Research Article

Interaction of Solitons for Sine-Gordon-Type Equations

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The subject of our consideration is a family of semilinear wave equations with a small parameter and nonlinearities which provide the existence of kink-type solutions (solitons). Using asymptotic analysis and numerical simulation, we demonstrate that solitons of the same type (kinks or antikinks) interact in the same manner as for the sine-Gordon equation. However, solitons of the different type preserve the shape after the interaction only in the case of two or three waves, and, moreover, under some additional conditions.

1. Introduction

We consider the semilinear wave equation

$$\varepsilon^2 (u_{tt} - u_{xx}) + F'(u) = 0, \quad x \in \mathbb{R}^1, \quad t > 0, \quad (1)$$

with some smooth nonlinearities $F(u)$ and the parameter $\varepsilon \rightarrow 0$. It is well known [1] that the unique completely integrable representative of the family (1) is the sine-Gordon equation, that is, (1) with $F'(u) = \sin(u)$ (see, e.g., [2, 3]).

At the same time, there are many nonlinearities $F(u)$ such that (1) admits exact traveling wave solutions of the soliton type:

$$u(x, t, \varepsilon) = \omega \left(\pm \beta \frac{x - Vt}{\varepsilon} \right),$$

$$\beta = (1 - V^2)^{-1/2}, \quad \omega(\eta) \in \mathcal{C}^\infty(\mathbb{R}),$$

$$\omega(\eta) \rightarrow 0 \quad \text{for } \eta \rightarrow -\infty, \quad \omega(\eta) \rightarrow 1 \quad \text{for } \eta \rightarrow +\infty. \quad (2)$$

Traditionally, solution (2) with the sign “plus” is called “kink,” whereas (2) with the sign “minus” is called “antikink.” It is easy to check that the conditions

$$(A) \quad F(z) \in \mathcal{C}^\infty(\mathbb{R}), \quad F(z) > 0 \text{ for } z \in (0, 1),$$

$$(B) \quad F^{(i)}(z_0) = 0, \quad i = 0, 1, \dots, k, \quad F^{(k+1)}(z_0) > 0,$$

where $z_0 = 0$ and $z_0 = 1$, and $k = 1$ or $k = 3$, are sufficient for the existence of kink/antikink solutions such that

$$|\omega'(\eta)| \leq c\eta^{-2} \quad \text{as } \eta \rightarrow \pm\infty. \quad (3)$$

Moreover, under the periodicity condition

$$(C) \quad F(z + 1) = F(z)$$

any combination of kink-antikink waves, that is,

$$u_\Sigma = \sum_{i=1}^N \omega \left(\pm \beta_i \frac{x - V_i t - x_i^0}{\varepsilon} \right), \quad x_{i+1}^0 - x_i^0 > 1, \quad 0 < t \ll 1, \quad (4)$$

will approximately sufficiently well the exact solution of the corresponding Cauchy problem. This brings up the question about the character of the interaction between the entities (2).

The first results of this topic have been obtained in [4–6], where the two-wave asymptotics have been constructed and then simulated numerically. In the present paper, we continue the cited investigations considering the interaction of three or more waves. The point is that there is a hypothesis (Danilov and Subochev [7], Boris Dubrovinsk's, private communication) that there are sufficiently many equations with sine-Gordon scenario of two solitary waves interaction, but three waves can interact in the same manner for the completely integrable equations only. In fact, the situation is more complicated. Our

main result consists in the conclusion that, in the leading term with respect to ε , two solitary waves (2) interact like sine-Gordon solitons but the stability of three or more waves depends on their parameters. Namely, structures with one kink and two antikinks or with one antikink and two kinks remain stable for special choice of the velocities, whereas interaction of four or more waves does not change the structure only for the entities of the same type (all of them are kinks or antikinks). Our main tool here is the numerical simulation.

The contents of the paper is the following. In Section 2 we present the asymptotics for the interaction of two solitary waves; the description of the finite difference scheme is contained in Section 3; in Section 4 we consider the numerical results.

