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Research Article

On Generalized Derivations of *BCI*-Algebras and Their Properties

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We introduce the concept of (f, g)-derivations of *BCI*-algebras and we investigate some fundamental properties and establish some results on (f, g)-derivations. Also, we treat to generalization of right derivation and left derivation of *BCI*-algebras and consider some related properties.

1. Basic Facts about *BCI*-Algebras

In 1966, Iséki introduced the concept of *BCI*-algebra, which is a generalization of the *BCK*-algebra, as an algebraic counterpart of the *BCI*-logic [1]; also see [2–6]. In this section, we summarize some basic concepts which will be used throughout the paper. For more details, we refer to the references in [7–12]. Let us recall the definition.

A *BCI*-algebra X is an abstract algebra $(X, *, 0)$ of type $(2, 0)$, satisfying the following conditions, for all $x, y, z \in X$:

$(BCI1)$ $((x * y) * (x * z)) * (z * y) = 0$;

$(BCI2)$ $(x * (x * y)) * y = 0$;

$(BCI3)$ $x * x = 0$;

$(BCI4)$ $x * y = 0$ and $y * x = 0$ imply that $x = y$.

A nonempty subset S of a *BCI*-algebra X is called a *subalgebra* of X if $x * y \in S$, for all $x, y \in S$. In any *BCI*-algebra X, one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. A *BCI*-algebra X satisfying $0 \leq x$, for all $x \in X$, is called a *BCK*-algebra. In any *BCK*/*BCI*-algebra X, the following properties are valid, for all $x, y, z \in X$:

(1) $x * 0 = x$;

(2) $(x * y) * z = (x * z) * y$;

(3) $x \leq y$ implies that $x * z \leq y * z, z * y \leq z * x$;

(4) $(x * z) * (y * z) \leq x * y$;

(5) $x * (x * (x * y)) = x * y$;

(6) $0 * (x * y) = (0 * x) * (0 * y)$;

(7) $x * 0 = 0$ implies that $x = 0$.

Let X be a *BCI*-algebra; the set $X_+ = \{x \in X : 0 \leq x\}$ is a subalgebra and is called the *BCK-part* of X. A *BCI*-algebra X is called *proper* if $X - X_+ \neq \emptyset$. If $X_+ = \{0\}$, then X is called a *p-semisimple* *BCI*-algebra. In any *BCI*-algebra X, the following properties are equivalent, for all $x, y \in X$: (1) X is p-semisimple, (2) $0 * (0 * x) = x$, (24) $x * y = 0 \Rightarrow x = y$, and (4) $y * (y * x) = x$. In any p-semisimple *BCI*-algebra X, the following properties are valid, for all $x, y, z \in X$: (1) $(x * z) * (y * z) = x * y$, (2) $x * (0 * y) = y * (0 * x)$, (3) $x * y = x * z$ implies that $y = z$, and (4) $y * x = z * x$ implies that $y = z$. For a *BCI*-algebra X, the set $G(X) = \{x \in X : 0 * x = x\}$ is called the *BCI − G part* of X. Note that $G(X) \cap X_+ = \{0\}$.

Let X be a p-semisimple *BCI*-algebra. We define addition "+" as $x + y = x * (0 * y)$, for all $x, y \in X$. Then, $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely, let $(X, +)$ be an abelian group with identity 0 and $x * y = x - y$. Then, X is a p-semisimple *BCI*-algebra and $x + y = x * (0 * y)$, for all $x, y \in X$ (see [2]).

Let X be a *BCI*-algebra; we define the binary operation $\wedge$ as $x \wedge y = y * (y * x)$, for all $x, y \in X$. In particular, we denote $a_x := x \wedge 0 = 0 * (0 * x)$. An element $a \in X$ is said to be an *initial element (p-atom)* of X, if $x \leq a$ implies $x = a$. We

denote by $L_p(X)$ the set of all initial elements (p-atoms) of X, indeed $L_p(X) = \{a \in X : x * a = 0 \Rightarrow x = a, \forall x \in X\}$, and we call it the *center* of X. Note that $L_p(X) = \{x \in X : a_x = x\}$, which is the p-semisimple part of X and X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$. Let X be a BCI-algebra with $L_p(X)$ as its center and $a \in L_p(X)$. Then, the set $V(a) = \{x \in X : a \leq x\}$ is called the *branch* of X with respect to a.

For any BCI-algebra X the following results are valid.

(1) If $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$, for all $a, b \in L_p(X)$.

(2) If $x \leq y$, then x, y are contained in the same branch of X.

(3) If $x, y \in V(a)$, for some $a \in L_p(X)$, then $x * y, y * x \in X_+$.

(4) If $a, b \in L_p(X)$, then $a * y = a * b$, for all $y \in V(b)$.

(5) $0 * x \in L_p(X)$, for all $x \in X$.

(6) $a_x \in L_p(X)$, for all $x \in X$. Indeed, $0 * (0 * a_x) = a_x$, for all $x \in X$ which implies that $a_x * y \in L_p(X)$, for all $y \in X$.

(7) $G(X) \subseteq L_p(X)$.

(8) $x * (x * a) = a$ and $a * x \in L_p(X)$, for all $x \in X$ and $a \in L_p(X)$.

A self-map f of a BCI-algebra X (i.e., a mapping of X into itself) is called an *endomorphism* of X if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that $f(0) = 0$. A subset I of a BCI-algebra X is called an *ideal* of X, if it satisfies

(1) $0 \in I$,

(2) $x * y \in I$ and $y \in I$ imply that $x \in I$, for all $x, y \in X$.

