$$

By using the operator $D_{\lambda,l,p}^{m,\alpha}$ defined by (7), we introduce the new classes $\mathcal{US}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$ and $\mathcal{UK}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$ as follows.

Definition 1. A function $f \in \mathcal{A}_p(n)$ is in the class $\mathcal{US}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$ if and only if f satisfies

$$\Re \left\{ p + \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))'}{D_{\lambda,l,p}^{m,\alpha} f(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))'}{D_{\lambda,l,p}^{m,\alpha} f(z)} - p \right) \right| + \beta, \quad (9)$$

where $z \in \mathbb{U}$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$ and $D_{\lambda,l,p}^{m,\alpha}$ is defined by (7).

Definition 2. A function $f \in \mathcal{A}_p(n)$ is in the class $\mathcal{UK}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$ if and only if f satisfies

$$\Re \left\{ p + \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))''}{(D_{\lambda,l,p}^{m,\alpha} f(z))'} - p \right) \right\} > \delta \left| \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))''}{(D_{\lambda,l,p}^{m,\alpha} f(z))'} - p \right) \right| + \beta, \quad (10)$$

where $z \in \mathbb{U}$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$, and $D_{\lambda,l,p}^{m,\alpha}$ is defined by (7).

Note that

$$f \in \mathcal{UK}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b) \iff \frac{1}{p} z f' \in \mathcal{US}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b). \quad (11)$$

Remark 3. (i) For $\delta = 0$, we have the classes

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{m,p,n}(0, \beta, b) &\equiv \mathcal{S}_{\alpha,\lambda,l}^{m,p,n}(\beta, b), \\ \mathcal{UK}_{\alpha,\lambda,l}^{m,p,n}(0, \beta, b) &\equiv \mathcal{K}_{\alpha,\lambda,l}^{m,p,n}(\beta, b) \end{aligned} \quad (12)$$

introduced and studied by Bulut [1]. In particular, we get the classes

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,0}^{m,1,1}(0, \beta, b) &\equiv \mathcal{S}_{\alpha,\lambda}^m(\beta, b), \\ \mathcal{UK}_{\alpha,\lambda,0}^{m,1,1}(0, \beta, b) &\equiv \mathcal{K}_{\alpha,\lambda}^m(\beta, b) \end{aligned} \quad (13)$$

introduced and studied by Bulut [2].

(ii) For $\delta = 0$ and $\beta = 0$, we have the classes

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{m,p,n}(0, 0, b) &\equiv \mathcal{S}_{\alpha,\lambda,l}^{m,p,n}(b), \\ \mathcal{UK}_{\alpha,\lambda,l}^{m,p,n}(0, 0, b) &\equiv \mathcal{K}_{\alpha,\lambda,l}^{m,p,n}(b). \end{aligned} \quad (14)$$

In particular, we set

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{0,p,n}(0, 0, b) &\equiv \mathcal{S}^{p,n}(b), \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,p,n}(0, 0, b) &\equiv \mathcal{K}^{p,n}(b) \end{aligned} \quad (15)$$

for the p -valently starlike and p -valently convex functions with complex order b ($b \in \mathbb{C} - \{0\}$), respectively.

(iii) For $m = 0$, we have the classes

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{0,p,n}(\delta, \beta, b) &\equiv \mathcal{S}^{p,n}(\delta, \beta, b) \\ &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ p + \frac{1}{b} \left(\frac{z f'(z)}{f(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left(\frac{z f'(z)}{f(z)} - p \right) \right| + \beta \right\}, \end{aligned} \quad (16)$$

$$\begin{aligned} \mathcal{UK}_{\alpha,\lambda,l}^{0,p,n}(\delta, \beta, b) &\equiv \mathcal{K}^{p,n}(\delta, \beta, b) \\ &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ p + \frac{1}{b} \left(1 + \frac{z f''(z)}{f'(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left(1 + \frac{z f''(z)}{f'(z)} - p \right) \right| + \beta \right\}. \end{aligned}$$

In particular, we set

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, b) &\equiv \mathcal{S}(\delta, \beta, b), \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, b) &\equiv \mathcal{K}(\delta, \beta, b). \end{aligned} \quad (17)$$

Also, we have the classes

$$\begin{aligned} \mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, 1) &\equiv \mathcal{SD}(\delta, \beta), \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, 1) &\equiv \mathcal{KD}(\delta, \beta) \end{aligned} \quad (18)$$

introduced and studied by Shams et al. [3].