2. Asymptotic Solution

For essentially nonintegrable interaction problems it is impossible to construct either explicit solutions (classical or weak) or asymptotics in the classical sense. However, it is possible to construct an asymptotic solution in the weak sense (see, e.g., [4, 5, 8–11] and references therein). The main advantage of this approach is the possibility of reducing the problem of describing nonlinear waves interaction to a qualitative analysis of some ordinary differential equations (instead of partial differential equations). This method takes into account the fact that kinks (as well as solitons [9, 10]) which are smooth for $\varepsilon > 0$ become nonsmooth in the limit as $\varepsilon \rightarrow 0$. So it is possible to treat such solutions as a mapping $\mathcal{C}^\infty((0, T); \mathcal{C}^\infty(\mathbb{R}_x^1))$ for $\varepsilon = \text{const} > 0$ and only as $\mathcal{C}((0, T); \mathcal{D}'(\mathbb{R}_x^1))$ uniformly in $\varepsilon \geq 0$. Accordingly, the remainder should be small in the weak sense. This rather trivial observation allowed to reach a progress for some old problems about nonlinear wave interaction for nonintegrable equations. As for the equations of the form (1), it should be noted that there is an obstacle to apply the standard $\mathcal{D}'$ construction. Indeed, in the $\mathcal{D}'$ sense, the differential terms of (1) are subordinated to the nonlinear term. Moreover, the left-hand side of (1) is of the value $O(\varepsilon^2)$ in the weak sense for any u of the form (4) and $t \ll 1$. Obviously, this prevents the construction of the correct asymptotics for the Cauchy problem. To overcome this obstacle, in [4] has been constructed a new definition of asymptotic solutions, which involves in the leading term the derivatives of u with arguments x/ε and t/ε .

Definition 1. A sequence $u(t, x, \varepsilon)$, belonging to $\mathcal{C}^\infty((0, T); \mathcal{C}^\infty(\mathbb{R}_x^1))$ for $\varepsilon = \text{const} > 0$ and belonging to $\mathcal{C}((0, T); \mathcal{D}'(\mathbb{R}_x^1))$ uniformly in ε , is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1) if the relation

$$2 \frac{d}{dt} \int_{-\infty}^{\infty} \varepsilon^2 u_t u_x \psi dx + \int_{-\infty}^{\infty} \{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\} \psi_x dx = O(\varepsilon^2) \quad (5)$$

holds uniformly in t for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

Here the right-hand side is a $\mathcal{C}^\infty$ -function for $\varepsilon = \text{const} > 0$ and a piecewise continuous function uniformly in $\varepsilon \geq 0$. The estimate is understood in the $\mathcal{C}(0, T)$ sense:

$$g(t, \varepsilon) = O(\varepsilon^k) \longleftrightarrow \max_{t \in [0, T]} |g(t, \varepsilon)| \leq c \varepsilon^k. \quad (6)$$

The left-hand side of (5) is the result of multiplication of (1) by $\psi(x)u_x$ and integration by parts in the case of smooth u . Therefore, the relation (5) is satisfied automatically for any exact solution. On the other hand, the relation (5) is just the orthogonality condition that appears for single-phase asymptotics [12, 13]. This condition both guarantees the first correction existence and allows to find an equation for the distorted kink's front motion.

Definition 2. A function $v(t, x, \varepsilon)$ is said to be of the value $O_{\mathcal{D}'}(\varepsilon^k)$ if the relation

$$\int_{-\infty}^{\infty} v(t, x, \varepsilon) \psi(x) dx = O(\varepsilon^k) \quad (7)$$

holds uniformly in t for any test function $\psi \in \mathcal{D}(\mathbb{R}_x^1)$.

Let us consider the interaction of two kinks,

$$u|_{t=0} = \sum_{i=1}^2 \omega \left(\beta_i \frac{x - x_i^0}{\varepsilon} \right), \quad (8)$$

$$\varepsilon \frac{\partial u}{\partial t} \Big|_{t=0} = - \sum_{i=1}^2 \beta_i V_i \omega' \left(\beta_i \frac{x - x_i^0}{\varepsilon} \right),$$

where $\beta_i = 1/\sqrt{1 - V_i^2}$, $|V_i| \in (0, 1)$, and the initial front positions x_i^0 are such that $x_2^0 - x_1^0 > 1$. Obviously, it is assumed that the trajectories $x = V_i t + x_i^0$ have a joint point $x = x^*$ at a time instant $t = t^*$.

The asymptotic ansatz for the problem (1), (8) has the following form:

$$u = \sum_{i=1}^2 \left\{ \omega \left(\beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\varepsilon} \right) + A_i(\tau) U \left(\beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\mu^2 \varepsilon} \right) \right\}. \quad (9)$$

Here $\Phi_i = \phi_{i0}(t) + \varepsilon \phi_{i1}(\tau)$, $\phi_{i0} = V_i t + x_i^0$ are the trajectories of the noninteracting kinks, $\tau = \psi_0(t)/\varepsilon$ denotes the “fast time,” $\psi_0(t) = \phi_{20}(t) - \phi_{10}(t)$. The phase corrections ϕ_{i1} are smooth functions such that

$$\phi_{i1} \rightarrow 0 \quad \text{as } \tau \rightarrow -\infty, \quad (10)$$

$$\phi_{i1} \rightarrow \phi_{i1}^\infty = \text{const}_i \quad \text{as } \tau \rightarrow +\infty,$$

with a rate not less than $1/|\tau|$. Furthermore, $A_i(\tau) \in \mathcal{C}^\infty$ are exponentially vanishing as $|\tau| \rightarrow \infty$ functions, μ is a sufficiently small parameter, $\varepsilon < \mu \ll 1$, and

$$U(\eta) = \frac{d^m U_0(\eta)}{d\eta^m}, \quad (11)$$

where $m \geq 1$ is an arbitrary number and $U_0(\eta) \in \mathcal{C}^\infty$ is a sufficiently fast vanishing function as $|\eta| \rightarrow \infty$.

