Research Article

Essentially $\lambda$-Hankel Operators

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The notion of essentially $\lambda$-Hankel operators is introduced on the space $H^2$. In addition to the discussion of some algebraic and topological properties of the set $\text{essHank}_\lambda$, the set of all essentially $\lambda$-Hankel operators on $H^2$, it is shown that an essentially Toeplitz Rhaly operator with determining sequence $\langle a_n \rangle$ is in $\text{essHank}_\lambda(\lambda \neq 0)$ if and only if $\lim_{n \to \infty} (n+1)|a_n| = 0$.

1. Introduction

The notion of Toeplitz operators was introduced by Toeplitz [1] in the year 1911. Hankel operators are the formal companions of Toeplitz operators. It is well known that Toeplitz and Hankel operators are characterized as solutions to the operator equations $S^*TS = T$ and $S^*H = HS$, respectively, where $S$ denotes the unilateral forward shift on $H^2$. The solutions of the operator equation $S^*XS = \lambda X$ (for an arbitrary complex number $\lambda$) were described by Sun in the year 1984 [2]. In the year 2002, Avendaño [3] introduced the notion of $\lambda$-Hankel operators as those operators $X$ which satisfy the operator equation $S^*X - XS = \lambda X$. In a different direction, Avendaño [4] studied the notion of Hankel operators in reference to the Calkin algebra $\mathcal{B}(H^2)/\mathcal{K}(H^2)$ and introduced the notion of essentially Hankel operators on $H^2$. Motivated by these developments, in this paper, we introduce and study the notion of essentially $\lambda$-Hankel operators on the space $H^2$.

For a fixed complex number $\lambda$, we denote the set of all essentially $\lambda$-Hankel operators on $H^2$ by $\text{essHank}_\lambda$. The set $\text{essHank}_\lambda$ is shown to be a norm-closed vector subspace of $\mathcal{B}(H^2)$ containing no essentially invertible operator. It is shown that for a general $\lambda$, $\text{essHank}_\lambda$ is neither an algebra of operators on $H^2$ nor a self-adjoint set. Although the set $\text{essHank}_\lambda$ contains no nonzero Toeplitz operators, it turns out to be invariant under multiplication by Toeplitz operators. It turns out that $\text{essToep} \cap \text{essHank}_\lambda$ is an algebra without identity, where $\text{essToep}$ denotes the class of all essentially Toeplitz operators on $H^2$. In particular, for purely imaginary $\lambda$, $\text{essToep} \cap \text{essHank}_\lambda$ is a $C^*$-algebra. We also show that if $\lambda \neq 0$, $\text{essToep} \cap \text{essHank}_\lambda$ contains no noncompact Rhaly operators.

We begin with the following.

Definition 1 (See [5]). A bounded linear operator $T$ on $H^2$ is said to be an essentially Toeplitz operator if it satisfies

$$S^*TS - T = K,$$

for some compact operator $K$ on $H^2$. The set of all essentially Toeplitz operators is denoted by $\text{essToep}$.

Definition 2 (See [4]). A bounded linear operator $X$ on $H^2$ is said to be an essentially Hankel operator if it satisfies

$$S^*X - XS = K,$$

for some compact operator $K$ on $H^2$. The set of all essentially Hankel operators on $H^2$ is denoted by $\text{essHank}$. For more details, one can refer to [4].

Definition 3 (See [3]). $X$ is said to be a $\lambda$-Hankel operator if it satisfies $S^*X - XS = \lambda X$.

Clearly, a 0-Hankel operator is just a Hankel operator.
Definition 4 (See [6]). Given a sequence $a = (a_n)$ of scalars, the Rhaly matrix (terraced matrix) $R_a$ [6] is defined as

$$R_a = \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$  

This means that Rhaly matrices are lower triangular matrices with constant row segments. It is known that if $|na_n|$ is bounded, then the Rhaly matrix $R_a$ represents a bounded linear operator on the space $\ell^2$ (identified with $H^2$) and $\|R_a\| \leq 2\sup_{n \in \mathbb{N}} |na_n|$.