(iv) For $\delta = 0$ and $m = 0$, we have the classes

$$\begin{aligned}\mathcal{US}_{\alpha,\lambda,l}^{0,p,n}(0, \beta, b) &\equiv \mathcal{S}^{p,n}(\beta, b) \\ &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ p + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - p \right) \right\} > \beta \right\}, \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,p,n}(0, \beta, b) &\equiv \mathcal{K}^{p,n}(\beta, b) \\ &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} > \beta \right\}\end{aligned}\quad (19)$$

of p -valently starlike and p -valently convex functions with complex order b ($b \in \mathbb{C} - \{0\}$) and type β ($0 \leq \beta < p$), respectively. In particular, we set

$$\begin{aligned}\mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(0, \beta, b) &\equiv \mathcal{S}(\beta, b), \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(0, \beta, b) &\equiv \mathcal{K}(\beta, b) \quad (b \in \mathbb{C} - \{0\}, 0 \leq \beta < 1).\end{aligned}\quad (20)$$

Note that for $b = 1$, we have

$$\mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(0, \beta, 1) \equiv \mathcal{S}^*(\beta), \quad \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(0, \beta, 1) \equiv \mathcal{K}(\beta) \quad (21)$$

which are the classes of starlike and convex functions type β ($0 \leq \beta < 1$), respectively.

(v) For $\delta = 1$, $m = 0$, $p = n = 1$, and $b = 1$, we have the classes

$$\begin{aligned}\mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(1, \beta, 1) &\equiv \mathcal{UST}(\beta), \\ \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, 1) &\equiv \mathcal{UCV}(\beta)\end{aligned}\quad (22)$$

of uniformly starlike and uniformly convex functions of order β ($0 \leq \beta < 1$), respectively, investigated by Rønning [4]. In particular, we get the classes

$$\mathcal{US}_{\alpha,\lambda,l}^{0,1,1}(1, 0, 1) \equiv \mathcal{UST}, \quad \mathcal{UK}_{\alpha,\lambda,l}^{0,1,1}(\delta, \beta, 1) \equiv \mathcal{UCV} \quad (23)$$

of uniformly starlike and uniformly convex functions, respectively, first defined by Goodman [5].

Remark 4. If we set $\delta = 0$ and $b = e^{-i\theta} \cos \theta$ ($|\theta| < \pi/2$) in Definitions 1 and 2, then we have the classes

$$\begin{aligned}\mathcal{S}_{\alpha,\lambda,l}^{m,p,n}(\beta, \theta) &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ e^{i\theta} \frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))'}{D_{\lambda,l,p}^{m,\alpha} f(z)} \right\} \right. \\ &\quad \left. > \beta \cos \theta, 0 \leq \beta < p \right\},\end{aligned}\quad (24)$$

$$\begin{aligned}\mathcal{K}_{\alpha,\lambda,l}^{m,p,n}(\beta, \theta) &= \left\{ f \in \mathcal{A}_p(n) : \Re \left\{ e^{i\theta} \left(1 + \frac{z(D_{\lambda,l,p}^{m,\alpha} f(z))''}{(D_{\lambda,l,p}^{m,\alpha} f(z))'} \right) \right\} \right. \\ &\quad \left. > \beta \cos \theta, 0 \leq \beta < p \right\},\end{aligned}\quad (25)$$

respectively. Also, we put

$$\mathcal{S}_{\alpha,\lambda,l}^{0,p,n}(\beta, \theta) \equiv \mathcal{S}^{p,n}(\beta, \theta), \quad \mathcal{K}_{\alpha,\lambda,l}^{0,p,n}(\beta, \theta) \equiv \mathcal{K}^{p,n}(\beta, \theta). \quad (26)$$

In particular, we set

$$\begin{aligned}\mathcal{S}_{\alpha,\lambda,l}^{0,1,1}(\beta, \theta) &\equiv \mathcal{S}^\theta_\beta \quad \left(|\theta| < \frac{\pi}{2}, 0 \leq \beta < 1 \right), \\ \mathcal{K}_{\alpha,\lambda,l}^{0,1,1}(\beta, \theta) &\equiv \mathcal{K}^\theta_\beta \quad \left(|\theta| < \frac{\pi}{2}, 0 \leq \beta < 1 \right)\end{aligned}\quad (27)$$

for the the class of θ -spirallike functions of order β [6], and the class of θ -Robertson functions of order β [7], respectively.