Let f be an endomorphism of a BCI-algebra X and let $L_p(X)$ be its center. Then, according to [13], we have the following results:

(1) $f(a) \in L_p(X)$, for all $a \in L_p(X)$,

(2) $f_x * f_y \in L_p(X)$ and $f_{x*y} = f_x * f_y$, for all $x, y \in X$, where $f_x = 0 * (0 * f(x))$,

(3) $f(a) = 0 * (0 * f(x))$, for all $x \in V(a)$.

A BCI-algebra X is called *commutative* if $x \leq y$ implies $x = x \wedge y = y * (y * x)$. It is called *branchwise commutative*, if $x \wedge y = y \wedge x$, for all $x, y \in V(a)$ and all $a \in L_p(X)$. Note that a BCI-algebra X is commutative if and only if it is branchwise commutative.

2. (f, g)-Derivations of BCI-Algebras

Recently greater interest has been developed in the derivation of BCI-algebras, introduced by Jun and Xin [14]. The notion was further explored in the form of f-derivations of BCI-algebras by Zhan and Liu [15]. We recall the following definition from [14]. Let X be a BCI-algebra. A *left-right derivation* (briefly, (l, r)-*derivation*) of X is a self-map d of X satisfying the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)), \quad \forall x, y \in X. \tag{1}$$

If d satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then d is called a *right-left derivation* (briefly, (r, l)-*derivation*) of X. Moreover, if d is both an (l, r)- and (r, l)-derivation, then it is called a *derivation*.

According to [15], a *left-right f-derivation* (briefly, $(l, r) - f$-*derivation*) of X is a self-map d of X satisfying the identity

$$d(x * y) = (d(x) * f(y)) \wedge (f(x) * d(y)), \quad \forall x, y \in X, \tag{2}$$

where f is an endomorphism of X. If d satisfies the identity $d(x * y) = (f(x) * d(y)) \wedge (d(x) * f(y))$, for all $x, y \in X$, then d is called a *right-left f-derivation* (briefly, $(r, l) - f$-*derivation*) of X. Moreover, if d is both an (l, r)- and (r, l)-f-derivation, then it is called an f-*derivation*.

Now, we introduce the notion of (f, g)-derivation of BCI-algebras.

Definition 1. Let X be a BCI-algebra. A *left-right (f, g)-derivation* (briefly, $(l, r) - (f, g)$-*derivation*) of X is a self-map d of X satisfying the identity

$$d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y)), \quad \forall x, y \in X, \tag{3}$$

where f, g are endomorphisms of X. If d satisfies the identity $d(x * y) = (f(x) * d(y)) \wedge (d(x) * g(y))$, for all $x, y \in X$, then d is called a *right-left (f, g)-derivation* (briefly, $(r, l) - (f, g)$-*derivation*) of X. Moreover, if d is both an (l, r)- and (r, l)-(f, g)-derivation, it is called an (f, g)-*derivation*.

Example 2. Let $X = \{0, a, b\}$ and a binary operation $*$ is defined as follows:

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

(4)

Then, it forms a p-semisimple BCI-algebra (see [16]). Define maps $d : X \to X$ and $f, g : X \to X$ by $d(x) = 0$, for all $x \in X$, $f = 0$ and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \\ a & \text{if } x = b. \end{cases} \tag{5}$$

Then, f and g are endomorphisms. It is easy to check that d is both (l, r)- and (r, l)-(f, g)-derivation of X. So, d is an (f, g)-derivation.

Now, we consider $h : X \to X$ by $h = I$. Then, h is an endomorphism and d is not an (l, r)-(h, g)-derivation, since $d(a * b) = 0$ but $(d(a) * h(b)) \wedge (g(a) * d(b)) = a \wedge b = a$. Also, d is not an (r, l)-(h, g)-derivation, since $d(a * b) = 0$ but $(h(a) * d(b)) \wedge (d(a) * g(b)) = a \wedge b = a$.

Example 3. Let $X = \{0, a, b, c, d, e\}$ be a BCI-algebra with the following Cayley table:

$$\begin{array}{c|cccccc} * & 0 & a & b & c & d & e \\ \hline 0 & 0 & 0 & b & b & b & b \\ a & a & 0 & b & b & b & b \\ b & b & b & 0 & 0 & 0 & 0 \\ c & c & b & a & 0 & 0 & 0 \\ d & d & b & a & a & 0 & a \\ e & e & b & a & a & a & 0 \end{array} \tag{6}$$

Define maps $d : X \to X$ and $f, g : X \to X$ by $g(x) = 0$, for all $x \in X$:

$$d(x) = \begin{cases} b & \text{if } x = 0, a \\ 0 & \text{otherwise,} \end{cases}$$
$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{otherwise.} \end{cases} \tag{7}$$

Then, f and g are endomorphisms. d is both derivation and f-derivation of X (see Example 2.2 of [15]). Also, it is easy to check that d is both (l, r)- and (r, l)-(f, g)-derivation of X. So, d is an (f, g)-derivation.

Example 4. Let both X and $d : X \to X$ be as in Example 3. Define $f, g : X \to X$ by $f = 0$ and $g = I$. Then, f, g are endomorphisms. d is not a f-derivation of X (see Example 2.3 of [15]). Also, d is not an (l, r)-(f, g)-derivation, since $d(b * c) = d(0) = b$ but $(d(b) * f(c)) \wedge (g(b) * d(c)) = 0 \wedge b = 0$. d is not an (r, l)-(f, g)-derivation, since $d(b * c) = d(0) = b$ but $(f(b) * d(c)) \wedge (d(b) * g(c)) = 0 \wedge (0 * c) = 0 \wedge b = 0$.