We now introduce the notion of essentially $\lambda$-Hankel operators on the space $H^2$ as follows.

Definition 5. For a fixed complex number $\lambda$, a bounded linear operator $X$ on $H^2$ is said to be an essentially $\lambda$-Hankel operator if it satisfies the operator equation

$$\begin{align*}
(S^* - \lambda I)X - XS &= K,
\end{align*}$$  

for some compact operator $K$ on $H^2$, $S$ denoting the unilateral forward shift on $H^2$.

We denote the set of all essentially $\lambda$-Hankel operators on $H^2$ by $\text{essHank}_\lambda$. Some basic facts and observations about $\text{essHank}_\lambda$ which follow from the definition itself are as follows:

(i) $\text{essHank}_\lambda \cap \mathcal{K}(H^2) = \mathcal{K}(H^2)$, where $\mathcal{K}(H^2)$ denotes the ideal of all compact operators on $H^2$.

(ii) Since the zero operator on $H^2$ is compact, every $\lambda$-Hankel operator is in $\text{essHank}_\lambda$.

(iii) If $\lambda = 0$, then $\text{essHank}_0 = \text{essHank}_0 = \text{essHank}$, where $\text{essHank}$ is the set of all essentially Hankel operators on $H^2$ introduced by Avendaño [4].

(iv) If $H$ is a Hankel operator, then $S^*H = HS$. Therefore, every Hankel operator is in $\text{essHank}_0$.

(v) We see that the operator $S^* - S - \lambda I$ cannot be compact on $H^2$, for any complex number $\lambda$. Therefore, if $X$ is a compact operator on $H^2$, then $X + K$ is in $\text{essHank}_\lambda$ for any $\lambda$, where $I$ denotes the identity operator on $H^2$.

(vi) For $X_1, X_2 \in \text{essHank}_\lambda$, $X_1X_2 \in \text{essHank}_\lambda$ if and only if

$$X_1(S^* - S - \lambda I)X_2 \in \mathcal{K}(H^2),$$  

(vii) from the definition itself, it is clear that if $X$ is a $\lambda$-Hankel operator and $K$ is a compact operator on $H^2$, then $X + K$ is in $\text{essHank}_\lambda$. That is, compact perturbations of $\lambda$-Hankel operators are in $\text{essHank}_\lambda$.

It was shown by Avendaño that the reverse implication is not true for the case $\lambda = 0$. Avendaño [4] proved that the Cesaro operator (i.e., the Rhaly operator corresponding to the sequence $\langle 1/n \rangle$) whose matrix with respect to the standard orthonormal basis is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$  

is in $\text{essHank}_0$ but is not expressible in $0$-Hankel plus compact form.

(viii) $\text{essHank}_\lambda$ is a norm-closed vector subspace of $\mathcal{B}(H^2)$, the space of all bounded linear operators on $H^2$.

Proof. For $\alpha, \beta \in \mathbb{C}$ and $X_1, X_2 \in \text{essHank}_\lambda$, we have

$$\begin{align*}
(S^* - \lambda I)(\alpha X_1 + \beta X_2) &= (\alpha X_1 + \beta X_2)S \\
&= (\alpha S^* - \lambda I)X_1 + (\beta S^* - \lambda I)X_2 \\
&= \alpha[X_1 + \beta X_2] \\
&= \alpha K_0 + \beta K_2,
\end{align*}$$  

where $K_1, K_2 \in \mathcal{K}(H^2)$. Therefore, $\alpha X_1 + \beta X_2 \in \text{essHank}_\lambda$.

