Research Article

The Relationship between Some Regular Subsemigroups of $\text{Hyp}_G(2)$

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The concept of regular subsemigroups plays an important role in the theory of semigroup. In this work, we study the relationship between some regular subsemigroups on the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

1. Introduction

Let S be a semigroup. The class of regular semigroups is one of the most important classes of semigroups. Recall that an element a in a semigroup S is said to be *regular* if there exists $b \in S$ such that $a = aba$. A semigroup S is said to be *regular* if its element is regular. An element $a \in S$ is called *idempotent* if $aa = a$. It is well-known that an idempotent element is an obvious example of a regular element. We denote the set of all idempotent elements of a semigroup S by $E(S)$. Next, we recall some definitions of special elements of a semigroup S .

Definition 1. An element a of a semigroup S is called *left (right) regular* if there exists $b \in S$ such that $ba^2 = a(a^2b = a)$ and a is called *intraregular* if $a \in Sa^2S$.

Definition 2. An element a of a semigroup S is called *coregular* if there is an element $b \in S$ such that $a = aba = bab$, b its *coinverse*. A semigroup is said to be *coregular* if each element of S is co-regular.

Then we have the following proposition.

Proposition 3. Let S be a semigroup and $a \in S$. Then a is co-regular element if and only if $a^3 = a$.

Proof. $(\Rightarrow)$ Let a be a co-regular element in S . Then there is an element $b \in S$ such that $a = aba = bab$. Thus $a = bab = b(aba)b = (bab)a(bab) = a^3$.

$(\Leftarrow)$ Suppose that $a^3 = a$. Then a is co-regular element. $\square$

Definition 4. An element a of a semigroup S is called *antiregular* if there exists $b \in S$ such that $aba = b$ and $bab = a$. The elements a and b are then called *anti-inverse*.

Proposition 5. Let S be a semigroup and $a \in S$. If a is antiregular element, then $a^5 = a$.

Proof. Let a be an antiregular element in S . Then there is an element $b \in S$ such that $aba = b$ and $bab = a$. Thus $a = bab = (aba)(bab)(aba) = a(bab)a(bab)a = a^5$. $\square$

Definition 6. An element a of a semigroup S is called *completely regular* if there exists $b \in S$ such that $aba = a$ and $ab = ba$.

Proposition 7 (see [1]). Let S be a semigroup and $a \in S$. Then a is completely regular element if and only if a is both left regular and right regular.

Remark 8. In general, for any semigroup S and $a \in S$, we have the following relationship:

a is co-regular $\Rightarrow a$ is antiregular $\Rightarrow a$ is completely regular $\Rightarrow a$ is left regular, right regular, and intraregular.

The consequent question is as follows: is there a semigroup such that all completely regular elements, left regular

elements, right regular elements, or intraregular elements are coregular elements?

As an example, consider the semigroup of all integers with the usual addition. We have that for any nonzero integers a and b , if $a = a + b + a$ and $a + b = b + a$, then $b = -a$ but $a = a + b + a \neq b + a + b = b$. Then a is completely regular but is not coregular.

In this paper, we are interested in the semigroup of generalized hypersubstitutions which is a generalization of hypersubstitutions.

Let $X := \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of symbols called *variables* and let $X_n := \{x_1, x_2, \dots, x_n\}$ be an n -element set. Let $(f_i)_{i \in I}$ be an indexed set which is disjoint from X . Each f_i is called an n_i -ary operation symbol, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns every f_i to the number n_i as its arity, written as $(n_i)_{i \in I}$ and is called a *type*.

An n -ary term of type τ is defined inductively as follows.

- (i) The variables $x_1, x_2, \dots, x_n$ are n -ary terms of type τ .
- (ii) If $t_1, t_2, \dots, t_{n_i}$ are n_i -ary terms of type τ , then $f_i(t_1, t_2, \dots, t_{n_i})$ is an n -ary term of type τ .

The smallest set which contains $x_1, x_2, \dots, x_n$ and is closed under finite application of (ii), denoted by $W_\tau(X_n)$. Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ and is called *the set of all terms of type τ* .