Theorem 5. *Let f, g be endomorphisms of BCI-algebra X and let d be a self-map of X defined by $d(x) = f_x$, for all $x \in X$. Then, d is an (l, r)-(f, g)-derivation of X.*

Proof. Suppose that $x, y \in X$. Then, we have

$$\begin{aligned} d(x * y) &= f_{x*y} \\ &= 0 * (0 * f(x * y)) = 0 * (0 * (f(x) * f(y))) \\ &= (0 * (0 * f(x))) * (0 * (0 * f(y))) \\ &= f_x * (0 * (0 * f(y))) \\ &= (0 * (0 * f_x)) * (0 * (0 * f(y))) \\ &= 0 * (0 * (f_x * f(y))) \\ &= 0 * (0 * ((0 * (0 * f(x))) * f(y))) \\ &= 0 * (0 * ((0 * f(y)) * (0 * f(x)))) \\ &= 0 * (0 * (0 * f(y * x))) \\ &= 0 * f(y * x) = 0 * (f(y) * f(x)) \\ &= (0 * f(y)) * (0 * f(x)) \\ &= (0 * (0 * f(x))) * f(y) = f_x * f(y). \end{aligned} \tag{8}$$

Since $f_x \in L_p(X)$, $f_x * f(y) \in L_p(X)$ and $f_x * f(y) = (f_x * f(y)) \wedge (g(x) * f_y)$. Hence, by (8), we get

$$\begin{aligned} d(x * y) &= (f_x * f(y)) \wedge (g(x) * f_y) \\ &= (d(x) * f(y)) \wedge (g(x) * d(y)). \end{aligned} \tag{9}$$

So, d is an (l, r)-(f, g)-derivation of X. □

Theorem 6. *Let d be a self-map of a BCI-algebra X. Then, the following properties hold.*

(1) *If d is an (l, r)-(f, g)-derivation of X, then $d(x) = d(x) \wedge g(x)$, for all $x \in X$.*

(2) *If d is an (r, l)-(f, g)-derivation of X, then $d(x) = f(x) \wedge d(x)$, for all $x \in X$ if and only if $d(0) = 0$.*

Proof. (1) Suppose that $x \in X$. Then, we have

$$\begin{aligned} d(x) &= d(x * 0) = (d(x) * f(0)) \wedge (g(x) * d(0)) \\ &= d(x) \wedge (g(x) * d(0)) \\ &= (g(x) * d(0)) * ((g(x) * d(0)) * d(x)) \\ &= (g(x) * d(0)) * ((g(x) * d(x)) * d(0)) \\ &\le g(x) * (g(x) * d(x)) = d(x) \wedge g(x). \end{aligned} \tag{10}$$

It is clear that $d(x) \wedge g(x) \le d(x)$. So, $d(x) = d(x) \wedge g(x)$.

(2) Suppose that d is an (r, l)-(f, g)-derivation of X. If $d(x) = f(x) \wedge d(x)$, for all $x \in X$, then $d(0) = f(0) \wedge d(0) = 0 \wedge d(0) = 0$. Conversely, let $d(0) = 0$. Then, $d(x) = d(x * 0) = (f(x) * d(0)) \wedge (d(x) * g(0)) = f(x) \wedge d(x)$. □

Corollary 7. *Let d be an (l, r)-(f, g)-derivation of a BCI-algebra X. Then, $d(0) \in L_p(X)$.*

Proof. By Theorem 6(1), $d(0) = d(0) \wedge g(0) = 0 * (0 * d(0))$. So, $d(0) \in L_p(X)$. □

Theorem 8. *Let X be a p-semisimple BCI-algebra and let d be an (l, r)-(f, g)-derivation X. Then,*

(1) *$d(a) = d(0) * (0 * f(a)) = d(0) + f(a)$, for all $a \in L_p(X)$;*

(2) *$d(a) \in L_p(X)$, for all $a \in L_p(X)$;*

(3) *$d(a + b) = d(a) + d(b) - d(0)$, for all $a, b \in L_p(X)$.*

Proof. (1) Let $a \in L_p(X)$. Then, $f(a) \in L_p(X)$. Hence, by Corollary 7, we have

$$\begin{aligned} d(a) &= d(0 * (0 * a)) \\ &= (d(0) * f(0 * a)) \wedge (g(0) * d(0 * a)) \\ &= (d(0) * f(0 * a)) \wedge (0 * d(0 * a)) \end{aligned}$$

$$\begin{aligned}
&= (0 * d(0 * a)) * \big((0 * d(0 * a)) \\
&\qquad * (d(0) * f(0 * a))\big) \\
&= (0 * d(0 * a)) * \big((0 * (d(0) * f(0 * a))) \\
&\qquad * d(0 * a)\big) \\
&= 0 * (0 * (d(0) * f(0 * a))) \\
&= 0 * (0 * (d(0) * (0 * f(a)))) \\
&= (0 * (0 * d(0))) * (0 * (0 * (0 * f(a)))) \\
&= (0 * (0 * d(0))) * (0 * f(a)) = d(0) * (0 * f(a)) \\
&= d(0) + f(a).
\end{aligned} \tag{11}$$

The proof of (2) and (3) follows directly from (1). □

Theorem 9. *Let d be an (r,l)-(f,g)-derivation of BCI-algebra X. Then,*

(1) *$d(a) \in G(X)$, for all $a \in G(X)$;*

(2) *$d(a) \in L_p(X)$, for all $a \in L_p(X)$;*

(3) *$d(a) = f(a) * d(0) = f(a) + d(0)$, for all $a \in L_p(X)$;*

(4) *$d(a+b) = d(a) + d(b) - d(0)$, for all $a, b \in L_p(X)$.*

Proof. (1) Suppose that $a \in G(X)$. Then, we have

$$\begin{aligned}
d(a) &= d(0 * a) \\
&= (f(0) * d(a)) \wedge (d(0) * g(a)) \\
&= (0 * d(a)) \wedge (d(0) * g(a)) = 0 * d(a).
\end{aligned} \tag{12}$$

So, $d(a) \in G(X)$.