The main result, which is known for the problem (1), (8), is the following.

Theorem 3 (see [4]). *Let the assumptions (A)–(C) hold. Set the additional assumptions*

$$(D) \ F(1/2 + z) = F(1/2 - z),$$

(E) *let the function $F(z)$ be such that the inequality*

$$\begin{aligned} & \int_{-\infty}^{\infty} F(\omega(\eta) + \omega(\theta\eta)) d\eta \\ & \leq \int_{-\infty}^{\infty} \left\{ \sqrt{F(\omega(\eta))} + \sqrt{F(\omega(\theta\eta))} \right\}^2 d\eta \end{aligned} \quad (12)$$

holds uniformly in $\theta \in (0, \infty)$. Then the interaction of kinks in the problem (1), (8) preserves the sine-Gordon scenario with accuracy $O_{\mathcal{D}'}(\varepsilon^2)$ in the sense of Definition 1. The weak asymptotic solution of (1), (8) has the form (9) with a special choice of the amplitudes A_i and of the parameter μ .

Remark 4. The symmetry (D) has been assumed to simplify the asymptotic analysis and it is not very important.

Remark 5. The sense of assumption (E) is the following. The phase corrections ϕ_{i1} are solutions of a 2×2 -dynamical system with a singularity whose support divides the phase plane into two parts with the possible exception of the point $(0, 0)$. Assumptions (10) are satisfied (consequently, the sine-Gordon scenario takes place) if and only if there exists a specific trajectory which goes from one half-plane to the other one through the point $(0, 0)$. When A_i in (9) are equal to zero, the existence of the trajectory implies the appearance of an additional very complicated assumption. This condition can be made more coarse and transformed into the simplest form (12). Such version can be treated as an admissible one since it is satisfied for the sine-Gordon equation for any velocities V_i , $i = 1, 2$. The same is true for the nonlinearity

$$F(u) = \sin^4(\pi u). \quad (13)$$

Taking into account a freedom in the choice of the amplitudes A_i , $i = 1, 2$, assumption (12) can be made weaker. However, the dynamical system with $A_i \neq 0$, $i = 1, 2$, is very complicated and its complete analysis remains undone.

Obviously, all stated above remains true for the antikink-antikink interaction.

Let us focus our attention in the kink-antikink interaction, that is, in (1) with initial data

$$\begin{aligned} u|_{t=0} &= \sum_{i=1}^2 \omega \left(S_i \beta_i \frac{x - x_i^0}{\varepsilon} \right), \\ \varepsilon \frac{\partial u}{\partial t} \Big|_{t=0} &= - \sum_{i=1}^2 S_i \beta_i V_i \omega' \left(S_i \beta_i \frac{x - x_i^0}{\varepsilon} \right), \end{aligned} \quad (14)$$

where $S_1 = 1$, $S_2 = -1$, and the notation β_i , V_i , x_i^0 is the same as in (8).

The asymptotic ansatz for the solution of the problem (1), (14) differs a little bit from (9), namely,

$$\begin{aligned} u &= \sum_{i=1}^2 \left\{ \omega \left(S_i \beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\varepsilon} \right) \right. \\ & \quad \left. + A_i(\tau) U \left(S_i \mu \beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\varepsilon} \right) \right\}, \end{aligned} \quad (15)$$

with the same notation and assumption (10).

Technically, the construction of (15) is similar to the kink-kink case. However, the resulting dynamical system for the phase corrections becomes much more complicated. Moreover, it is impossible to simplify the additional assumption, which appears here also, without loss of the adequacy. For this reason we do not present the explicit form of the additional condition but state only the existence of the weak asymptotics (15) under some restrictions for V_1 and V_2 . We refer the readers to [5] for the explicit statement.

Finally we note that there is a correspondence between weak asymptotic solutions and energy relations for (1).

Theorem 6. *Let the assumptions of Theorem 3 hold. Then two kinks (9) preserve mod $O_{\mathcal{D}'}(\varepsilon^2)$ their forms after the interaction if and only if they satisfy the conservation law*

$$\frac{d}{dt} \int_{-\infty}^{\infty} u_t u_x dx = 0 \quad (16)$$

and the energy relation

$$\begin{aligned} & 2 \frac{d}{dt} \int_{-\infty}^{\infty} x \varepsilon^2 u_t u_x dx \\ & + \int_{-\infty}^{\infty} \{ (\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u) \} dx = 0. \end{aligned} \quad (17)$$

Similar conclusion is true for the kink-antikink pair (15) [5].