The concept of a hypersubstitution was introduced first by Denecke et al. in 1991; see [2]. A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ which assigns every n_i -ary operation symbol f_i to an n_i -ary term. The set of all hypersubstitutions of type τ is denoted by $\text{Hyp}(\tau)$. Every $\sigma \in \text{Hyp}(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ by the following steps:

- (i) $\hat{\sigma}[x] := x$, for any variable $x \in X$,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

A binary operation $\circ_h$ on $\text{Hyp}(\tau)$ is defined by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for every $\sigma_1, \sigma_2 \in \text{Hyp}(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the hypersubstitution where $\sigma_{\text{id}}(f_i) = f_i(x_1, \dots, x_{n_i})$. Then $(\text{Hyp}(\tau); \circ_h, \sigma_{\text{id}})$ is a monoid with σ_{id} as an identity element.

In 2000, Leeratanavalee and Denecke generalized the concepts of a hypersubstitution, a hyperidentity, and a solid variety to a generalized hypersubstitution, a strong hyperidentity, and a strongly solid variety, respectively; see [3]. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ from the set of all n_i -ary operation symbols into the set of all terms built up by elements of the alphabet $X := \{x_1, x_2, \dots\}$ and operation symbols from $\{f_i \mid i \in I\}$ which does not necessarily preserve the arity.

We denote the set of all generalized hypersubstitutions of type τ by $\text{Hyp}_G(\tau)$. To define a binary operation on $\text{Hyp}_G(\tau)$, we defined first the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

- (i) if $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$,

- (ii) if $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$,

- (iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)). \quad (1)$$

Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ by the following steps:

- (i) $\hat{\sigma}[x] := x \in X$,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i -ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

We define a binary operation $\circ_G$ on $\text{Hyp}_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$. Let σ_{id} be the hypersubstitution mapping which maps each n_i -ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. Then $(\text{Hyp}_G(\tau); \circ_G, \sigma_{\text{id}})$ is a monoid and the monoid $(\text{Hyp}(\tau); \circ_h, \sigma_{\text{id}})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(\text{Hyp}_G(\tau); \circ_G, \sigma_{\text{id}})$.

Throughout this present paper, we fix the type $\tau = (2)$. By σ_t , we denote the generalized hypersubstitution which maps f to the term $t \in W_{(2)}(X)$.

Proposition 9 (see [3]). *For arbitrary $t, t_1, t_2, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ one has*

- (i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \hat{\sigma}[t_2], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, t_2, \dots, t_n)]$,
- (ii) $(\hat{\sigma}_1 \circ \hat{\sigma}_2)^\wedge = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

2. Main Results

Let $\tau = (2)$ be a fixed type with the binary operation symbol f . Let $\sigma_t \in \text{Hyp}_G(2)$, the set of all variables occurring in t is denoted by $\text{var}(t)$

- $R_1 := \{\sigma_t \mid t = f(x_2, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin \text{var}(t')\}$,
- $R_2 := \{\sigma_t \mid t = f(t', x_1) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin \text{var}(t')\}$,
- $R_3 := \{\sigma_t \mid t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin \text{var}(t')\}$,
- $R_4 := \{\sigma_t \mid t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin \text{var}(t')\}$,
- $R_5 := \{\sigma_t \mid t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}\}$,
- $R_6 := \{\sigma_t \mid \text{var}(t) \cap \{x_1, x_2\} = \emptyset\}$.

In [4], Puninagool and Leeratanavalee showed that $\bigcup_{i=1}^6 R_i$ is a set of all regular elements in $\text{Hyp}_G(2)$ and $(\bigcup_{i=3}^6 R_i) \setminus \{\sigma_{f(x_2, x_1)}\} = E(\text{Hyp}_G(2))$.

In [5], Sudsanit et al. proved that the set of all left regular elements and the set of all right regular elements in $\text{Hyp}_G(2)$ are the same; that is, $E(\text{Hyp}_G(2)) \cup \{\sigma_{f(x_2, x_1)}\}$.