(2) Suppose that $a \in L_p(X)$. Then, we have

$$\begin{aligned}
d(a) &= d(0 * (0 * a)) \\
&= (f(0) * d(0 * a)) \wedge (d(0) * g(0 * a)) \\
&= 0 * d(0 * a) \in L_p(X).
\end{aligned} \tag{13}$$

(3) Suppose that $a \in L_p(X)$. Then, we have

$$\begin{aligned}
d(a) &= d(a * 0) = (f(a) * d(0)) \wedge (d(a) * g(0)) \\
&= f(a) * d(0) = f(a) * (0 * d(0)) = f(a) + d(0).
\end{aligned} \tag{14}$$

(4) The proof follows from (3). □

Definition 10. An (f,g)-derivation d of a BCI-algebra X is called *regular* if $d(0) = 0$. If $d(0) \neq 0$, then d is called *irregular*.

Theorem 11. *Let X be a commutative BCI-algebra and let d be a regular (r,l)-(f,g)-derivation of X. Then, both $f(x)$ and $d(x)$ belong to the same branch, for all $x \in X$.*

Proof. Suppose that $x \in X$. Then, we have

$$\begin{aligned}
0 = d(0) &= d(a_x * x) = (f(a_x) * d(x)) \wedge (d(a_x) * g(x)) \\
&= (f_x * d(x)) \wedge (d(a_x) * g(x)) = f_x * d(x).
\end{aligned} \tag{15}$$

Hence, $f_x \leq d(x)$ and so $d(x) \in V(f_x)$. Also, we have $f(x) \in V(f_x)$, since $f_x \leq f(x)$. This completes the proof. □

Theorem 12. *Let d be a regular (r,l)-(f,g)-derivation of BCI-algebra X. Then,*

(1) *$d(x) \leq f(x)$, for all $x \in X$;*

(2) *$d(x) * f(y) \leq f(x) * d(y)$, for all $x, y \in X$.*

Proof. (1) By Theorem 6 (2), $d(x) = f(x) \wedge d(x) \leq f(x)$, for all $x \in X$.

(2) By part (1), $d(x) * f(y) \leq f(x) * f(y) \leq f(x) * d(y)$, for all $x \in X$. □

Theorem 13. *Every (r,l)-(f,g)-derivation ((l,r)-(f,g)-derivation) of a BCK-algebra X is regular.*

Proof. Suppose that X is a BCK-algebra and d is an (r,l)-(f,g)-derivation of X. Then, for all $x \in X$, we have

$$\begin{aligned}
d(0) &= d(0 * x) = (f(0) * d(x)) \wedge (d(0) * g(x)) \\
&= (0 * d(x)) \wedge (d(0) * g(x)) = 0 \wedge (d(0) * g(x)) = 0.
\end{aligned} \tag{16}$$

So, d is regular.

Now, suppose that d is an (l,r)-(f,g)-derivation of X. Then, $d(0) = d(0 * x) = (d(0) * f(x)) \wedge (g(0) * d(x)) = (d(0) * f(x)) \wedge 0 = 0$. □

Theorem 14. *Let f be an epimorphism and let g be an endomorphism of a BCI-algebra X. Also, let d be an (r,l)-(f,g)-derivation of X and $a \in X$ such that $a * d(x) = 0$ and $f(a) * d(x) = 0$, for all $x \in X$. Then, d is regular. Moreover, X is a BCK-algebra.*

Proof. For all $x \in X$, we have

$$\begin{aligned}
0 &= a * d(a * x) = a * ((f(a) * d(x)) \wedge (d(a) * g(x))) \\
&= a * (0 \wedge (d(a) * g(x))) = a.
\end{aligned} \tag{17}$$

Hence, $d(0) = d(0 * 0) = (f(0) * d(0)) \wedge d(0) = 0 \wedge d(0) = 0$.

By Theorem 12(1), $0 * f(x) \leq 0 * d(x) = 0$. So, $(0 * f(x)) * 0 = 0$, for all $x \in X$. Then, $0 * f(x) = 0$, for all $x \in X$. Therefore, $f(X)$ is a BCK-algebra. This implies that X is a BCK-algebra, since f is an epimorphism. □

Theorem 15. *Let f be an epimorphism and let g be an endomorphism of a BCI-algebra X. Also, let d be an (f,g)-derivation of X and $a \in X$ such that $d(x) * a = 0$ and $d(x) * f(a) = 0$, for all $x \in X$. Then, d is regular. Moreover, X is a BCK-algebra.*

Proof. For all $x \in X$, we have

$$\begin{aligned} 0 &= d(x * a) * a = ((d(x) * f(a)) \wedge (g(x) * d(a))) * a \\ &= (0 \wedge (g(x) * d(a))) * a = 0 * a. \end{aligned} \tag{18}$$

So,

$$\begin{aligned} d(0) &= d(0 * a) = (d(0) * f(a)) \wedge (g(0) * d(a)) \\ &= 0 \wedge (0 * d(a)) = 0. \end{aligned} \tag{19}$$

By Theorem 12(1), we get

$$\begin{aligned} 0 * f(x) &\leq 0 * d(x) = (d(x) * a) * d(x) \\ &= (d(x) * d(x)) * a = 0 * a = 0. \end{aligned} \tag{20}$$

Thus, $(0 * f(x)) * 0 = 0$, for all $x \in X$. So, $0 * f(x) = 0$, for all $x \in X$. Hence, $f(X)$ is a BCK-algebra. This implies that X is a BCK-algebra, since f is an epimorphism. □

Theorem 16. *Let X be a p-semisimple BCI-algebra, and let d and d' be (l,r)-(f,g)-derivations (resp., (r,l)-(f,g)-derivations) of X. Also, let $f \circ f = f$. Then, $d \circ d'$ is also an (l,r)-(f,g)-derivation (resp., (r,l)-(f,g)-derivation) of X.*

