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Research Article

Essentially λ-Hankel Operators

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The notion of essentially λ-Hankel operators is introduced on the space H^2. In addition to the discussion of some algebraic and topological properties of the set essHank$_\lambda$, the set of all essentially λ-Hankel operators on H^2, it is shown that an essentially Toeplitz Rhaly operator with determining sequence $\langle a_n \rangle$ is in essHank$_\lambda$ $(\lambda \neq 0)$ if and only if $\lim_{n\to\infty}(n+1)|a_n| = 0$.

1. Introduction

The notion of Toeplitz operators was introduced by Toeplitz [1] in the year 1911. Hankel operators are the formal companions of Toeplitz operators. It is well known that Toeplitz and Hankel operators are characterized as solutions to the operator equations $S^*TS = T$ and $S^*H = HS$, respectively, where S denotes the unilateral forward shift on H^2. The solutions of the operator equation $S^*XS = \lambda X$ (for an arbitrary complex number λ) were described by Sun in the year 1984 [2]. In the year 2002, Avendaño [3] introduced the notion of λ-Hankel operators as those operators X which satisfy the operator equation $S^*X - XS = \lambda X$. In a different direction, Avendaño [4] studied the notion of Hankel operators in reference to the Calkin algebra $\mathscr{B}(H^2)/\mathscr{K}(H^2)$ and introduced the notion of essentially Hankel operators on H^2. Motivated by these developments, in this paper, we introduce and study the notion of essentially λ-Hankel operators on the space H^2.

For a fixed complex number λ, we denote the set of all essentially λ-Hankel operators on H^2 by essHank$_\lambda$. The set essHank$_\lambda$ is shown to be a norm-closed vector subspace of $\mathscr{B}(H^2)$ containing no essentially invertible operator. It is shown that for a general λ, essHank$_\lambda$ is neither an algebra of operators on H^2 nor a self-adjoint set. Although the set essHank$_\lambda$ contains no nonzero Toeplitz operators, it turns out to be invariant under multiplication by Toeplitz operators. It turns out that essToep $\cap$ essHank$_\lambda$ is an algebra without identity, where essToep denotes the class of all essentially Toeplitz operators on H^2. In particular, for purely imaginary λ, essToep $\cap$ essHank$_\lambda$ is a C^*-algebra. We also show that if $\lambda \neq 0$, essToep $\cap$ essHank$_\lambda$ contains no noncompact Rhaly operators.

We begin with the following.

Definition 1 (See [5]). A bounded linear operator T on H^2 is said to be an essentially Toeplitz operator if it satisfies

$$S^*TS - T = K, \tag{1}$$

for some compact operator K on H^2. The set of all essentially Toeplitz operators is denoted by essToep.

Definition 2 (See [4]). A bounded linear operator X on H^2 is said to be an essentially Hankel operator if it satisfies

$$S^*X - XS = K, \tag{2}$$

for some compact operator K on H^2. The set of all essentially Hankel operators on H^2 is denoted by essHank. For more details, one can refer to [4].

Definition 3 (See [3]). X is said to be a λ-Hankel operator if it satisfies $S^*X - XS = \lambda X$.

Clearly, a 0-Hankel operator is just a Hankel operator.

Definition 4 (See [6]). Given a sequence $a = \langle a_n \rangle$ of scalars, the Rhaly matrix (terraced matrix) R_a [6] is defined as

$$R_a = \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_2 & 0 & 0 & \cdots \\ a_3 & a_3 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}. \tag{3}$$

This means that Rhaly matrices are lower triangular matrices with constant row segments. It is known that [6] if $\{na_n\}$ is bounded, then the Rhaly matrix R_a represents a bounded linear operator on the space ℓ^2 (identified with H^2) and $\|R_a\| \leq 2\sup_{n\in\mathbb{N}}|na_n|$.

We now introduce the notion of essentially λ-Hankel operators on the space H^2 as follows

Definition 5. For a fixed complex number λ, a bounded linear operator X on H^2 is said to be an essentially λ-Hankel operator if it satisfies the operator equation

$$(S^* - \lambda I) X - XS = K, \tag{4}$$

for some compact operator K on H^2, S denoting the unilateral forward shift on H^2.

