Research Article

Sufficient Conditions and Applications for Carathéodory Functions

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Let P be the class of functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ which are analytic in U . By using the method of differential subordinations, we give some sufficient conditions for Carathéodory function, that is, if $p(z) \in P$ satisfies $p(z) \neq 0$ ($0 < |z| < 1$) and $p(z)^m + \lambda(zp'(z)/(p(z))^\mu) < h(z)$, where $h(z) = ((1+az)/(1+bz))^m + \lambda(a-b)z/(1+bz)^{2-\mu}(1+az)^\mu$, $-1 \leq b < a \leq 1$, $\lambda > 0$, $m, \mu \in \{0, 1, 2\}$, $m + \mu \leq 2$, then $p(z) < (1+az)/(1+bz)$. Some useful consequences of this result are also given.

1. Introduction

Let A be the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in $U = \{z : |z| < 1\}$. We denote by S the subclass of A consisting of univalent functions and by S^* and K the usual subclasses of S whose members are starlike (with respect to the origin) and close-to-convex, respectively. Finally, we denote by R the family of functions $f \in A$ which satisfy the condition $\operatorname{Re} f'(z) > 0$ ($z \in U$). It is well known that $R \subset K$.

Let P be the class of functions of the following form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (2)$$

which are analytic in U . If $p(z) \in P$ satisfies $\operatorname{Re} p(z) > 0$ ($z \in U$), then we say that $p(z)$ is the Carathéodory function. For Carathéodory functions, Nunokawa et al. [1] have shown some sufficient conditions applying the differential inequalities. In the present paper, using the method of differential subordinations, we derive certain conditions under which we have $p(z) < (1+az)/(1+bz)$, where $-1 \leq b < a \leq 1$. Our results generalize or improve some results due to [1–4].

To prove our results, we need the following lemma due to Miller and Mocanu [5].

Lemma 1. Let $g(z)$ be analytic and univalent in U and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. Set

$$Q(z) = zg'(z)\varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z) \quad (3)$$

and suppose that

- (i) $Q(z)$ is starlike univalent in U , and
- (ii) $\operatorname{Re}(zh'(z)/Q(z)) = \operatorname{Re}\{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0$ ($z \in U$).

If $p(z)$ is analytic in U , with $p(0) = g(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \quad (4)$$

then $p(z) < g(z)$ and $g(z)$ is the best dominant of (4).

2. Main Results

Theorem 2. Let $-1 \leq b < a \leq 1$, $\lambda > 0$, $m, \mu \in \{0, 1, 2\}$, and $m + \mu \leq 2$. If $p(z) \in P$ satisfies $p(z) \neq 0$ ($0 < |z| < 1$) and

$$p(z)^m + \lambda \frac{zp'(z)}{(p(z))^\mu} < h(z), \quad (5)$$

where

$$h(z) = \left(\frac{1+az}{1+bz} \right)^m + \frac{\lambda(a-b)z}{(1+bz)^{2-\mu}(1+az)^\mu}, \quad (6)$$

then $p(z) < (1+az)/(1+bz)$ and $(1+az)/(1+bz)$ is the best dominant of (5).

Proof. Let $-1 \leq b < a \leq 1$, $\lambda > 0$, $m, \mu \in \{0, 1, 2\}$,

$$D = \begin{cases} C & \mu = 0, \\ C \setminus \{0\} & \mu = 1, 2, \end{cases} \quad (7)$$

and choose

$$g(z) = \frac{1+az}{1+bz}, \quad \theta(w) = w^m, \quad \varphi(w) = \frac{\lambda}{w^\mu}. \quad (8)$$

Then $g(z)$ is analytic and univalent in U , $g(0) = p(0) = 1$, $p(U) \subset D$, $\theta(w)$ and $\varphi(w)$ satisfy the conditions of the lemma. The function

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{\lambda(a-b)z}{(1+bz)^{2-\mu}(1+az)^\mu} \quad (9)$$

is univalent and starlike in U because

$$\begin{aligned} \operatorname{Re} \frac{zQ'(z)}{Q(z)} &= 1 - (2-\mu) \operatorname{Re} \frac{bz}{1+bz} - \mu \operatorname{Re} \frac{az}{1+az} \\ &= -1 + (2-\mu) \operatorname{Re} \frac{1}{1+bz} + \mu \operatorname{Re} \frac{1}{1+az} \\ &> -1 + (2-\mu) \frac{1}{1+|b|} + \mu \frac{1}{1+|a|} \geq 0 \quad (z \in U). \end{aligned} \quad (10)$$