Proof. Suppose that d and d' are (l,r)-(f,g)-derivations of X. Then, for all $x, y \in X$,

$$\begin{aligned} &(d \circ d')(x * y) \\ &\quad = d(d'(x * y)) \\ &\quad = d((d'(x) * f(y)) \wedge (g(x) * d'(y))) \\ &\quad = d(d'(x) * f(y)) \\ &\quad = (d(d'(x)) * f(f(y))) \wedge (g(d'(x)) * d(f(y))) \\ &\quad = (d \circ d')(x) * f(y) \\ &\quad = ((d \circ d')(x) * f(y)) \wedge (g(x) * (d \circ d')(y)). \end{aligned} \tag{21}$$

So, $d \circ d'$ is an (l,r)-(f,g)-derivation of X. Now, let d, d' be (r,l)-(f,g)-derivations of X. Then, for all $x, y \in X$, we have

$$\begin{aligned} &(d \circ d')(x * y) \\ &\quad = d((f(x) * d'(y)) \wedge (d'(x) * g(y))) \\ &\quad = d(f(x) * d'(y)) \\ &\quad = (f(f(x)) * d(d'(y))) \wedge (d(f(x)) * g(d'(y))) \\ &\quad = f(x) * d \circ d'(y) \\ &\quad = (f(x) * (d \circ d')(y)) \wedge ((d \circ d')(x) * g(y)). \end{aligned} \tag{22}$$

So, $d \circ d'$ is an (r,l)-(f,g)-derivation of X. □

Theorem 17. *Let X be a p-semisimple BCI-algebra, and let d and d' be, respectively, (r,l)-(f,g)-derivation and (l,r)-(f,g)-derivation of X such that $f \circ d = d \circ f$, $d' \circ f = f \circ d'$. Then, $d \circ d' = d' \circ d$.*

Proof. For all $x, y \in X$, we have

$$\begin{aligned} &(d \circ d')(x * y) \\ &\quad = d((d'(x) * f(y)) \wedge (g(x) * d'(y))) \\ &\quad = d(d'(x) * f(y)) \\ &\quad = (f(d'(x)) * d(f(y))) \wedge (d(d'(x)) * g(f(y))) \\ &\quad = f(d'(x)) * d(f(y)) = f \circ d'(x) * d \circ f(y). \end{aligned} \tag{23}$$

Also, for all $x, y \in X$,

$$\begin{aligned} &(d' \circ d)(x * y) \\ &\quad = d'((f(x) * d(y)) \wedge (d(x) * g(y))) \\ &\quad = d'(f(x) * d(y)) \\ &\quad = (d'(f(x)) * f(d(y))) \wedge (g(f(x)) * d'(d(y))) \\ &\quad = d'(f(x)) * f(d(y)) = d' \circ f(x) * f \circ d(y) \\ &\quad = f \circ d'(x) * d \circ f(y). \end{aligned} \tag{24}$$

By using (23) and (24), we obtain $d \circ d'(x * y) = d' \circ d(x * y)$, for all $x, y \in X$. By putting $y = 0$, we get $d \circ d'(x) = d' \circ d(x)$, for all $x \in X$. □

Definition 18 (see [16]). Let X be a BCI-algebra and let d, d' be two self-maps of X. We define $d \bullet d' : X \to X$ by $(d \bullet d')(x) = d(x) * d'(x)$, for all $x \in X$.

Theorem 19. *Let X be a p-semisimple BCI-algebra and let d, d' be (f,g)-derivations of X. Then,*

(1) $(f \circ d') \bullet (d \circ f) = (d \circ f) \bullet (f \circ d')$,

(2) $(d \circ d') \bullet (f \circ f) = (f \circ f) \bullet (d \circ d')$.

Proof. (1) Since d is an (r,l)-(f,g)-derivation and d' is an (l,r)-(f,g)-derivation of X, for all $x, y \in X$, we have

$$\begin{aligned} &(d \circ d')(x * y) \\ &\quad = d((d'(x) * f(y)) \wedge (g(x) * d'(y))) \\ &\quad = d(d'(x) * f(y)) \\ &\quad = (f(d'(x)) * d(f(y))) \wedge (d(d'(x)) * g(f(y))) \\ &\quad = f \circ d'(x) * d \circ f(y). \end{aligned} \tag{25}$$

Also, since d is an (l,r)-(f,g)-derivation and d' is an (r,l)-(f,g)-derivation of X, for all $x, y \in X$, we have

$$\begin{aligned}(d\circ d')(x*y) &= d((f(x)*d'(y))\wedge(d'(x)*g(y)))\\ &= d(f(x)*d'(y))\\ &= (d(f(x))*f(d'(y)))\wedge(g(f(x))*d(d'(y)))\\ &= d(f(x))*f(d'(y)) = d\circ f(x)*f\circ d'(y).\end{aligned} \tag{26}$$

From (25) and (26), we get $f\circ d'(x)*d\circ f(y) = d\circ f(x)*f\circ d'(y)$, for all $x, y \in X$. By putting $x = y$, $(f\circ d' \bullet d\circ f)(x) = (d\circ f \bullet f\circ d')(x)$, for all $x \in X$.