3. Finite Differences Scheme

The actual numerical simulation for (1) is realized for a finite x -interval, $x \in [0, L]$. For this reason we simulate the Cauchy problem by the following mixed problem:

$$\begin{aligned} \varepsilon^2 (u_{tt} - u_{xx}) + F'(u) &= 0, \quad x \in (0, L), \quad t \in (0, T), \\ u|_{x=0} &= v_\ell, \quad u|_{x=L} = v_r, \end{aligned} \quad (18)$$

$$u|_{t=0} = u^0 \left(\frac{x}{\varepsilon} \right), \quad \varepsilon \frac{\partial u}{\partial t} \Big|_{t=0} = u^1 \left(\frac{x}{\varepsilon} \right),$$

where u^0 is a combination of kinks and antikinks of the form (4) and u^1 denotes its time derivative calculated at $t = 0$, $v_\ell = u^0|_{x=0}$, $v_r = u^0|_{x=L}$. To simulate by (18) the interaction phenomena, we assume that L , T , and the initial front positions x_i^0 , $i = 1, 2$, are such that the intersection point of the solitary wave fronts belongs to $Q_T = (0, T) \times (0, L)$. Furthermore, let L , T , and x_i^0 be such that uniformly in $t \leq T$

$$|u_\Sigma|_{x \in [0, \delta]} - v_\ell| \leq c\varepsilon^2, \quad |u_\Sigma|_{x \in [L-\delta, L]} - v_r| \leq c\varepsilon^2 \quad (19)$$

for some sufficiently small $\delta > 0$. Since it is impossible to create any finite difference scheme for the problem (18), which remains stable uniformly in $\varepsilon \rightarrow 0$ and $t \in (0, T)$, $T = \text{const}$, we will treat ε as a small but fixed constant. However, we will fix any relation between ε and finite differences scheme parameters.

To create a finite differences scheme for (18) we should choose appropriate approximations for the differential terms and for the nonlinear term. Let us do it separately.

3.1. Preliminary Nonlinear "Scheme". As usually, we define a mesh $Q_{T,\tau,h} = \{(x_i, t_j) = (ih, j\tau), i = 0, \dots, I, j = 0, \dots, J\}$ over Q_T and denote

$$\begin{aligned} y_i^j &= u(x_i, t_j), & y_{it}^j &= \frac{y_i^{j+1} - y_i^j}{\tau}, & y_{i\bar{t}}^j &= \frac{y_i^j - y_i^{j-1}}{\tau}, \\ y_{ix}^j &= \frac{y_{i+1}^j - y_i^j}{h}, & y_{i\bar{x}}^j &= \frac{y_i^j - y_{i-1}^j}{h}, & y_{i\bar{t}\bar{t}}^j &= (y_{it}^j)_t, \\ & & y_{ix\bar{x}}^j &= (y_{ix}^j)_{\bar{x}}. \end{aligned} \quad (20)$$

Let us consider the following system of nonlinear equations:

$$\begin{aligned} \varepsilon^2 (y_{i\bar{t}\bar{t}}^j - y_{ix\bar{x}}^{j+1}) + F'(y_i^{j+1}) &= 0, \\ i &= 1, \dots, I-1, \quad j = 2, 3, \dots, \\ y_0^j &= v_\ell, \quad y_I^j = v_r, \quad j = 0, 1, \dots, \\ y_i^0 &= u^0\left(\frac{x_i}{\varepsilon}\right), \quad \varepsilon y_{it}^0 = \tilde{u}^1\left(\frac{x_i}{\varepsilon}, \tau\right), \quad i = 0, \dots, I, \end{aligned} \quad (21)$$

where $\tilde{u}^1(x_i/\varepsilon, \tau)$ is such that last equality in (18) is approximated with accuracy $O(\tau^2)$. Obviously, the local approximation accuracy of (21) is $O(\tau^2 + h^2)$.

To simplify the notation, we will write

$$y := y_i^j, \quad \hat{y} := y_i^{j+1}, \quad \check{y} := y_i^{j-1}. \quad (22)$$

So the short form of (21) is the following:

$$\varepsilon^2 (y_{i\bar{t}\bar{t}} - \hat{y}_{x\bar{x}}) + F'(\hat{y}) = 0. \quad (23)$$

Our first result consists in obtaining the boundedness condition for the problem (21) solution.