Also, if $\langle X_n \rangle \to X$ in $\mathcal{B}(H^2)$, where each $X_n$ is in $\text{essHank}_\lambda$, then $\langle (S^* - \lambda I)X_n \rangle \to (S^* - \lambda I)X$ in $\mathcal{B}(H^2)$. As each $X_n \in \text{essHank}_\lambda$ and $\mathcal{B}(H^2)$ is a uniformly closed subspace of $\mathcal{B}(H^2)$, it follows that $(S^* - \lambda I)X - XS \in \mathcal{K}(H^2)$. Thus, $X \in \text{essHank}_\lambda$. Hence, the conclusion is clear.

We see that if $X \in \text{essHank}_\lambda$, then

$$S^*X - XS = K, \quad \text{where } K \in \mathcal{K}(H^2).$$  

On taking adjoints on both the sides, we get

$$S^*X^* - X^*S = -K^*, \quad \text{where } K^* \in \mathcal{K}(H^2).$$  

Therefore, $X^* \in \text{essHank}_\lambda$.

Thus, $\text{essHank}_\lambda$ is a self-adjoint set. We show that for a general complex number $\lambda$, this is not the case. That is, $\text{essHank}_\lambda$ is not a self-adjoint set in general. For this we begin with the following.

Theorem 6. If $\lambda$ and $\mu$ are distinct complex numbers, then

$$\text{essHank}_\lambda \cap \text{essHank}_\mu = \emptyset.$$  

Proof. Let $X \in \text{essHank}_\lambda \cap \text{essHank}_\mu$. Then,

$$\begin{align*}
(S^* - \lambda I)X - XS &= K_1, \\
(S^* - \mu I)X - XS &= K_2,
\end{align*}$$  

where $K_1, K_2 \in \mathcal{K}(H^2)$. Subtracting the previous two equations we have

$$\begin{align*}
(\lambda - \mu)X &= \frac{1}{2}(K_1 - K_2), \\
(\lambda - \mu)X &= \frac{1}{2}(K_2 - K_1),
\end{align*}$$  

where $K_1, K_2 \in \mathcal{K}(H^2)$.
where \( \lambda - \mu \neq 0 \). This implies that \( X \in \mathcal{H}(H^2) \). Therefore, \( \text{essHank}_\lambda \cap \text{essHank}_\mu \subseteq \mathcal{H}(H^2) \). Reverse inclusion is obvious by the definition.

**Theorem 7.** If \( X \in \text{essHank}_\lambda \), then \( X^* \in \text{essHank}_\mu \), where \( \mu = -\bar{\lambda} \).

*Proof.* Let \( X \in \text{essHank}_\lambda \). Then,

\[
(S^* - \lambda I) X - X S = K, \tag{13}
\]

for some compact operator \( K \) on \( H^2 \). Taking adjoints on both sides of (13), we obtain

\[
X^* (S - \bar{\lambda} I) - S^* X^* = K^*, \tag{14}
\]

where \( K^* \in \mathcal{H}(H^2) \). Thus, \([S^* - (\bar{\lambda} - I)]X^* - X^* S \in \mathcal{H}(H^2) \). This means that \( X^* \in \text{essHank}_\mu \), where \( \mu = -\bar{\lambda} \). \( \square \)

**Corollary 8.** A necessary condition for a noncompact operator \( X \) in \( \text{essHank}_\lambda \), \( \lambda \neq 0 \), to be self-adjoint is that \( \lambda \) is purely imaginary.

*Proof.* Let \( X \in \text{essHank}_\lambda \), where \( X \) is a noncompact operator on \( H^2 \). Then, \( X^* \in \text{essHank}_\mu \), where \( \mu = -\bar{\lambda} \). Now, if \( X = X^* \), then \( X, X^* \in \text{essHank}_\lambda \cap \text{essHank}_\mu \). As \( X \) is non-compact, we have \( \lambda = \mu \). That is, \( \lambda + \bar{\lambda} = 0 \). That is, \( \lambda \) is purely imaginary. \( \square \)

**Theorem 9.** Let \( X \in \text{essHank}_\lambda \). Then, \( 0 \in \sigma_e(X) \), where \( \sigma_e(X) \) denotes the essential spectrum of the operator \( X \).