We now define the following general integral operators.

Definition 5. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. One defines the following general integral operators:

$$\begin{aligned}\mathcal{J}_{p,\eta}^{m,k} : \mathcal{A}_p(n)^\eta &\longrightarrow \mathcal{A}_p(n), \\ \mathcal{J}_{p,\eta}^{m,k}(f_1, \dots, f_\eta) &= \mathcal{F}_{p,\eta,m,k},\end{aligned}\quad (28)$$

$$\mathcal{F}_{p,\eta,m,k}(z) = \int_0^z pt^{p-1} \prod_{j=1}^\eta \left(\frac{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)}{t^p} \right)^{k_j} dt, \quad (29)$$

$$\begin{aligned}\mathcal{J}_{p,\eta}^{m,k} : \mathcal{A}_p(n)^\eta &\longrightarrow \mathcal{A}_p(n), \\ \mathcal{J}_{p,\eta}^{m,k}(f_1, \dots, f_\eta) &= \mathcal{G}_{p,\eta,m,k},\end{aligned}\quad (30)$$

$$\mathcal{G}_{p,\eta,m,k}(z) = \int_0^z pt^{p-1} \prod_{j=1}^\eta \left(\frac{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{pt^{p-1}} \right)^{k_j} dt, \quad (31)$$

where $z \in \mathbb{U}$, $f_j \in \mathcal{A}_p(n)$, and $D_{\lambda,l,p}^{m_j,\alpha}$ is defined by (7) for $1 \leq j \leq \eta$.

Remark 6. (i) For $\eta = 1$, $m_1 = m$, $k_1 = k$, and $f_1 = f$, we have the integral operators

$$\mathcal{F}_{p,m,k}(z) = \int_0^z pt^{p-1} \left(\frac{D_{\lambda,l,p}^{m,\alpha} f(t)}{t^p} \right)^k dt, \quad (32)$$

$$\mathcal{G}_{p,m,k}(z) = \int_0^z pt^{p-1} \left(\frac{(D_{\lambda,l,p}^{m,\alpha} f(t))'}{pt^{p-1}} \right)^k dt. \quad (33)$$

(ii) For $\alpha = 0$, $\lambda = 1$, and $l = 0$, we have the integral operators

$$\begin{aligned} \mathcal{F}_{p,n,m,k}(z) &= \int_0^z pt^{p-1} \left(\frac{D^{m_1} f_1(t)}{t^p} \right)^{k_1} \cdots \left(\frac{D^{m_n} f_n(t)}{t^p} \right)^{k_n} dt, \\ \mathcal{G}_{p,n,m,k}(z) &= \int_0^z pt^{p-1} \left(\frac{(D^{m_1} f_1(t))'}{pt^{p-1}} \right)^{k_1} \cdots \left(\frac{(D^{m_n} f_n(t))'}{pt^{p-1}} \right)^{k_n} dt \end{aligned} \quad (34)$$

introduced by Saltık et al. [8].

(iii) For $m_1 = \cdots = m_\eta = 0$, we have the integral operators

$$\begin{aligned} \mathcal{F}_p(z) &= \int_0^z pt^{p-1} \left(\frac{f_1(t)}{t^p} \right)^{k_1} \cdots \left(\frac{f_n(t)}{t^p} \right)^{k_n} dt, \\ \mathcal{G}_p(z) &= \int_0^z pt^{p-1} \left(\frac{f_1'(t)}{pt^{p-1}} \right)^{k_1} \cdots \left(\frac{f_n'(t)}{pt^{p-1}} \right)^{k_n} dt \end{aligned} \quad (35)$$

introduced by Frasin [9].