Lemma 7 (see [6]). *Let ε be a sufficiently small constant and let*

$$\frac{\tau}{\varepsilon^2} \leq \text{const}. \quad (24)$$

Suppose that system (21) is solvable for any $j = 2, \dots, J$. Then uniformly in j

$$\begin{aligned} &\| \varepsilon y_t \|^2 + \| \varepsilon \hat{y}_x \|^2 + 2 \left\| \sqrt{F(\hat{y})} \right\|^2 \\ &\quad + \frac{\tau}{\varepsilon^2} \left\{ \left\| \varepsilon^2 y_{i\bar{t}\bar{t}} \right\|^2(j) + \left\| \varepsilon^2 y_{x\bar{x}} \right\|^2(j) \right\} \\ &\leq \left\{ \left\| \varepsilon y_t^0 \right\|^2 + \left\| \varepsilon y_x^1 \right\|^2 + 2 \left\| \sqrt{F(y^1)} \right\|^2 \right\} e^{ct_j\tau/\varepsilon^2} (1 + O(\tau)), \end{aligned} \quad (25)$$

where $\| \cdot \|$ and $\| \cdot \| (j)$ are the $\mathcal{L}^2$ norms, namely,

$$\| f \|^2 = h \sum_{i=1}^{I-1} |f_i|^2, \quad \| f \| (j)^2 = \tau \sum_{k=1}^j \| f^k \|^2. \quad (26)$$

Here and in what follows c denotes a const > 0 which does not depend on h, τ , and ε .

As a consequence of this lemma and the identity

$$y_i^j = y_i^0 + \tau \sum_{k=0}^{j-1} y_{it}^k, \quad (27)$$

we obtain the inequality

$$\begin{aligned} \| y^j \|^2 &\leq 2 \| y^0 \|^2 + 2 t_j \tau \sum_{k=0}^{j-1} \| y_t^k \|^2 \\ &\leq 2 \| y^0 \|^2 + 2 \frac{t_j^2}{\varepsilon^2} c_0, \end{aligned} \quad (28)$$

where $c_0 > 0$ denotes the right-hand side in (25). Obviously, this estimate is very rough. However, it can be improved a little for the specific initial data (8) and (14).

Lemma 8 (see [6]). *Let the assumptions of Lemma 7 be satisfied. Then for the initial data $u^0(x_i/\varepsilon)$, $\tilde{u}^1(x_i/\varepsilon, \tau)$, which approximate the Cauchy data (8) or (14), the following estimate holds uniformly in j :*

$$\sqrt{\varepsilon} \left\{ \| y_t^j \| + \| y_x^j \| + \| y^j \| \right\} \leq c. \quad (29)$$

3.2. Linearization. Now we should verify the solvability of (21) for any fixed $j \geq 1$, that is, of the equation

$$y^{j+1} - \tau^2 y_{x\bar{x}}^{j+1} + \frac{\tau^2}{\varepsilon^2} F'(y^{j+1}) = G^j, \quad G^j = y^j + \tau y_{it}^j, \quad (30)$$

as well as select a way to linearize the nonlinearity. To this aim let us construct the sequence of functions $\varphi(s) := \{\varphi_0(s), \dots, \varphi_I(s)\}$, $s \geq 0$, such that $\varphi(0) = y^j$ and $\varphi(s)$ for $s \geq 1$ satisfies the equation

$$\begin{aligned} \varphi(s) - \tau^2 \varphi_{x\bar{x}}(s) + \frac{\tau^2}{\varepsilon^2} \left\{ F'(\varphi(s-1)) + F''(\varphi(s-1)) \right. \\ \left. \times (\varphi(s) - \varphi(s-1)) \right\} = G^j, \end{aligned} \quad (31)$$

$$\varphi_0(s) = v_\ell, \quad \varphi_I(s) = v_r.$$

The solvability of the algebraic system (31) is obvious for sufficiently small τ and $\tau/\varepsilon^2 \leq \text{const}$. To simplify the notation we write $\varphi := \varphi(s)$, $\bar{\varphi} := \varphi(s-1)$, $\bar{\bar{\varphi}} := \varphi(s-2)$. Let also define

$$w := \varphi - \bar{\varphi}, \quad \bar{w} := \bar{\varphi} - \bar{\bar{\varphi}}. \quad (32)$$

Lemma 9 (see [6]). *Let assumption (24) be satisfied and let τ be sufficiently small. Then*

$$\begin{aligned} & \|\varphi\|^2 + \frac{\tau^2}{\varepsilon^2} \|\varepsilon \varphi_x\|^2 \\ & \leq (1 + c\tau) \left\{ \|y^j\|^2 + c\sqrt{\tau} \left(\|y^j\|^2 + \|\varepsilon y_t^j\|^2 \right) \right\} \end{aligned} \quad (33)$$

$$+ c \left\{ \|w\|^2 + \tau^2 \|w_x\|^2 \right\}^2,$$

$$g(1) \leq c\tau^{3/2}, \quad g(s) \leq c\tau g^2(s-1) \quad \text{for } s > 1, \quad (34)$$

where

$$g(s) := \|w(s)\|^2 + \tau^2 \|w_x(s)\|^2. \quad (35)$$

Thus we immediately conclude that the terms of the w -sequence vanish very rapidly,

$$\|w(1)\|^2 \leq c\tau^{3/2}, \quad \|w(2)\|^2 \leq c\tau^4, \quad \|w(3)\|^2 \leq c\tau^9, \dots \quad (36)$$