We denote the set of all essentially λ-Hankel operators on H^2 by essHank_λ. Some basic facts and observations about essHank_λ which follow from the definition itself are as follows:

(i) $\text{essHank}_\lambda \cap \mathscr{K}(H^2) = \mathscr{K}(H^2)$, where $\mathscr{K}(H^2)$ denotes the ideal of all compact operators on H^2,

(ii) since the zero operator on H^2 is compact, every λ-Hankel operator is in essHank_λ,

(iii) if $\lambda = 0$, then $\text{essHank}_\lambda = \text{essHank}_0 = \text{essHank}$, where essHank is the set of all essentially Hankel operators on H^2 introduced by Avendaño [4],

(iv) If H is a Hankel operator, then $S^*H = HS$. Therefore, every Hankel operator is in essHank_0.

(v) we see that the operator $S^* - S - \lambda I$ cannot be compact on H^2, for any complex number λ. Therefore $I \notin \text{essHank}_\lambda$, for any λ, where I denotes the identity operator on H^2,

(vi) for $X_1, X_2 \in \text{essHank}_\lambda$, $X_1X_2 \in \text{essHank}_\lambda$ if and only if

$$X_1 (S^* - S - \lambda I) X_2 \in \mathscr{K}(H^2), \tag{5}$$

(vii) from the definition itself, it is clear that if X is a λ-Hankel operator and K is a compact operator on H^2, then $X + K$ is in essHank_λ. That is, compact perturbations of λ-Hankel operators are in essHank_λ. It was shown by Avendaño that the reverse implication is not true for the case $\lambda = 0$. Avendaño [4] proved that the Cesaro operator (i.e., the Rhaly operator corresponding to the sequence $\langle 1/n \rangle$) whose matrix with respect to the standard orthonormal basis is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{6}$$

is in essHank_0 but is not expressible in 0-Hankel plus compact form,

(viii) essHank_λ is a norm-closed vector subspace of $\mathscr{B}(H^2)$, the space of all bounded linear operators on H^2.

Proof. For α, β in $\mathbb{C}$ and $X_1, X_2 \in \text{essHank}_\lambda$, we have

$$\begin{aligned} &(S^* - \lambda I)(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2) S \\ &\quad = \alpha [(S^* - \lambda I) X_1 - X_1 S] + \beta [(S^* - \lambda I) X_2 - X_2 S] \\ &\quad = \alpha K_1 + \beta K_2, \end{aligned} \tag{7}$$

where $K_1, K_2 \in \mathscr{K}(H^2)$. Therefore, $\alpha X_1 + \beta X_2 \in \text{essHank}_\lambda$. Also, if $\langle X_n \rangle \to X$ in $\mathscr{B}(H^2)$, where each X_n is in essHank_λ, then $\langle (S^* - \lambda I)X_n - X_n S \rangle \to (S^* - \lambda I)X - XS$ in $\mathscr{B}(H^2)$. As each $X_n \in \text{essHank}_\lambda$ and $\mathscr{K}(H^2)$ is a uniformly closed subspace of $\mathscr{B}(H^2)$, it follows that $(S^* - \lambda I)X - XS \in \mathscr{K}(H^2)$. Thus, $X \in \text{essHank}_\lambda$. Hence, the conclusionis clear. □

We see that if $X \in \text{essHank}_0$, then

$$S^*X - XS = K, \quad \text{where } K \in \mathscr{K}(H^2). \tag{8}$$

On taking adjoints on both the sides, we get

$$S^*X^* - X^*S = -K^*, \quad \text{where } K^* \in \mathscr{K}(H^2). \tag{9}$$

Therefore, $X^* \in \text{essHank}_0$.

Thus, essHank_0 is a self-adjoint set. We show that for a general complex number λ, this is not the case. That is, essHank_λ is not a self-adjoint set in general. For this we begin with the following.

Theorem 6. *If λ and μ are distinct complex numbers, then*

$$\text{essHank}_\lambda \cap \text{essHank}_\mu = \mathscr{K}(H^2). \tag{10}$$

Proof. Let $X \in \text{essHank}_\lambda \cap \text{essHank}_\mu$. Then,

$$\begin{aligned} (S^* - \lambda I) X - XS &= K_1, \\ (S^* - \mu I) X - XS &= K_2, \end{aligned} \tag{11}$$

where $K_1, K_2 \in \mathscr{K}(H^2)$. Subtracting the previous two equations we have

$$(\lambda - \mu) X \in \mathscr{K}(H^2), \tag{12}$$

where $\lambda - \mu \neq 0$. This implies that $X \in \mathscr{K}(H^2)$. Therefore, $\text{essHank}_\lambda \cap \text{essHank}_\mu \subseteq \mathscr{K}(H^2)$. Reverse inclusion is obvious by the definition. □