Further, we have

$$\begin{aligned} \theta(g(z)) + Q(z) &= \left(\frac{1+az}{1+bz} \right)^m + \frac{\lambda(a-b)z}{(1+bz)^{2-\mu}(1+az)^\mu} \\ &= h(z), \\ \frac{zh'(z)}{Q(z)} &= \frac{m}{\lambda} \left(\frac{1+az}{1+bz} \right)^{m+\mu-1} + \frac{zQ'(z)}{Q(z)} \quad (z \in U). \end{aligned} \quad (11)$$

Since $m + \mu \leq 2$, from (11) it is easy to know that

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0 \quad (z \in U). \quad (12)$$

Hence the function $h(z)$ is close-to-convex and univalent in U . Now it follows from (5)–(12) that

$$\begin{aligned} \theta(p(z)) + zp'(z)\varphi(p(z)) &< \theta(g(z)) \\ &+ zg'(z)\varphi(g(z)) = h(z). \end{aligned} \quad (13)$$

Therefore, by virtue of the lemma, we conclude that $p(z) < g(z)$ and $g(z)$ is the best dominant of (5). The proof of the theorem is complete. $\square$

Making use of the theorem, we can obtain a number of interesting results.

Corollary 3. Let $\beta > 0$, $\alpha > 0$. If $p(z) \in P$ satisfies

$$\begin{aligned} (p(z))^2 + \frac{\beta}{\alpha} zp'(z) &< \frac{\alpha + 2(1+\beta)z + \alpha z^2}{\alpha(1-z)^2} \\ &= h_1(z) \quad (z \in U), \end{aligned} \quad (14)$$

then $\operatorname{Re} p(z) > 0$.

Proof. Let $a = 1$, $b = -1$, $\lambda = \beta/\alpha$, $\beta > 0$, $\alpha > 0$, $m = 2$, and $\mu = 0$ in the theorem, then we have

$$\begin{aligned} g(z) &= \frac{1+z}{1-z}, \quad \theta(w) = w^2, \\ \varphi(w) &= \frac{\beta}{\alpha}, \quad Q(z) = \frac{2\beta z}{\alpha(1-z)^2} \end{aligned} \quad (15)$$

satisfy the conditions of the lemma. Note that

$$\operatorname{Re} \frac{zh_1'(z)}{Q(z)} = \left(\frac{2\alpha}{\beta} + 1 \right) \operatorname{Re} \frac{1+z}{1-z} > 0 \quad (z \in U). \quad (16)$$

Hence, similar to the proof of the theorem, we conclude that $p(z) < g(z)$, that is, $\operatorname{Re} p(z) > 0$. $\square$

Remark 4. Note that Corollary 3 was also proved by Nunokawa et al. [1] using another method.

Taking $a = 1$, $b = -1$, $\lambda = \beta/\alpha$, $\alpha > 0$, $\beta > 0$, and $m = \mu = 1$ in the theorem, we have the following.

Corollary 5. Let $\alpha > 0$, $\beta > 0$, $p(z) \in P$ satisfies $p(z) \neq 0$ ($0 < |z| < 1$), and

$$\alpha p(z) + \beta \frac{zp'(z)}{p(z)} < \alpha \frac{1+z}{1-z} + \frac{2\beta z}{1-z^2} = h_2(z) \quad (z \in U), \quad (17)$$

then $\operatorname{Re} p(z) > 0$.

Remark 6. Note that the function

$$w = h_2(z) = \alpha \frac{1+z}{1-z} + \frac{2\beta z}{1-z^2} \quad (\alpha > 0, \beta > 0) \quad (18)$$

maps U onto the w -plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq \sqrt{\beta(2\alpha + \beta)}$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -\sqrt{\beta(2\alpha + \beta)}$. Hence Corollary 5 coincides with the result obtained by Nunokawa et al. [1, Theorem 2] using another method.

Letting $f(z) \in A$, $p(z) = f'(z)$, $\alpha = 1$ in Corollary 5, we have the following.

Corollary 7. If $f(z) \in A$, $f'(z) \neq 0$ ($0 < |z| < 1$), and

$$f'(z) + \beta \frac{zf''(z)}{f'(z)} \neq ib \quad (z \in U), \quad (19)$$

where $\beta > 0$ is real and $|b| \geq \sqrt{\beta(2 + \beta)}$, then $f(z) \in R$.

Remark 8. Lewandowski et al. [3] proved if $f(z) \in A$, $f'(z) \neq 0$ ($0 < |z| < 1$), $\beta > 0$, and

$$\operatorname{Re} \left\{ f'(z) + \beta \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in U), \quad (20)$$

then $f(z) \in R$. We see that Corollary 7 improves this result.