By virtue of (33), the terms of the φ -sequence are bounded uniformly in s

$$\|\varphi(s)\|^2 \leq \|y^j\|^2 (1 + O(\sqrt{\tau})). \quad (37)$$

Furthermore, for any $n > 0$

$$\begin{aligned} & \|\varphi(s+n) - \varphi(s)\| \\ & \leq \sum_{i=1}^n \|w_{s+i}\| \leq \|w_{s+1}\| \sum_{i=1}^{\infty} \frac{\|w_{s+i}\|}{\|w_{s+1}\|} \leq c \|w_{s+1}\|. \end{aligned} \quad (38)$$

This implies the main statement of this subsection.

Theorem 10 (see [6]). *Let assumption (24) be satisfied and $\varepsilon = \text{const}$. Then for sufficiently small τ the sequence φ converges in the $\mathcal{L}_h^2$ sense to the solution of (30). Moreover,*

$$\|y^{j+1} - \varphi(2)\| \leq c\tau^{9/2}. \quad (39)$$

3.3. Algorithm for the Numerical Simulation. Since the accuracy $O(\tau^{9/2})$ is much less than the accuracy of the finite differences scheme (21), we obtain the following algorithm for the numerical simulation of the problem (18) solution:

For any fixed $j = 1, 2, \dots, [T/\tau]$, $T = \text{const}$, we

- (i) define $\varphi(0) := y^j$,
- (ii) calculate $\varphi(s)$, $s = 1, 2$, accordingly with (31),
- (iii) define $y^{j+1} := \varphi(2)$, redefine $j := j + 1$, and go back to (i).

By virtue of the estimates (29) and (39), this algorithm allows to calculate bounded in $\mathcal{L}^2(Q_{T,h,\tau})$ numerical solution of the problem (18).

Note that this result can be improved. Moreover, it turns out that the algorithm is absolutely stable. To prove this we state firstly the proposition.

Lemma 11 (see [6]). *Let assumption (24) be satisfied and $\varepsilon = \text{const}$. Then uniformly in s*

$$\|\varepsilon \varphi_t(s)\| \leq \text{const}, \quad \|\varepsilon \varphi_x(s)\| \leq \text{const}. \quad (40)$$

Moreover, uniformly in j

$$\|\varepsilon^2 y_{xt}^j\| + \|\varepsilon^2 y_{xx}^{j+1}\| \leq \frac{c}{\sqrt{\varepsilon}},$$

$$\|\varepsilon y_t^{j+1} - \varepsilon \varphi_t(s)\| \leq c\varepsilon^{p(s)}, \quad \|\varepsilon y_x^{j+1} - \varepsilon \varphi_x(s)\| \leq c\varepsilon^{p(s)}, \quad (41)$$

with some $p(s)$ that tend to infinity as $s \rightarrow \infty$.

An immediate consequence of the Lemmas 7–11 is the following.

Theorem 12 (see [6]). *Let the assumption (24) be satisfied and $\varepsilon = \text{const}$. Then the solution of the above described finite differences scheme converges to the solution of the problem (18) as $\tau, h \rightarrow 0$ in the $W_2^1(Q_T)$ sense.*

Finally, in view of the boundedness of the sequence $\varphi^j(s)$, $s = 1, 2$, $j = 2, 3, \dots$, it is easy to establish our last statement.

Theorem 13 (see [6]). *Under the assumptions of Theorem 12 the above described finite differences scheme is stable in the $W_2^1(\Omega_{T,\tau,h})$ sense.*

4. Results of Numerical Simulation

The numerical algorithm has been realized as a program and tested using the sine-Gordon equation in the cases of one, two, three, and four solitary waves. Next we used the nonlinearities (13) and the following:

$$F(u) = 2 - \cos(2\pi u) - \cos(4\pi u), \quad (42)$$

$$F(u) = 1 + \sin\left(2\pi\left(u + \frac{5}{4}\right)\right) \cos\left(4\pi\left(u + \frac{5}{4}\right)\right). \quad (43)$$

Obviously, these functions satisfy the assumptions (A)–(D). Note also that functions (13), (42), and (43) for $u \in (0, 1)$ have one, three, and five critical points respectively. At the same time, for (13) $F^{(i)}(0) = 0$ if $i = 0, \dots, 3$ and $F^{(4)}(0) > 0$. Therefore, the explicit kink type solution for this case, that is,

$$\omega(\eta) = \frac{1}{\pi} \operatorname{arccot}\left(-\sqrt{2}\pi\eta\right), \quad (44)$$

tends to zero as η^{-1} when $\eta \rightarrow -\infty$. Conversely, for functions (42), and (43) $F^{(i)}(0) = 0$ if $i = 0, 1$ and $F^{(2)}(0) > 0$.