2. Sufficient Conditions for the Operator $\mathcal{F}_{p,\eta,m,k}$

Theorem 7. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta_j \geq 0$, $0 \leq \beta_j < p$, and $f_j \in \mathcal{U}\mathcal{S}_{\alpha,\lambda,l}^{m_j,p,n}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \quad (36)$$

then the integral operator $\mathcal{F}_{p,\eta,m,k}$, defined by (29), is in the class $\mathcal{K}^{p,n}(\tau, b)$, where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \quad (37)$$

Proof. From the definition (29), we observe that $\mathcal{F}_{p,\eta,m,k}(z) \in \mathcal{A}_p(n)$. On the other hand, it is easy to see that

$$(\mathcal{F}_{p,\eta,m,k}(z))' = pz^{p-1} \prod_{j=1}^{\eta} \left(\frac{D_{\lambda,l,p}^{m_j,\alpha} f_j(z)}{z^p} \right)^{k_j}. \quad (38)$$

Differentiating (38) logarithmically and multiplying by z , we obtain

$$\frac{z(\mathcal{F}_{p,\eta,m,k}(z))''}{(\mathcal{F}_{p,\eta,m,k}(z))'} = (p-1) + \sum_{j=1}^{\eta} k_j \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \quad (39)$$

or equivalently

$$1 + \frac{z(\mathcal{F}_{p,\eta,m,k}(z))''}{(\mathcal{F}_{p,\eta,m,k}(z))'} - p = \sum_{j=1}^{\eta} k_j \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right). \quad (40)$$

Then, by multiplying (40) with $1/b$, we have

$$\begin{aligned} &\frac{1}{b} \left(1 + \frac{z(\mathcal{F}_{p,\eta,m,k}(z))''}{(\mathcal{F}_{p,\eta,m,k}(z))'} - p \right) \\ &= \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \end{aligned} \quad (41)$$

or equivalently

$$\begin{aligned} &p + \frac{1}{b} \left(1 + \frac{z(\mathcal{F}_{p,\eta,m,k}(z))''}{(\mathcal{F}_{p,\eta,m,k}(z))'} - p \right) \\ &= \sum_{j=1}^{\eta} k_j \left(p + \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \right) \\ &\quad + p - p \sum_{j=1}^{\eta} k_j. \end{aligned} \quad (42)$$

Since $f_j \in \mathcal{U}\mathcal{S}_{\alpha,\lambda,l}^{m_j,p,n}(\delta_j, \beta_j, b)$ ($1 \leq j \leq \eta$), we get

$$\begin{aligned} & \Re \left\{ p + \frac{1}{b} \left(1 + \frac{z(\mathcal{F}_{p,\eta,m,k}(z))''}{(\mathcal{F}_{p,\eta,m,k}(z))'} - p \right) \right\} \\ &= \sum_{j=1}^{\eta} k_j \Re \left\{ p + \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \right\} \\ &+ p - p \sum_{j=1}^{\eta} k_j \\ &> \sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \right| \\ &+ p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \end{aligned} \quad (43)$$

Since

$$\sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right) \right| > 0, \quad (44)$$

the integral operator $\mathcal{F}_{p,\eta,m,k} \in \mathcal{K}^{p,n}(\tau, b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \quad (45)$$

□

Corollary 8. In Theorem 7, letting

- (i) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$ ($1 \leq j \leq \eta$), one obtains [10, Theorem 1].
- (ii) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$, $\beta_j = \beta \in [0, 1)$ ($1 \leq j \leq \eta$), one obtains [11, Theorem 1].
- (iii) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$, $\beta_j = 0$ ($1 \leq j \leq \eta$), one obtains [12, Theorem 1].

Putting $\eta = 1$, $m_1 = m$, $k_1 = k$, $\beta_1 = \beta$, $\delta_1 = \delta$, and $f_1 = f$ in Theorem 7, we have the following.

Corollary 9. Let $m \in \mathbb{N}_0$ and $k > 0$. Also let $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$, and $f \in \mathcal{U}\mathcal{S}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$. If

$$0 \leq p + k(\beta - p) < p, \quad (46)$$

then the integral operator $\mathcal{F}_{p,m,k}$, defined by (32), is in the class $\mathcal{K}^{p,n}(\mu, b)$, where

$$\mu = p + k(\beta - p). \quad (47)$$

Putting $p = n = 1$ and $m = 0$ in Corollary 9, we have the following.