(2) Since d and d' are (l,r)-(f,g)-derivations, then for all $x, y \in X$, we have

$$\begin{aligned}(d\circ d')(x*y) &= d((d'(x)*f(y))\wedge(g(x)*d'(y)))\\ &= d(d'(x)*f(y))\\ &= (d(d'(x))*f(f(y)))\wedge(g(d'(x))*d(f(y)))\\ &= d\circ d'(x)*f\circ f(y).\end{aligned} \tag{27}$$

On the other hand, since d and d' are (r,l)-(f,g)-derivations, then for all $x, y \in X$, we have

$$\begin{aligned}(d\circ d')(x*y) &= d((f(x)*d'(y))\wedge(d'(x)*g(y)))\\ &= d(f(x)*d'(y))\\ &= (f(f(x))*d(d'(y)))\wedge(d(f(x))*g(d'(y)))\\ &= f\circ f(x)*d\circ d'(y).\end{aligned} \tag{28}$$

By (27) and (28), for all $x, y \in X$,

$$d\circ d'(x)*f\circ f(y) = f\circ f(x)*d\circ d'(y). \tag{29}$$

By putting $x = y$, we get $(d\circ d' \bullet f\circ f)(x) = (f\circ f \bullet d\circ d')(x)$, for all $x \in X$. □

Theorem 20. *Let X be a commutative p-semisimple BCI-algebra and let d, d' be (f,g)-derivations of X. Then, $d = d'$ if $d(x)$ and $d'(x)$ lie in the same branch.*

Proof. For all $x, y \in X$, we have

$$(d\wedge d')(x*y) = d(x*y)\wedge d'(x*y) = d(x*y). \tag{30}$$

On the other hand,

$$\begin{aligned}(d\wedge d')(x*y) &= d(x*y)\wedge d'(x*y)\\ &= d'(x*y)\wedge d(x*y) = d'(x*y).\end{aligned} \tag{31}$$

By (30) and (31), $d(x*y) = d'(x*y)$, for all $x, y \in X$. By putting $y = 0$, we obtain $d(x) = d'(x)$, for all $x \in X$. □

We denote by $\mathrm{Der}(X)$ the set of all (f,g)-derivations on X.

Definition 21 (see [16]). Let $d, d' \in \mathrm{Der}(X)$. The binary operation $\wedge$ on $\mathrm{Der}(X)$ is defined as $(d\wedge d')(x) = d(x)\wedge d'(x)$, for all $x \in X$.

Theorem 22. *Let X be a p-semisimple BCI-algebra and let d, d' be (l,r)-(f,g)-derivations (resp., (r,l)-(f,g)-derivations) of X. Then, $d\wedge d'$ is also an (l,r)-(f,g)-derivation (resp., (r,l)-(f,g)-derivation) of X.*

Proof. Suppose that d, d' are (l,r)-(f,g)-derivations of X and $x, y \in X$. Then, we have

$$\begin{aligned}(d\wedge d')(x*y) &= d(x*y)\wedge d'(x*y) = d(x*y)\\ &= (d(x)*f(y))\wedge(g(x)*d(y)) = d(x)*f(y)\\ &= (d(x)\wedge d'(x))*f(y) = (d\wedge d')(x)*f(y)\\ &= ((d\wedge d')(x)*f(y))\wedge(g(x)*(d\wedge d')(y)).\end{aligned} \tag{32}$$

So, $d\wedge d'$ is an (l,r)-(f,g)-derivation of X.

Suppose that d, d' are (r,l)-(f,g)-derivations of X and $x, y \in X$. Then,

$$\begin{aligned}(d\wedge d')(x*y) &= d(x*y)\wedge d'(x*y) = d(x*y)\\ &= f(x)*d(y) = f(x)*(d(y)\wedge d'(y))\\ &= f(x)*(d\wedge d')(y)\\ &= (f(x)*(d\wedge d')(y))\wedge((d\wedge d')(x)*g(y)).\end{aligned} \tag{33}$$

So, $d\wedge d'$ is an (r,l)-(f,g)-derivation of X. □

Theorem 23. *Let X be a p-semisimple BCI-algebra. Then, $(Der(X),\wedge)$ is a semigroup.*

Proof. By Theorem 22, $d\wedge d' \in \mathrm{Der}(X)$. Suppose that $d, d', d'' \in \mathrm{Der}(X)$. Then,

$$\begin{aligned}(d\wedge(d'\wedge d''))(x*y) &= d(x*y)\wedge(d'\wedge d'')(x*y)\\ &= d(x*y).\end{aligned} \tag{34}$$

Table 1

f	g	(f,g)-derivations
d_1	d_1	d_1
	d_2	d_1
d_2	d_1	d_1
	d_2	d_1, d_2

Table 2

f	g	(f,g)-derivations
d_1	d_1	d_1
	d_2	d_1
d_2	d_1	d_1
	d_2	d_1, d_2, d_3

Also, we have

$$\begin{aligned}\left(\left(d\wedge d'\right)\wedge d''\right)(x*y) &= \left(d\wedge d'\right)(x*y)\\ &= d(x*y)\wedge d'(x*y)\\ &= d(x*y).\end{aligned} \tag{35}$$

Therefore, $d\wedge(d'\wedge d'') = (d\wedge d')\wedge d''$. This completes the proof. □

At the end of this section, we classify (f,g)-derivations on BCK-algebras of order 3.

There is only one BCK-algebra of order 2. It is called B_{2-1} and it is as follows:

$*$	0	a
0	0	0
a	a	0

(36)

There are only two endomorphisms on B_{2-1}. They are as $d_1 = 0$ and $d_2 = I$. We have Table 1, and so

$\wedge$	d_1	d_2
d_1	d_1	d_1
d_2	d_1	d_2

(37)

There are only three BCK-algebras of order 3. In the following, we classify (f,g)-derivations on them.