FIGURE 1: Evolution of the kink-kink pair, $V_1 > 0, V_2 < 0$.

Therefore, the kink type solutions for these cases tend to zero with exponential rates when $\eta \rightarrow -\infty$. However, their explicit form remains unknown; that is, why we simulate them solving numerically the Cauchy problem

$$\frac{d\omega}{d\eta} = \sqrt{2F(\omega)}, \quad \eta > 0, \quad \omega|_{\eta=0} = \frac{1}{2} \quad (45)$$

and, by virtue of the condition (C), define ω with negative argument as $\omega(\eta) = 1 - \omega(-\eta)$. To calculate the solution of (45) we use the Runge-Kutta method of the forth order with the mesh step $h_\eta = 0.01$.

4.1. Two-Wave Interaction. In accordance with the asymptotic analysis, two solitons interact preserving the shape; see Figures 1 and 2 for the nonlinearity (13) and $\varepsilon = 0.1$ (here and in what follows we numerate the waves from the left to the right). Moreover, this remains true independently of the wave parameters V_1, V_2 for all nonlinearities under consideration. Thus we can conclude that the additional conditions, which appear for the asymptotics, are restrictions of the asymptotic method only.

4.2. Three-Wave Interaction. There are 32 combinations of three solitons (kinks and antikinks) with trajectories which intersect at one instant of time. In view of the symmetry $x \rightarrow -x, t \rightarrow -t$ for (1), we can reduce this number to 16, considering only admissible combinations of three kinks and of two kinks and one antikink. Next we reduce the number of combinations to 12 using the symmetry $t \rightarrow -t$. All these combinations have been analyzed numerically with the same result for each of the nonlinearities (13), (42), (43): three kinks preserve the shape after the interaction for all admissible velocities, whereas there are only two stable combinations for two kinks and one antikink. The corresponding numerical results are depicted in Figures 3–5 for (43) with $\varepsilon = 0.1$. In Figure 6 we present the stable structure of one kink and two antikinks which is dual to Figure 4 in the sense the symmetry $x \rightarrow -x, t \rightarrow -t$. So that, there are 12 stable structures of kinks (K) and antikinks (A):

FIGURE 2: Evolution of the kink-antikink pair, $V_1 > 0, V_2 < 0$.

- (1) KKK: $(V_1 > 0, V_2 > 0, V_3 > 0), (V_1 > 0, V_2 > 0, V_3 < 0), (V_1 > 0, V_2 < 0, V_3 < 0), (V_1 < 0, V_2 < 0, V_3 < 0)$;
- (2) KKA: $(V_1 > 0, V_2 < 0, V_3 < 0)$;
- (3) KAA: $(V_1 > 0, V_2 > 0, V_3 < 0)$;
- (4) AAK: $(V_1 > 0, V_2 < 0, V_3 < 0)$;
- (5) AKK: $(V_1 > 0, V_2 > 0, V_3 < 0)$;
- (6) AAA: $(V_1 > 0, V_2 > 0, V_3 > 0), (V_1 > 0, V_2 > 0, V_3 < 0), (V_1 > 0, V_2 < 0, V_3 < 0), (V_1 < 0, V_2 < 0, V_3 < 0)$.

4.3. Interaction of Four or More Waves. We obtain again that the interaction of waves of the same type does not destroy the structure; see, for example, Figure 7 with 6 kinks, $V_i > 0, i = 1, 2, 3$, and $V_j < 0, j = 4, 5, 6$. As for waves of different types, we did not find any stable combination. In fact, even for four waves the number of possible combinations is too large to simulate each of them, so we checked a part of the combinations only. However, we guess that there are no stable structures for the following reason: each combination of four waves can be considered either as a union of a triplet with an additional wave on its left, or as a union of a triplet with an additional wave on its right. It turns out that one of these triplets should be unstable since the list of stable triplets is too scanty. Thus the total combination is unstable. We illustrate this in Figure 8 for the interaction of two kinks and two antikinks. Indeed, such combination can be treated as the union of the stable triplet depicted in the Figure 4 and the antikink on his right. On the contrary, the same combination is the union of the kink and the kink-antikink-antikink triplet with $V_i < 0, i = 1, 2, 3$, which is unstable.

5. Conclusion

Summarizing all stated above, we can deduce that there exists a family of nonlinearities such that kink-kink and kink-antikink pairs preserve the sine-Gordon scenario of interaction at least in the leading term in the asymptotic

FIGURE 3: Evolution of the kink-triplet, $V_1 > 0, V_2 < 0, V_3 < 0$.