Theorem 7. *If $X \in \text{essHank}_\lambda$, then $X^* \in \text{essHank}_\mu$, where $\mu = -\overline{\lambda}$.*

Proof. Let $X \in \text{essHank}_\lambda$. Then,

$$(S^* - \lambda I) X - XS = K, \tag{13}$$

for some compact operator K on H^2. Taking adjoints on both sides of (13), we obtain

$$X^* \left(S - \overline{\lambda} I\right) - S^* X^* = K^*, \tag{14}$$

where $K^* \in \mathscr{K}(H^2)$. Thus, $[S^* - (-\overline{\lambda})I]X^* - X^*S \in \mathscr{K}(H^2)$. This means that $X^* \in \text{essHank}_\mu$, where $\mu = -\overline{\lambda}$. □

Corollary 8. *A necessary condition for a noncompact operator X in essHank_λ, $\lambda \neq 0$, to be self-adjoint is that λ is purely imaginary.*

Proof. Let $X \in \text{essHank}_\lambda$, where X is a noncompact operator on H^2. Then, $X^* \in \text{essHank}_\mu$, where $\mu = -\overline{\lambda}$. Now, if $X = X^*$, then $X, X^* \in \text{essHank}_\lambda \cap \text{essHank}_\mu$. As X is non-compact, we have $\lambda = \mu$. That is, $\lambda + \overline{\lambda} = 0$. That is, λ is purely imaginary. □

Theorem 9. *Let $X \in \text{essHank}_\lambda$. Then, $0 \in \sigma_e(X)$, where $\sigma_e(X)$ denotes the essential spectrum of the operator X.*

Proof. Let $X \in \text{essHank}_\lambda$. Then,

$$(S^* - \lambda I) X - XS = K, \tag{15}$$

where K is a compact operator on H^2.
Case (i). $\lambda = 0$: In this case, X satisfies

$$S^* X - XS = K, \tag{16}$$

where $K \in \mathscr{K}(H^2)$. Clearly, X cannot be Fredholm, for if X is a Fredholm operator of index n, then S^*X and $XS + K$ are Fredholm operators with indices $n+1$ and $n-1$, respectively, leading to $1 = -1$, which is absurd. Therefore, $0 \in \sigma_e(X)$.
Case (ii). $\lambda \neq 0$: In this case, if X is essentially invertible, then $(S^* - \lambda I) - XSX^{-1}$ is a compact operator on H^2. This leads to the essential similarly of $S^* - \lambda I$ and S, which is a contradiction as $\sigma_e(S) = \sigma_e(S^*) = \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle in the complex plane. So, $0 \in \sigma_e(X)$ in this case also. □

In the next theorem, we show that there is no nonzero Toeplitz operator in essHank_λ. For this we need the following lemmas.

Lemma 10. *A nonzero Toeplitz operator cannot be a λ-Hankel operator.*

Proof. Let T be a nonzero Toeplitz operator. Then,

$$S^* T S = T. \tag{17}$$

If possible, suppose that T is λ-Hankel also. Then,

$$(S^* - \lambda I) T = TS. \tag{18}$$

From (17) and (18), it follows that

$$(\lambda T + TS) S = T. \tag{19}$$

That is,

$$Te_{n+2} = Te_n - \lambda Te_{n+1}, \tag{20}$$

for all $n \geq 0$. This means that T is finite dimensional and hence a compact operator on H^2. But nonzero Toeplitz operators are never compact. Thus, we have a contradiction and the conclusion follows. □

Lemma 11. *If T is a nonzero Toeplitz operator, then so is $S^*T - TS$, S denoting the unilateral forward shift on H^2.*

Using the previous two lemmas, we now prove that the set essHank_λ contains no nonzero Toeplitz operator.

Theorem 12. $\text{essHank}_\lambda \cap \mathscr{J} = \{0\}$, *where $\mathscr{J}$ denotes the set of all Toeplitz operators on H^2.*

Proof. Let $T \in \text{essHank}_\lambda \cap \mathscr{J}$. Then,

$$\begin{gathered} S^* T S = T, \\ (S^* - \lambda I) T - TS = K, \end{gathered} \tag{21}$$

for some compact operator K on H^2. Since T is a Toeplitz operator, $S^*T - TS$ is also a Toeplitz operator on H^2. It follows that $(S^* - \lambda I)T - TS$ is a Toeplitz operator on H^2. As a nonzero Toeplitz operator cannot be compact, we must have $(S^* - \lambda I)T - TS = 0$. That is, T is a λ-Hankel operator on H^2. Now, using Lemma 10, we get that $T = 0$. Hence,

$$\text{essHank}_\lambda \cap \mathscr{J} = \{0\}. \tag{22}$$

□

In the next theorem, we show that the set essHank_λ is invariant under multiplication by Toeplitz operators.