Corollary 10. Let $k > 0$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < 1$, and $f \in \mathcal{S}(\delta, \beta, b)$. If

$$0 \leq 1 + k(\beta - 1) < 1, \quad (48)$$

then the integral operator

$$\mathcal{F}_k(z) = \int_0^z \left(\frac{f(t)}{t} \right)^k dt \quad (49)$$

is convex of complex order b ($b \in \mathbb{C} - \{0\}$) and type $\rho = 1 + k(\beta - 1)$; that is, $\mathcal{F}_k \in \mathcal{K}(\rho, b)$.

Putting $p = n = 1$, $m_j = 0$ ($1 \leq j \leq \eta$), and $b = 1$ in Theorem 7, we have the following.

Corollary 11. Let $\eta \in \mathbb{N}$, $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$, $\delta_j \geq 0$, $0 \leq \beta_j < 1$, and $f_j \in \mathcal{S}\mathcal{D}(\delta_j, \beta_j)$ for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1) < 1, \quad (50)$$

then the integral operator

$$\mathcal{F}(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{k_1} \dots \left(\frac{f_n(t)}{t} \right)^{k_n} dt \quad (51)$$

is in the class $\mathcal{K}(\sigma)$, where

$$\sigma = 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1). \quad (52)$$

Theorem 12. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $|\theta| < \pi/2$, $0 \leq \beta_j < p$, and $f_j \in \mathcal{S}_{\alpha,\lambda,l}^{m_j,p,n}(\beta_j, \theta)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \quad (53)$$

then the integral operator $\mathcal{F}_{p,\eta,m,k}$, defined by (29), is in the class $\mathcal{K}^{p,n}(\tau, \theta)$, where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \quad (54)$$

Proof. From (24), (25), and (40), we get the desired result. □

Putting $\eta = 1$, $m_1 = m$, $k_1 = k$, $\beta_1 = \beta$ and $f_1 = f$ in Theorem 12, we have the following.

Corollary 13. Let $m \in \mathbb{N}_0$ and $k > 0$. Also let $|\theta| < \pi/2$, $0 \leq \beta < p$, and $f \in \mathcal{S}_{\alpha,\lambda,l}^{m,p,n}(\beta, \theta)$. If

$$0 \leq p + k(\beta - p) < p, \quad (55)$$

then the integral operator $\mathcal{F}_{p,m,k}$, defined by (32), is in the class $\mathcal{K}^{p,n}(\mu, \theta)$, where

$$\mu = p + k(\beta - p). \quad (56)$$

Putting $p = n = 1$ and $m = 0$ in Corollary 13, we have the following.

Corollary 14. Let $k > 0$, $|\theta| < \pi/2$, $0 \leq \beta < 1$, and $f \in \mathcal{S}_\beta^\theta$. If

$$0 \leq 1 + k(\beta - 1) < 1, \quad (57)$$

then the integral operator $\mathcal{F}_k$, defined by (49), is in the class $\mathcal{K}_\rho^\theta$, where

$$\rho = 1 + k(\beta - 1). \quad (58)$$

Putting $p = n = 1$ and $m_j = 0$ ($1 \leq j \leq \eta$) in Theorem 12, we have the following.

Corollary 15. Let $\eta \in \mathbb{N}$, $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$, $0 \leq \beta_j < 1$, and $f_j \in \mathcal{S}_{\beta_j}^\theta$ ($|\theta| < \pi/2$) for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (59)$$

then the integral operator $\mathcal{F}$, defined by (51), is in the class $\mathcal{K}_\sigma^\theta$, where

$$\sigma = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1). \quad (60)$$

Theorem 16. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta_j \geq 0$, and $0 \leq \beta_j < p$ for $1 \leq j \leq \eta$. If

$$\left| \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'}{D_{\lambda,l,p}^{m_j,\alpha} f_j(t)} - p \right| > -\frac{p + \sum_{j=1}^{\eta} k_j(\beta_j - p)}{(1/|b|) \sum_{j=1}^{\eta} k_j \delta_j} \quad (61)$$

for all $1 \leq j \leq \eta$, then the integral operator $\mathcal{F}_{p,\eta,m,k}$, defined by (29), is p -valently convex of complex order b ($b \in \mathbb{C} - \{0\}$); that is, $\mathcal{F}_{p,\eta,m,k} \in \mathcal{K}^{p,n}(b)$.