*Proof.* Let \( X \in \text{essHank}_\lambda \). Then,

\[
(S^* - \lambda I) X - X S = K, \tag{15}
\]

where \( K \) is a compact operator on \( H^2 \).

Case (i). \( \lambda = 0 \): In this case, \( X \) satisfies

\[
S^* X - X S = K, \tag{16}
\]

where \( K \in \mathcal{H}(H^2) \). Clearly, \( X \) cannot be Fredholm, for if \( X \) is a Fredholm operator of index \( n \), then \( S^* X \) and \( XS + K \) are Fredholm operators with indices \( n + 1 \) and \( n - 1 \), respectively, leading to \( 1 = -1 \), which is absurd. Therefore, \( 0 \in \sigma_e(X) \).

Case (ii). \( \lambda \neq 0 \): In this case, if \( X \) is essentially invertible, then \( (S^* - \lambda I) - XSX^{-1} \) is a compact operator on \( H^2 \). This leads to the essential similarly of \( S^* - \lambda I \) and \( S \), which is a contradiction as \( \sigma_e(S) = \sigma_e(S^*) = \mathbb{T} \), where \( \mathbb{T} \) denotes the unit circle in the complex plane. So, \( 0 \in \sigma_e(X) \) in this case also. \( \square \)

In the next theorem, we show that there is no nonzero Toeplitz operator in \( \text{essHank}_\lambda \). For this we need the following lemmas.

**Lemma 10.** A nonzero Toeplitz operator cannot be a \( \lambda \)-Hankel operator.

*Proof.* Let \( T \) be a nonzero Toeplitz operator. Then,

\[
S^* TS = T. \tag{17}
\]

If possible, suppose that \( T \) is a \( \lambda \)-Hankel also. Then,

\[
(S^* - \lambda I) T = TS. \tag{18}
\]

From (17) and (18), it follows that

\[
(\lambda T + TS) S = T. \tag{19}
\]

That is,

\[
T e_{n+2} = T e_n - \lambda T e_{n+1}, \tag{20}
\]

for all \( n \geq 0 \). This means that \( T \) is finite dimensional and hence a compact operator on \( H^2 \). But nonzero Toeplitz operators are never compact. Thus, we have a contradiction and the conclusion follows. \( \square \)

**Lemma 11.** If \( T \) is a nonzero Toeplitz operator, then so is \( S^* T - TS \), \( S \) denoting the unilateral forward shift on \( H^2 \).

Using the previous two lemmas, we now prove that the set \( \text{essHank}_\lambda \) contains no nonzero Toeplitz operator.

**Theorem 12.** \( \text{essHank}_\lambda \cap \mathcal{J} = \{0\} \), where \( \mathcal{J} \) denotes the set of all Toeplitz operators on \( H^2 \).

*Proof.* Let \( T \in \text{essHank}_\lambda \cap \mathcal{J} \). Then,

\[
S^* TS = T, \tag{21}
\]

\[
(S^* - \lambda I) T - TS = K, \tag{22}
\]

for some compact operator \( K \) on \( H^2 \). Since \( T \) is a Toeplitz operator, \( S^* T - TS \) is also a Toeplitz operator on \( H^2 \). It follows that \( (S^* - \lambda I) T - TS \) is a Toeplitz operator on \( H^2 \). As a nonzero Toeplitz operator cannot be compact, we must have \( (S^* - \lambda I) T - TS = 0 \). That is, \( T \) is a \( \lambda \)-Hankel operator on \( H^2 \). Now, using Lemma 10, we get that \( T = 0 \). Hence,

\[
\text{essHank}_\lambda \cap \mathcal{J} = \{0\}. \tag{23}
\]

In the next theorem, we show that the set \( \text{essHank}_\lambda \) is invariant under multiplication by Toeplitz operators.

**Theorem 13.** If \( X \in \text{essHank}_\lambda \) and \( T_\phi \) is any Toeplitz operator on \( H^2 \), then \( XT_\phi \) and \( T_\phi X \) both are in \( \text{essHank}_\lambda \).