Let $B_{3-1} = \{0,a,b\}$. Consider the following table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

(38)

There are only two endomorphisms on B_{3-1}. They are as $d_1 = 0$ and $d_2 = I$. Set

$$d_3 = \begin{cases} 0 & \text{if } x = 0, a\\ a & \text{if } x = b;\end{cases} \tag{39}$$

then we have Table 2, and so

$\wedge$	d_1	d_2	d_3
d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_3
d_3	d_1	d_3	d_3

(40)

Let $B_{3-2} = \{0,a,b\}$. Consider the following table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

(41)

There are only four endomorphisms on B_{3-2}. They are as $d_1 = 0$, $d_2 = I$,

$$\begin{aligned} d_3 &= \begin{cases} 0 & \text{if } x = 0, a\\ a & \text{if } x = b,\end{cases}\\ d_4 &= \begin{cases} 0 & \text{if } x = 0, a\\ b & \text{if } x = b.\end{cases}\end{aligned} \tag{42}$$

Set

$$d_5 = \begin{cases} 0 & \text{if } x = 0, b\\ a & \text{if } x = a;\end{cases} \tag{43}$$

then we have Table 3, and so

$\wedge$	d_1	d_2	d_3	d_4	d_5
d_1	d_1	d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_3	d_4	d_5
d_3	d_1	d_3	d_3	d_1	d_1
d_4	d_1	d_4	d_3	d_4	d_1
d_5	d_1	d_5	d_1	d_1	d_5

(44)

Let $B_{3-3} = \{0,a,b\}$. Consider the following table:

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

(45)

There are only seven endomorphisms on B_{3-3}. They are as $d_1 = 0$, $d_2 = I$,

$$\begin{aligned} d_3 &= \begin{cases} 0 & \text{if } x = 0, a\\ a & \text{if } x = b,\end{cases}\\ d_4 &= \begin{cases} 0 & \text{if } x = 0, a\\ b & \text{if } x = b,\end{cases}\\ d_5 &= \begin{cases} 0 & \text{if } x = 0, b\\ a & \text{if } x = a,\end{cases}\\ d_6 &= \begin{cases} 0 & \text{if } x = 0, b\\ b & \text{if } x = a,\end{cases}\\ d_7 &= \begin{cases} 0 & \text{if } x = 0\\ a & \text{if } x = b\\ b & \text{if } x = a.\end{cases}\end{aligned} \tag{46}$$

Table 3

f	g	(f,g)-derivations	f	g	(f,g)-derivations
d_1	d_1	d_1	d_3	d_1	d_1
	d_2	d_1		d_2	d_1
	d_3	d_1		d_3	d_1, d_3
	d_4	d_1		d_4	d_1
d_2	d_1	d_1	d_4	d_1	d_1
	d_2	d_1, d_2, d_4, d_5		d_2	d_1, d_4
	d_3	d_1		d_3	d_1, d_3
	d_4	d_1, d_4		d_4	d_1, d_4

Obviously, if we set $f = d_1 = 0$, then for all $g = d_i$, $1 \le i \le 7$, we have only one (f,g)-derivation and it is $d_1 = 0$. Also, we have Table 4, and so

$$\begin{array}{c|ccccccc} \wedge & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \\ \hline d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 \\ d_2 & d_1 & d_2 & d_1 & d_4 & d_5 & d_1 & d_1 \\ d_3 & d_1 & d_1 & d_3 & d_1 & d_1 & d_1 & d_3 \\ d_4 & d_1 & d_4 & d_1 & d_4 & d_1 & d_1 & d_1 \\ d_5 & d_1 & d_5 & d_1 & d_1 & d_5 & d_1 & d_1 \\ d_6 & d_1 & d_1 & d_1 & d_1 & d_1 & d_6 & d_6 \\ d_7 & d_1 & d_1 & d_3 & d_1 & d_1 & d_6 & d_7 \end{array} \tag{47}$$

3. *(F, G)*-Derivations of *BCI*-Algebras

Let X be a BCI-algebra. A *right derivation* of X is a self-map D of X satisfying the identity

$$\begin{aligned} &D(x * y) \\ &\quad = (x * D(y)) \wedge (y * D(x)), \quad \forall x, y \in X. \end{aligned} \tag{48}$$

If D satisfies the identity $D(x * y) = (D(x) * y) \wedge (D(y) * x)$, for all $x, y \in X$, then D is called a *left derivation* of X.

Let X be a BCI-algebra. A *right F-derivation* of X is a self-map D of X satisfying the identity

$$\begin{aligned} &D(x * y) \\ &\quad = (F(x) * D(y)) \wedge (F(y) * D(x)), \quad \forall x, y \in X, \end{aligned} \tag{49}$$

where F is an endomorphism of X. If D satisfies the identity $D(x * y) = (D(x) * F(y)) \wedge (D(y) * F(x))$, for all $x, y \in X$, then D is called a *left F-derivation* of X. Moreover, if D is both right and left F-derivation, then D is called an *F-derivation* of X; see [13].

The notion of left (F,G)-derivation of BCI-algebras is introduced in [17]. In this section, we introduce the notion of right (F,G)-derivation and give some examples and propositions to explain the theory of left (F,G)-derivation and right (F,G)-derivation in BCI-algebras.

Definition 24. Let X be a BCI-algebra. A *right (F,G)-derivation* of X is a self-map D of X satisfying the identity

$$\begin{aligned} &D(x * y) \\ &\quad = (F(x) * D(y)) \wedge (G(y) * D(x)), \quad \forall x, y \in X, \end{aligned} \tag{50}$$

where F and G are endomorphisms of X. If D satisfies the identity $D(x * y) = (D(x) * F(y)) \wedge (D(y) * G(x))$, for all $x, y \in X$, then D is called a *left (F,G)-derivation* of X. Moreover, if D is both right and left (F,G)-derivation, then it is said that D is an *(F,G)-derivation* of X.

Example 25. Let X, d, f, and g be as in Example 3. Define $D = d$, $F = f$, and $G = g = 0$. It is easy to see that D is both right (F,G)-derivation and left (F,G)-derivation. So, D is an (F,G)-derivation.