FIGURE 6: Evolution of the antikink-antikink-kink triplet, $V_1 > 0, V_2 < 0, V_3 < 0$.

FIGURE 4: Evolution of the kink-kink-antikink triplet, $V_1 > 0, V_2 < 0, V_3 < 0$.

FIGURE 7: Interaction of 6 kinks for the nonlinearity (43).

FIGURE 5: Evolution of the antikink-kink-kink triplet, $V_1 > 0, V_2 > 0, V_3 < 0$.

FIGURE 8: Evolution of four waves for the nonlinearity (43), $V_1 > 0, V_2 < 0, V_3 < 0, V_4 < 0$.

sense. Apparently, this family can be specified by assumptions (A)–(D).

As for multiwave interactions, the situation is more complicated. Apparently, a sufficiently large number of solitons of the same type interact preserving the shape, whereas there are only four stable combinations of kinks and antikinks. In fact, this is rather unexpected, since single kink and antikink waves have the same properties. The second strange phenomenon is that there does not appear any perturbation of the radiation type, comparable with the collision of solitons for the KdV-type equations [11]. Let us note also that our algorithm (see Section 3) allows to eliminate “radiation due to discreteness effects” which appears for the trivial linearization; see [14].

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References

- [1] A. V. Zhiber and V. V. Sokolov, “Exactly integrable hyperbolic equations of Liouville type,” *Uspekhi Matematicheskikh Nauk*, vol. 56, no. 1, pp. 63–106, 2001, English translation in *Russian Mathematical Surveys*, vol. 56, no. 1, pp. 61–101, 2001.
- [2] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, “Method for solving the sine-Gordon equation,” *Physical Review Letters*, vol. 30, pp. 1262–1264, 1973.
- [3] V. E. Zakharov, L. A. Takhtadjan, and L. D. Faddeev, “Complete description of solutions of the “sine-Gordon” equation,” *Doklady Akademii Nauk SSSR*, vol. 219, pp. 1334–1337, 1974, English translation in *Soviet Physics—Doklady*, vol. 19, pp. 824–826, 1974.
- [4] D. A. Kulagin and G. A. Omel’yanov, “Interaction of kinks for semilinear wave equations with a small parameter,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 2, pp. 347–378, 2006.
- [5] M. G. Garcia and G. A. Omel’yanov, “Kink-antikink interaction for semi-linear wave equations with a small parameter,” *Electronic Journal of Differential Equations*, vol. 2009, no. 45, pp. 1–26, 2009.
- [6] G. A. Omel’yanov and I. Segundo-Caballero, “Asymptotic and numerical description of the kink/antikink interaction,” *Electronic Journal of Differential Equations*, vol. 2010, no. 150, pp. 1–19, 2010.
- [7] V. G. Danilov and P. Subochev, “Wave solutions of semilinear parabolic equations,” *Theoretical and Mathematical Physics*, vol. 89, no. 1, pp. 1029–1046, 1991.
- [8] V. G. Danilov and V. M. Shelkovich, “Generalized solutions of nonlinear differential equations and the Maslov algebras of distributions,” *Integral Transforms and Special Functions*, vol. 6, no. 1–4, pp. 171–180, 1998.
- [9] V. G. Danilov, G. A. Omel’yanov, and V. M. Shelkovich, “Weak asymptotics method and interaction of nonlinear waves,” in *Asymptotic Methods for Wave and Quantum Problems*, M. V. Karasev, Ed., vol. 208 of *American Mathematical Society Translations 2*, pp. 33–164, American Mathematical Society, Providence, RI, USA, 2003.
- [10] V. G. Danilov and G. A. Omel’yanov, “Weak asymptotics method and the interaction of infinitely narrow δ -solitons,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 4, pp. 773–799, 2003.
- [11] M. G. Garcia and G. A. Omel’yanov, “Interaction of solitary waves for the generalized KdV equation,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 8, pp. 3204–3218, 2012.
- [12] V. P. Maslov and G. A. Omel’yanov, “Asymptotic soliton-form solutions of equations with small dispersion,” *Uspekhi Matematicheskikh Nauk*, vol. 36, no. 3, pp. 63–126, 1981, English translation in *Russian Mathematical Surveys*, vol. 36, no. 3, pp. 73–149, 1981.
- [13] V. P. Maslov and G. A. Omel’yanov, *Geometric Asymptotics For Non-Linear PDE, I.*, vol. 202 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 2001.
- [14] J. F. Currie, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, “Numerical simulation of sine-Gordon soliton dynamics in the presence of perturbations,” *Physical Review B*, vol. 15, no. 12, pp. 5567–5580, 1977.