Proof. From (43) and (61), we easily get that the integral operator $\mathcal{F}_{p,\eta,m,k}$ is p -valently convex of complex order b . $\square$

3. Sufficient Conditions for the Operator $\mathcal{G}_{p,\eta,m,k}$

Theorem 17. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta_j \geq 0$, $0 \leq \beta_j < p$, and $f_j \in \mathcal{U}\mathcal{K}_{\alpha,\lambda,l}^{m_j,p,n}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j(\beta_j - p) < p, \quad (62)$$

then the integral operator $\mathcal{G}_{p,\eta,m,k}$, defined by (31), is in the class $\mathcal{K}^{p,n}(\tau, b)$, where

$$\tau = p + \sum_{j=1}^{\eta} k_j(\beta_j - p). \quad (63)$$

Proof. From the definition (31), we observe that $\mathcal{G}_{p,\eta,m,k}(z) \in \mathcal{A}_p(n)$. On the other hand, it is easy to see that

$$(\mathcal{G}_{p,\eta,m,k}(z))' = pz^{p-1} \prod_{j=1}^{\eta} \left(\frac{(D_{\lambda,l,p}^{m_j,\alpha} f_j(z))'}{pz^{p-1}} \right)^{k_j}. \quad (64)$$

Differentiating (64) logarithmically and multiplying by z , we obtain

$$\begin{aligned} & \frac{z(\mathcal{G}_{p,\eta,m,k}(z))''}{(\mathcal{G}_{p,\eta,m,k}(z))'} \\ &= (p-1) + \sum_{j=1}^{\eta} k_j \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - (p-1) \right) \end{aligned} \quad (65)$$

or equivalently

$$\begin{aligned} & 1 + \frac{z(\mathcal{G}_{p,\eta,m,k}(z))''}{(\mathcal{G}_{p,\eta,m,k}(z))'} - p \\ &= \sum_{j=1}^{\eta} k_j \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - (p-1) \right). \end{aligned} \quad (66)$$

Then, by multiplying (66) with $1/b$, we have

$$\begin{aligned} & \frac{1}{b} \left(1 + \frac{z(\mathcal{G}_{p,\eta,m,k}(z))''}{(\mathcal{G}_{p,\eta,m,k}(z))'} - p \right) \\ &= \sum_{j=1}^{\eta} k_j \frac{1}{b} \left(\frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - (p-1) \right) \end{aligned} \quad (67)$$

or equivalently

$$\begin{aligned} & p + \frac{1}{b} \left(1 + \frac{z(\mathcal{G}_{p,\eta,m,k}(z))''}{(\mathcal{G}_{p,\eta,m,k}(z))'} - p \right) \\ &= \sum_{j=1}^{\eta} k_j \left(p + \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - p \right) \right) \\ &+ p - p \sum_{j=1}^{\eta} k_j. \end{aligned} \quad (68)$$

Since $f_j \in \mathcal{U}\mathcal{K}_{\alpha,\lambda,l}^{m_j,p,n}(\delta_j, \beta_j, b)$ ($1 \leq j \leq \eta$), we get

$$\begin{aligned} & \Re \left\{ p + \frac{1}{b} \left(1 + \frac{z(\mathcal{G}_{p,\eta,m,k}(z))''}{(\mathcal{G}_{p,\eta,m,k}(z))'} - p \right) \right\} \\ &= \sum_{j=1}^{\eta} k_j \Re \left\{ p + \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - p \right) \right\} \\ &+ p - p \sum_{j=1}^{\eta} k_j \\ &> \sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - p \right) \right| \\ &+ p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \end{aligned} \quad (69)$$

Since

$$\sum_{j=1}^{\eta} k_j \delta_j \left| \frac{1}{b} \left(1 + \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - p \right) \right| > 0, \quad (70)$$

the integral operator $\mathcal{G}_{p,\eta,m,k} \in \mathcal{K}^{p,n}(\tau, b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \quad (71)$$

□

Corollary 18. In Theorem 17, letting

- (i) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$ ($1 \leq j \leq \eta$), one obtains [10, Theorem 3].
- (ii) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$, $\beta_j = \beta \in [0, 1)$ ($1 \leq j \leq \eta$), one obtains [11, Theorem 3].
- (iii) $p = n = 1$, $\delta_j = 0$, and $m_j = 0$, $\beta_j = 0$ ($1 \leq j \leq \eta$), one obtains [12, Theorem 2].