*Proof.* Let \( T_\phi \) be a Toeplitz operator on the space \( H^2 \). Then, we have

\[
S^* T_\phi S = T_\phi. \tag{24}
\]

Since \( S \) is essentially unitary, we have the commutator of \( T_\phi \) with \( S \), and that with \( S^* \) is compact on \( H^2 \). Now, let \( X \in \text{essHank}_\lambda \). Then,
(i) consider
\[(XT_\phi)S = (XS)T_\phi \pmod{\mathcal{H}(H^2)} \] (24)
Therefore,
\[(XT_\phi)S - (S^* - \lambda I)(XT_\phi) \in \mathcal{H}(H^2). \] (25)

(ii) Consider
\[(S^* - \lambda I)(T_\phi X) \]
\[= S^*T_\phi X - \lambda T_\phi X \]
\[= T_\phi(S^*X - \lambda T_\phi X) \pmod{\mathcal{H}(H^2)} \] (26)
Therefore,
\[(T_\phi X)S - (S^* - \lambda I)(T_\phi X) \in \mathcal{H}(H^2). \] (27)
Hence, the conclusion is clear.

More generally, we have the following.

**Theorem 14.** If \(X \in \text{essHank}_\lambda\) and \(T \in \text{essToep}\), then \(XT, TX \in \text{essHank}_\lambda\).

**Proof.** Let \(X \in \text{essHank}_\lambda\) and \(T \in \text{essToep}\). Then,
\[(S^* - \lambda I)(X) - XS \in \mathcal{H}(H^2), \]
\[ST - TS \in \mathcal{H}(H^2). \] (28)

Now,
\[(S^* - \lambda I)(XT) - (XT)S \]
\[= (S^* - \lambda I)(XT) - X(ST) \pmod{\mathcal{H}(H^2)} \] (29)
\[= [(S^* - \lambda I)X - XS]T \pmod{\mathcal{H}(H^2)} \]
\[\in \mathcal{H}(H^2). \]
Therefore, \(XT \in \text{essHank}_\lambda\).

Also,
\[(S^* - \lambda I)(TX) - (TX)S \]
\[= TS^*X - \lambda TXT - TXS \pmod{\mathcal{H}(H^2)} \] (30)
\[= T[(S^* - \lambda I)X - XS] \pmod{\mathcal{H}(H^2)} \]
\[\in \mathcal{H}(H^2). \]
Therefore, \(TX \in \text{essHank}_\lambda\).

We mention here that the previous result was proved for the special case \(\lambda = 0\) by Avendaño [7]. It is easy to see that \(\text{essToep}\) is a \(C^*\)-algebra. This fact together with the previous theorem gives us that \(\text{essToep} \cap \text{essHank}_\lambda\) is an algebra of operators on \(H^2\) as shown in the following.

**Theorem 15.** \(\text{essToep} \cap \text{essHank}_\lambda\) is an algebra of operators on \(H^2\) with no identity.

**Proof.** Since \(\text{essToep}\) is a \(C^*\)-algebra and \(\text{essHank}_\lambda\) is a vector subspace of \(\mathcal{B}(H^2)\), it follows that \(\text{essToep} \cap \text{essHank}_\lambda\) is a vector subspace of \(\mathcal{B}(H^2)\). Also, if \(X_1, X_2 \in \text{essToep} \cap \text{essHank}_\lambda\), then using Theorem 14, we get that \(X_1X_2 \in \text{essToep} \cap \text{essHank}_\lambda\). Thus, \(\text{essToep} \cap \text{essHank}_\lambda\) is an algebra of operators on \(H^2\). As \(I \notin \text{essHank}_\lambda\), the theorem is proved.

**Remark 16.** We have seen that if \(\lambda\) is purely imaginary, then \(\text{essHank}_\lambda\) is a self-adjoint set. Therefore, for purely imaginary \(\lambda\), \(\text{essToep} \cap \text{essHank}_\lambda\) turns to be a \(C^*\)-algebra.