Corollary 9. Let $f(z) \in A$ with $f(z)f'(z) \neq 0$ ($0 < |z| < 1$). If

$$\frac{(zf(z))''}{f'(z)} - 2 \frac{zf'(z)}{f(z)} < \frac{(1-\delta)z}{1+(1-\delta)z} = h_3(z) \quad (21)$$

for some δ ($0 \leq \delta < 1$), then $f(z) \in S$ and

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \delta \quad (z \in U). \quad (22)$$

Proof. Letting $f(z) \in A$, $f(z)f'(z) \neq 0$ ($0 < |z| < 1$), then $p(z) = z^2 f'(z)/f^2(z) \in P$,

$$\frac{zp'(z)}{p(z)} = \frac{(zf(z))''}{f'(z)} - 2 \frac{zf'(z)}{f(z)}. \quad (23)$$

Taking $m = 0$, $\mu = \lambda = 1$ and $a = 1 - \delta$ ($0 \leq \delta < 1$), $b = 0$ in the theorem, it follows from (5), (6), and (21) that $|(z^2 f'(z))/(f^2(z)) - 1| < 1 - \delta$ ($z \in U$), which implies that $f(z) \in S$ (see [6]). $\square$

Remark 10. Frassin and Darus [2] have shown that if $f(z) \in A$ and

$$\left| \frac{(zf(z))''}{f'(z)} - 2 \frac{zf'(z)}{f(z)} \right| < \frac{1-\delta}{2-\delta} \quad (z \in U) \quad (24)$$

for some δ ($0 \leq \delta < 1$), then $|(z^2 f'(z)/f^2(z)) - 1| < 1 - \delta$ ($z \in U$).

For $0 \leq \delta < 1$, the function

$$h_3(z) = \frac{(1-\delta)z}{1-(1-\delta)z} \quad (25)$$

is analytic and convex univalent in U . Since

$$h_3(U) = \left\{ w : \left| w + \frac{(1-\delta)^2}{\delta(2-\delta)} \right| < \frac{1-\delta}{\delta(2-\delta)} \right\} \quad (0 < \delta < 1),$$

$$h_3(U) = \left\{ w : \operatorname{Re} w < \frac{1}{2} \right\} \quad (\delta = 0), \quad (26)$$

the disk $|w| < (1-\delta)/(2-\delta)$ is properly contained in $h_3(U)$. Therefore, in view of (21), we see that Corollary 9 is better than the result given in [2].

Letting $f(z) \in A$, $p(z) = f'(z) \in P$, $m = 1$, $\mu = 0$, $a = 1 - 2\alpha$, $0 \leq \alpha < 1$, and $b = -1$ in the theorem, we have the following.

Corollary 11. If $f(z) \in A$, $\lambda > 0$, $0 \leq \alpha < 1$, and

$$f'(z) + \lambda z f''(z) < \frac{1+(1-2\alpha)z}{1-z} + \lambda \frac{2(1-\alpha)z}{(1-z)^2} = h_4(z) \quad (z \in U), \quad (27)$$

then $\operatorname{Re} f'(z) > \alpha$.

For the univalent function $h_4(z)$, we now find the image $h_4(U)$ of the unit disk U .

Let $h_4(e^{i\theta}) = u + iv$, where u and v are real. We have

$$u = \alpha - \frac{(1-\alpha)\lambda}{1-\cos\theta}, \quad v = (1-\alpha) \frac{\sin\theta}{1-\cos\theta}. \quad (28)$$

Elimination of θ yields

$$v^2 = \frac{2\alpha(1-\alpha) - (1-\alpha)^2\lambda}{\lambda} - \frac{2(1-\alpha)}{\lambda}u. \quad (29)$$

Therefore, we conclude that

$$h_4(U) = \left\{ w = u + iv : v^2 > -\frac{2(1-\alpha)}{\lambda} \times \left(u + \frac{(1-\alpha)\lambda}{2} - \alpha \right) \right\}. \quad (30)$$

Remark 12. Nunokawa and Hoshino [4] have proved that if $f(z) \in A$, $\lambda > 0$ and

$$\operatorname{Re} \left\{ f'(z) + \lambda z f''(z) \right\} > -\frac{\lambda}{2}, \quad (z \in U), \quad (31)$$

then $\operatorname{Re} f'(z) > 0$. For $\alpha = 0$, $h_4(U)$ properly contains the half plane $\operatorname{Re} w > -\lambda/2$. Hence Corollary 11 with $\alpha = 0$ is better than the result given in [4].

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