Putting $\eta = 1$, $m_1 = m$, $k_1 = k$, $\beta_1 = \beta$, $\delta_1 = \delta$, and $f_1 = f$ in Theorem 17, we have the following.

Corollary 19. Let $m \in \mathbb{N}_0$ and $k > 0$. Also let $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < p$ and $f \in \mathcal{U}\mathcal{K}_{\alpha,\lambda,l}^{m,p,n}(\delta, \beta, b)$. If

$$0 \leq p + k(\beta - p) < p, \quad (72)$$

then the integral operator $\mathcal{G}_{p,m,k}$, defined by (33), is in the class $\mathcal{K}^{p,n}(\mu, b)$, where

$$\mu = p + k(\beta - p). \quad (73)$$

Putting $p = n = 1$ and $m = 0$ in Corollary 19, we have the following.

Corollary 20. Let $k > 0$, $b \in \mathbb{C} - \{0\}$, $\delta \geq 0$, $0 \leq \beta < 1$, and $f \in \mathcal{K}(\delta, \beta, b)$. If

$$0 \leq 1 + k(\beta - 1) < 1, \quad (74)$$

then the integral operator

$$\mathcal{G}_k(z) = \int_0^z (f'(t))^k dt \quad (75)$$

is convex of complex order b ($b \in \mathbb{C} - \{0\}$) and type $\rho = 1 + k(\beta - 1)$; that is, $\mathcal{G}_k \in \mathcal{K}(\rho, b)$.

Putting $p = n = 1$, $m_j = 0$ ($1 \leq j \leq \eta$), and $b = 1$ in Theorem 17, we have the following.

Corollary 21. Let $\eta \in \mathbb{N}$, $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$, $\delta_j \geq 0$, $0 \leq \beta_j < 1$, and $f_j \in \mathcal{K}\mathcal{D}(\delta_j, \beta_j)$ for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1) < 1, \quad (76)$$

then the integral operator

$$\mathcal{G}(z) = \int_0^z (f_1'(t))^{k_1} \cdots (f_\eta'(t))^{k_\eta} dt \quad (77)$$

is in the class $\mathcal{K}(\sigma)$, where

$$\sigma = 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1). \quad (78)$$

Theorem 22. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $|\theta| < \pi/2$, $0 \leq \beta_j < p$, and $f_j \in \mathcal{K}_{\alpha,\lambda,l}^{m_j,p,n}(\beta_j, \theta)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \quad (79)$$

then the integral operator $\mathcal{G}_{p,\eta,m,k}$, defined by (31), is in the class $\mathcal{K}^{p,n}(\tau, \theta)$, where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \quad (80)$$

Proof. From (25) and (66), we get the desired result. □

Putting $\eta = 1$, $m_1 = m$, $k_1 = k$, $\beta_1 = \beta$, and $f_1 = f$ in Theorem 22, we have the following.

Corollary 23. Let $m \in \mathbb{N}_0$ and $k > 0$. Also let $|\theta| < \pi/2$, $0 \leq \beta < p$, and $f \in \mathcal{K}_{\alpha,\lambda,l}^{m,p,n}(\beta, \theta)$. If

$$0 \leq p + k(\beta - p) < p, \quad (81)$$

then the integral operator $\mathcal{G}_{p,m,k}$, defined by (33), is in the class $\mathcal{K}^{p,n}(\mu, \theta)$, where

$$\mu = p + k(\beta - p). \quad (82)$$

Putting $p = n = 1$ and $m = 0$ in Corollary 23, we have the following.

Corollary 24. Let $k > 0$, $|\theta| < \pi/2$, $0 \leq \beta < 1$, and $f \in \mathcal{K}_\beta^\theta$. If

$$0 \leq 1 + k(\beta - 1) < 1, \quad (83)$$

then the integral operator $\mathcal{G}_k$, defined by (75), is in the class $\mathcal{K}_\rho^\theta$, where

$$\rho = 1 + k(\beta - 1). \quad (84)$$

Putting $p = n = 1$ and $m_j = 0$ ($1 \leq j \leq \eta$) in Theorem 22, we have the following.