**Theorem 17.** If \(X \in \text{essHank}_\lambda\), then
\[S\left(\text{Ker} X\right) \subseteq \text{Ker} (X + K), \] (31)
for some compact operator \(K\) on \(H^2\).

**Proof.** Let \(X \in \text{essHank}_\lambda\). Then,
\[(S^* - \lambda I)X - XS = K, \] (32)
for some compact operator \(K\) on \(H^2\).

If \(f \in \text{Ker} X\), then \(X f = 0\). From (32) it follows that \((XS + K)f = 0\). That is, \((X + KS^*)f = 0\). This implies that \((X + K)Sf = 0\), where \(K = KS^* \in \mathcal{H}(H^2)\). Thus, \(Sf \in \text{Ker} (X + K^*)\), where \(K^* \in \mathcal{H}(H^2)\). Hence, the result is clear.

**Corollary 18.** If \(X \in \text{essHank}_\lambda\), then
\[S\left(\text{Range} X\right)^\perp \subseteq \left[\text{Range} (X + K)\right]^\perp, \] (33)
for some compact operator \(K\) on \(H^2\).

**Proof.** Let \(X \in \text{essHank}_\lambda\). Then, \(X^* \in \text{essHank}_\mu\), where \(\mu = -\lambda\). Applying Theorem 17 to \(X^*\), we get
\[S\left(\text{Ker} X^*\right) \subseteq \text{Ker} (X^* + K), \] (34)
for some compact operator \(K\) on \(H^2\). This means that
\[S\left(\text{Ker} X^*\right) \subseteq \text{Ker} [\left(\text{Ker} X^*\right)^*], \] (35)
Therefore,
\[S\left(\text{Range} X\right)^\perp \subseteq \left[\text{Range} (X + K^*)\right]^\perp, \] (36)
where \(K^* \in \mathcal{H}(H^2)\). So, the conclusion follows.
Remark 19. It is known that $[3, 7]$ kernel of a $\lambda$-Hankel operator is an invariant subspace of $S$, and closure of the range of a $\lambda$-Hankel operator is an invariant subspace of $S^\ast$. We mention here that these facts about $\lambda$-Hankel operators follow as deductions to Theorem 17 and Corollary 18. For if $X$ is $\lambda$-Hankel then $K = 0$ in Theorem 17 and Corollary 18 leading to $S(KerX) \subseteq KerX$ and $S^\ast(Range X) \subseteq Range X$, respectively.

It is known that a Rhaly operator $R_\alpha$ is essentially Toeplitz if and only if it is essentially Hankel. That is, $R_\alpha \in essToep$ if and only if $R_\alpha \in essHank$. We show that this is not the case if $\lambda \neq 0$. In fact, an essentially Toeplitz Rhaly operator is in $essHank_\lambda(\lambda \neq 0)$ if and only if it is compact. Precisely, we have the following.

Theorem 20. Let $R_\alpha \in essToep$ be a Rhaly operator with determining sequence $a = \langle a_n \rangle \in \ell^2$. Then $R_\alpha \in essHank_\lambda (\lambda \neq 0)$ if and only if $\lim_{n \to \infty} (n+1)|a_n| = 0$.

Proof. Let $R_\alpha \in essToep$. Then, $R_\alpha \in essHank_0 [4]$. Now if $R_\alpha \in essHank_\lambda (\lambda \neq 0)$, then we have $R_\alpha \in essHank_0 \cap essHank_\lambda$. As $\lambda \neq 0$, we have $R_\alpha \in \mathcal{R}(H^2)$, by Theorem 6. It is known that $[8]$ a Rhaly operator $R_\alpha$ with determining sequence $a = \langle a_n \rangle$ is compact if and only if $\lim_{n \to \infty}(n+1)|a_n| = 0$. The desired result follows. \qed

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