Theorem 13. *If $X \in \text{essHank}_\lambda$ and T_ϕ is any Toeplitz operator on H^2, then XT_ϕ and $T_\phi X$ both are in essHank_λ.*

Proof. Let T_ϕ be a Toeplitz operator on the space H^2. Then, we have

$$S^* T_\phi S = T_\phi. \tag{23}$$

Since S is essentially unitary, we have the commutator of T_ϕ with S, and that with S^* is compact on H^2. Now, let $X \in \text{essHank}_\lambda$. Then,

(i) consider

$$\begin{aligned}\left(XT_{\phi}\right)S &= (XS)\,T_{\phi} \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &= \left[\left(S^{*}-\lambda I\right)X\right]T_{\phi} \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right).\end{aligned} \tag{24}$$

Therefore,

$$\left(XT_{\phi}\right)S-\left(S^{*}-\lambda I\right)\left(XT_{\phi}\right)\in\mathscr{K}\left(H^{2}\right). \tag{25}$$

(ii) Consider

$$\begin{aligned}&\left(S^{*}-\lambda I\right)\left(T_{\phi}X\right)\\ &\quad = S^{*}T_{\phi}X-\lambda T_{\phi}X\\ &\quad = T_{\phi}S^{*}X-\lambda T_{\phi}X \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad = T_{\phi}\left(\lambda X+XS\right)-\lambda T_{\phi}X \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad = \left(T_{\phi}X\right)S \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right).\end{aligned} \tag{26}$$

Therefore,

$$\left(T_{\phi}X\right)S-\left(S^{*}-\lambda I\right)\left(T_{\phi}X\right)\in\mathscr{K}\left(H^{2}\right). \tag{27}$$

Hence, the conclusion is clear. □

More generally, we have the following.

Theorem 14. *If $X\in\operatorname{essHank}_{\lambda}$ and $T\in\operatorname{essToep}$, then XT, $TX\in\operatorname{essHank}_{\lambda}$.*

Proof. Let $X\in\operatorname{essHank}_{\lambda}$ and $T\in\operatorname{essToep}$. Then,

$$\begin{gathered}\left(S^{*}-\lambda I\right)X-XS\in\mathscr{K}\left(H^{2}\right),\\ ST-TS\in\mathscr{K}\left(H^{2}\right)\end{gathered} \tag{28}$$

Now,

$$\begin{aligned}&\left(S^{*}-\lambda I\right)(XT)-(XT)S\\ &\quad = \left(S^{*}-\lambda I\right)(XT)-X(ST) \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad = \left[\left(S^{*}-\lambda I\right)X-XS\right]T \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad \in\mathscr{K}\left(H^{2}\right).\end{aligned} \tag{29}$$

Therefore, $XT\in\operatorname{essHank}_{\lambda}$.

Also,

$$\begin{aligned}&\left(S^{*}-\lambda I\right)(TX)-(TX)S\\ &\quad = TS^{*}X-\lambda TX-TXS \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad = T\left[\left(S^{*}-\lambda I\right)X-XS\right] \quad \left(\operatorname{mod}\mathscr{K}\left(H^{2}\right)\right)\\ &\quad \in\mathscr{K}\left(H^{2}\right).\end{aligned} \tag{30}$$

Therefore, $TX\in\operatorname{essHank}_{\lambda}$.

We mention here that the previous result was proved for the special case $\lambda=0$ by Avendaño [7]. It is easy to see that essToep is a C^{*}-algebra. This fact together with the previous theorem gives us that $\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$ is an algebra of operators on H^{2} as shown in the following. □

Theorem 15. *$\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$ is an algebra of operators on H^{2} with no identity.*

Proof. Since essToep is a C^{*}-algebra and $\operatorname{essHank}_{\lambda}$ is a vector subspace of $\mathscr{B}(H^{2})$, it follows that $\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$ is a vector subspace of $\mathscr{B}(H^{2})$. Also, if $X_{1},X_{2}\in\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$, then using Theorem 14, we get that $X_{1}X_{2}\in\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$. Thus, $\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$ is an algebra of operators on H^{2}. As $I\notin\operatorname{essHank}_{\lambda}$, the theorem is proved. □

Remark 16. We have seen that if λ is purely imaginary, then $\operatorname{essHank}_{\lambda}$ is a self-adjoint set. Therefore, for purely imaginary λ, $\operatorname{essToep}\cap\operatorname{essHank}_{\lambda}$ turns to be a C^{*}-algebra.