Furthermore, Boonmee and Leeratanavalee showed that the set of all intraregular elements in $\text{Hyp}_G(2)$ is also $E(\text{Hyp}_G(2)) \cup \{\sigma_{f(x_2, x_1)}\}$; see [6].

Proposition 10. *Let $\sigma_t \in \text{Hyp}_G(2)$. Then the following conditions are equivalent:*

- (a) σ_t is coregular,
- (b) σ_t is antiregular,
- (c) σ_t is completely regular,
- (d) σ_t is left regular,
- (e) σ_t is right regular,
- (f) σ_t is intraregular.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obtained by Remark 8.

(d) $\Rightarrow$ (e) $\Rightarrow$ (f) is a consequence of the fact that the set of all left regular, right regular, and intraregular elements in $\text{Hyp}_G(2)$ are the same.

(f) $\Rightarrow$ (a) Let σ_t be intraregular; then $\sigma_t \in E(\text{Hyp}_G(2)) \cup \{\sigma_{f(x_2, x_1)}\}$. So $\sigma_t = \sigma_t^3$ and then σ_t is a coregular element. $\square$

Corollary 11. *$E(\text{Hyp}_G(2)) \cup \{\sigma_{f(x_2, x_1)}\}$ is the set of all coregular elements, antiregular elements, completely regular elements, left regular elements, right regular elements, and intraregular elements in $\text{Hyp}_G(2)$.*

Moreover, the next result describes the relationship between some regular subsemigroups of this semigroup.

Let S be a semigroup. A nonempty subset T of S is called a *subsemigroup* of S if $T^2 \subseteq T$. A nonempty subsemigroup T of S is called a *regular subsemigroup* if, for any element a of T , there exists an element $b \in T$ such that $a = aba$.

Proposition 12. *Let R be a regular subsemigroup of $\text{Hyp}_G(2)$. Then the following conditions are equivalent:*

- (a) R is coregular,
- (b) R is antiregular,
- (c) R is completely regular,
- (d) R is left regular,
- (e) R is right regular,
- (f) R is intraregular.

Proof. (a) $\Rightarrow$ (b) Let $\sigma_t \in R$ be such that $\sigma_t^3 = \sigma_t$. Then σ_t is an antiregular element.

(b) $\Rightarrow$ (c) Let σ_t be an antiregular element in R . By using Proposition 5, we have $\sigma_t = \sigma_t^5 = \sigma_t \circ_G \sigma_t^3 \circ_G \sigma_t$ and $\sigma_t^3 \circ_G \sigma_t = \sigma_t \circ_G \sigma_t^3$. Since $\sigma_t^3 = \sigma_t$ for all $\sigma_t \in R$, so R is completely regular.

(c) $\Rightarrow$ (d) It is obviously by definition.

(d) $\Rightarrow$ (e) Since $\sigma_t^3 = \sigma_t \circ_G \sigma_t^2 = \sigma_t^2 \circ_G \sigma_t = \sigma_t$ for all $\sigma_t \in R$, so R is right regular.

(e) $\Rightarrow$ (f) Let $\sigma_t \in R$. Since R is right regular, so there exists $\sigma_s \in R$ such that $\sigma_t^2 \circ_G \sigma_s = \sigma_t$. Since $\sigma_t = \sigma_t^2 \circ_G \sigma_s = \sigma_t \circ_G \sigma_t \circ_G \sigma_s = \sigma_t \circ_G (\sigma_t^2 \circ_G \sigma_s) \circ_G \sigma_s = \sigma_t \circ_G \sigma_t^2 \circ_G \sigma_s^2 \in R \circ_G \sigma_t^2 \circ_G R$, so σ_t is intraregular.

(f) $\Rightarrow$ (a) Let $\sigma_t \in R \subseteq E(\text{Hyp}_G(2)) \cup \{\sigma_{f(x_2, x_1)}\}$. Then $\sigma_t = \sigma_t^3$. So R is coregular. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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