Corollary 25. Let $\eta \in \mathbb{N}$, $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$, $0 \leq \beta_j < 1$, and $f_j \in \mathcal{K}_{\beta_j}^\theta$ ($|\theta| < \pi/2$) for $1 \leq j \leq \eta$. If

$$0 \leq 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1) < 1, \quad (85)$$

then the integral operator $\mathcal{G}$, defined by (77), is in the class $\mathcal{K}_\sigma^\theta$, where

$$\sigma = 1 + \sum_{j=1}^{\eta} k_j(\beta_j - 1). \quad (86)$$

Theorem 26. Let $\eta \in \mathbb{N}$, $m = (m_1, \dots, m_\eta) \in \mathbb{N}_0^\eta$, and $k = (k_1, \dots, k_\eta) \in \mathbb{R}_+^\eta$. Also let $b \in \mathbb{C} - \{0\}$, $\delta_j \geq 0$, and $0 \leq \beta_j < p$ for $1 \leq j \leq \eta$. If

$$\left| 1 + \frac{z(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))''}{(D_{\lambda,l,p}^{m_j,\alpha} f_j(t))'} - p \right| > -\frac{p + \sum_{j=1}^{\eta} k_j(\beta_j - p)}{(1/|b|) \sum_{j=1}^{\eta} k_j \delta_j} \quad (87)$$

for all $1 \leq j \leq \eta$, then the integral operator $\mathcal{G}_{p,\eta,m,k}$, defined by (31), is p -valently convex of complex order b ($b \in \mathbb{C} - \{0\}$); that is, $\mathcal{G}_{p,\eta,m,k} \in \mathcal{K}^{p,n}(b)$.

Proof. From (69) and (87), we easily get that the integral operator $\mathcal{G}_{p,\eta,m,k}$ is p -valently convex of complex order b . $\square$

References

- [1] S. Bulut, "The generalization of the generalized Al-Oboudi differential operator," *Applied Mathematics and Computation*, vol. 215, no. 4, pp. 1448–1455, 2009.
- [2] S. Bulut, "Coefficient inequalities for certain subclasses of analytic functions defined by using a general derivative operator," *Kyungpook Mathematical Journal*, vol. 51, no. 3, pp. 241–250, 2011.
- [3] S. Shams, S. R. Kulkarni, and J. M. Jahangiri, "Classes of uniformly starlike and convex functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 55, no. 53–56, pp. 2959–2961, 2004.
- [4] F. Rønning, "On starlike functions associated with parabolic regions," *Annales Universitatis Mariae Curie-Skłodowska. A. Mathematica*, vol. 45, pp. 117–122, 1991.
- [5] A. W. Goodman, "On uniformly convex functions," *Annales Polonici Mathematici*, vol. 56, no. 1, pp. 87–92, 1991.
- [6] R. J. Libera, "Univalent α -spiral functions," *Canadian Journal of Mathematics*, vol. 19, pp. 449–456, 1967.
- [7] P. N. Chichra, "Regular functions $f(z)$ for which $zf'(z)$ is α -spiral-like," *Proceedings of the American Mathematical Society*, vol. 49, pp. 151–160, 1975.
- [8] G. Saltık, E. Deniz, and E. Kadioğlu, "Two new general p -valent integral operators," *Mathematical and Computer Modelling*, vol. 52, no. 9–10, pp. 1605–1609, 2010.
- [9] B. A. Frasin, "New general integral operators of p -valent functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 4, article 109, p. 9, 2009.
- [10] S. Bulut, "A note on the paper of Breaz and Güney," *Journal of Mathematical Inequalities*, vol. 2, no. 4, pp. 549–553, 2008.
- [11] D. Breaz and H. Ö. Güney, "The integral operator on the classes $\mathcal{S}_\alpha^*(b)$ and $\mathcal{C}_\alpha(b)$," *Journal of Mathematical Inequalities*, vol. 2, no. 1, pp. 97–100, 2008.
- [12] D. Breaz, M. K. Aouf, and N. Breaz, "Some properties for integral operators on some analytic functions with complex order," *Acta Mathematica. Academiae Paedagogicae Nyíregyháziensis. New Series*, vol. 25, no. 1, pp. 39–43, 2009.