Theorem 17. *If $X\in\operatorname{essHank}_{\lambda}$, then*

$$S\left(\operatorname{Ker}X\right)\subseteq\operatorname{Ker}\left(X+K\right), \tag{31}$$

for some compact operator K on H^{2}.

Proof. Let $X\in\operatorname{essHank}_{\lambda}$. Then,

$$\left(S^{*}-\lambda I\right)X-XS=K, \tag{32}$$

for some compact operator K on H^{2}.

If $f\in\operatorname{Ker}X$, then $Xf=0$. From (32) it follows that $(XS+K)f=0$. That is, $(X+KS^{*})Sf=0$. This implies that $(X+K^{'})Sf=0$, where $K^{'}=KS^{*}\in\mathscr{K}(H^{2})$. Thus, $Sf\in\operatorname{Ker}(X+K^{'})$, where $K^{'}\in\mathscr{K}(H^{2})$. Hence, the result is clear. □

Corollary 18. *If $X\in\operatorname{essHank}_{\lambda}$, then*

$$S\left(\overline{\operatorname{Range}\ X}\right)^{\perp}\subseteq\left[\overline{\operatorname{Range}\ \left(X+K\right)}\right]^{\perp}, \tag{33}$$

for some compact operator K on H^{2}.

Proof. Let $X\in\operatorname{essHank}_{\lambda}$. Then, $X^{*}\in\operatorname{essHank}_{\mu}$, where $\mu=-\overline{\lambda}$. Applying Theorem 17 to X^{*}, we get

$$S\left(\operatorname{Ker}X^{*}\right)\subseteq\operatorname{Ker}\left(X^{*}+K\right), \tag{34}$$

for some compact operator K on H^{2}. This means that

$$S\left(\operatorname{Ker}X^{*}\right)\subseteq\operatorname{Ker}\left[\left(X+K^{*}\right)^{*}\right]. \tag{35}$$

Therefore,

$$S\left(\overline{\operatorname{Range}\ X}\right)^{\perp}\subseteq\left[\overline{\operatorname{Range}\ \left(X+K^{*}\right)}\right]^{\perp}, \tag{36}$$

where $K^{*}\in\mathscr{K}(H^{2})$. So, the conclusion follows. □

Remark 19. It is known that [3, 7] kernel of a λ-Hankel operator is an invariant subspace of S, and closure of the range of a λ-Hankel operator is an invariant subspace of S^*. We mention here that these facts about λ-Hankel operators follow as deductions to Theorem 17 and Corollary 18. For if X is λ-Hankel then $K = 0$ in Theorem 17 and Corollary 18 leading to $S(\mathrm{Ker}X) \subseteq \mathrm{Ker}X$ and $S^*(\overline{\mathrm{Range}\ X}) \subseteq \overline{\mathrm{Range}\ X}$, respectively.

It is known that a Rhaly operator R_a is essentially Toeplitz if and only if it is essentially Hankel. That is, $R_a \in \mathrm{essToep}$ if and only if $R_a \in \mathrm{essHank}_0$. We show that this is not the case if $\lambda \neq 0$. In fact, an essentially Toeplitz Rhaly operator is in $\mathrm{essHank}_\lambda (\lambda \neq 0)$ if and only if it is compact. Precisely, we have the following.

Theorem 20. *Let $R_a \in \mathrm{essToep}$ be a Rhaly operator with determining sequence $a = \langle a_n \rangle \in \ell^2$. Then $R_a \in \mathrm{essHank}_\lambda$ $(\lambda \neq 0)$ if and only if $\lim_{n\to\infty}(n+1)|a_n| = 0$.*

Proof. Let $R_a \in \mathrm{essToep}$. Then, $R_a \in \mathrm{essHank}_0$ [4]. Now if $R_a \in \mathrm{essHank}_\lambda$ $(\lambda \neq 0)$, then we have $R_a \in \mathrm{essHank}_0 \cap \mathrm{essHank}_\lambda$. As $\lambda \neq 0$, we have $R_a \in \mathscr{K}(H^2)$, by Theorem 6. It is known that [8] a Rhaly operator R_a with determining sequence $a = \langle a_n \rangle$ is compact if and only if $\lim_{n\to\infty}(n+1)$ $|a_n| = 0$. The desired result follows. □

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