Example 26. Let X and d be as in Example 3. Define $D = d$, $F = 0$, and $G = I$. D is not a right (F,G)-derivation, since $D(b * c) = b$ but $(F(b) * D(c)) \wedge (G(c) * D(b)) = 0 \wedge c = 0$. Also, D is not a left (F,G)-derivation, since $D(b * c) = b$ but $(D(b) * F(c)) \wedge (D(c) * G(b)) = 0 \wedge b = 0$.

Example 27. Let $X = \{0, a, b\}$ and the binary operation $*$ is defined as follows:

$$\begin{array}{c|ccc} * & 0 & a & b \\ \hline 0 & 0 & 0 & 0 \\ a & a & 0 & a \\ b & b & b & 0 \end{array} \tag{51}$$

Then, $(X, *)$ is a commutative BCK-algebra. Define maps $D, F, G : X \to X$ by $D(x) = 0$, for all $x \in X$, $F = I$ and

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \\ a & \text{if } x = b. \end{cases} \tag{52}$$

Then, F and G are endomorphisms. It is easy to check that D is a left (F,G)-derivation. D is not a right (F,G)-derivation, since $D(b*a) = 0$ but $(F(b)*D(a)) \wedge (G(a)*D(b)) = b*0 = b$.

Definition 28. An (F,G)-derivation D of a BCI-algebra X is called *regular* if $D(0) = 0$. If $D(0) \neq 0$, then D is called *irregular*.

Theorem 29. *Every right (F,G)-derivation of a BCK-algebra is regular.*

Proof. Suppose that X is a BCK-algebra and D is a right (F,G)-derivation of X. Then,

$$D(0) = D(0 * 0) = (F(0) * D(0)) \wedge (G(0) * D(0)) = 0. \tag{53}$$

So, D is regular. □

Theorem 30. *Let D be a right (F,G)-derivation of a BCI-algebra X. Then, $D(x) \in G(X)$, for all $x \in G(X)$.*

Proof. Suppose that $x \in G(X)$. Then,

$$\begin{aligned} D(x) &= D(0 * x) = (F(0) * D(x)) \wedge (G(x) * D(0)) \\ &\le F(0) * D(x) = 0 * D(x) \in L_p(X). \end{aligned} \tag{54}$$

Thus, $D(x) = 0 * D(x)$. So, $D(x) \in G(X)$. □

Table 4

f	g	(f,g)-derivations	f	g	(f,g)-derivations	f	g	(f,g)-derivations
	d_1	d_1		d_1	d_1		d_1	d_1
	d_2	d_1, d_2, d_4, d_5		d_2	d_1, d_4		d_2	d_1
	d_3	d_1		d_3	d_1		d_3	d_1
d_2	d_4	d_1, d_4	d_4	d_4	d_1, d_4	d_6	d_4	d_1
	d_5	d_1, d_5		d_5	d_1		d_5	d_1
	d_6	d_1		d_6	d_1		d_6	d_1, d_6
	d_7	d_1		d_7	d_1		d_7	d_1, d_6
	d_1	d_1		d_1	d_1		d_1	d_1
	d_2	d_1		d_2	d_1, d_5		d_2	d_1
	d_3	d_1, d_3		d_3	d_1		d_3	d_1, d_3
d_3	d_4	d_1	d_5	d_4	d_1	d_7	d_4	d_1
	d_5	d_1		d_5	d_1, d_5		d_5	d_1
	d_6	d_1		d_6	d_1		d_6	d_1, d_6
	d_7	d_1, d_3		d_7	d_1		d_7	d_1, d_3, d_6, d_7

Theorem 31. *Let D be a right (F,G)-derivation of a BCI-algebra X. Then, the following hold:*

(1) *$D(x) \in L_p(X)$, for all $x \in X$;*

(2) *$F(y) * (F(y) * D(x)) = D(x)$, for all $x, y \in X$;*

(3) *$D(x) * F(y) = 0 * (F(y) * D(x))$, for all $x, y \in X$;*

(4) *$D(x) * F(y) \in L_p(X)$, for all $x, y \in X$.*

Proof. (1) For all $x \in X$, we have

$$\begin{aligned} D(x) &= D(x * 0) = (F(x) * D(0)) \wedge (G(0) * D(x)) \\ &= (0 * D(x)) * ((0 * D(x)) * (F(x) * D(0))) \\ &= (0 * D(x)) * ((0 * (F(x) * D(0))) * D(x)) \\ &\leq 0 * (0 * (F(x) * D(0))). \end{aligned} \tag{55}$$

So, $D(x) \in L_p(X)$.

(2), (3), and (4) are clear. □

Theorem 32. *Let X be a p-semisimple BCI-algebra. Then*

(1) *if D is an (l,r)-(F,G)-derivation of X, then D is a left (F,G)-derivation of X;*

(2) *if D is an (r,l)-(F,G)-derivation of X, then D is a right (F,G)-derivation of X.*

Proof. (1) Suppose that $x, y \in X$ and D is an (l,r)-(F,G)-derivation. Then,

$$\begin{aligned} &D(x * y) \\ &\quad = (D(x) * F(y)) \wedge (G(x) * D(y)) \leq D(x) * F(y). \end{aligned} \tag{56}$$

So, $D(x * y) = D(x) * F(y) = (D(x) * F(y)) \wedge (D(y) * G(x))$, which implies that D is a left (F,G)-derivation of X.

(2) Suppose that $x, y \in X$ and D is an (r,l)-(F,G)-derivation. Then,

$$\begin{aligned} D(x * y) &= (F(x) * D(y)) \wedge (D(x) * G(y)) \\ &\leq F(x) * D(y). \end{aligned} \tag{57}$$

So, $D(x * y) = F(x) * D(y) = (F(x) * D(y)) \wedge (G(y) * D(x))$, which implies that D is a right (F,G)-derivation of X. □

Theorem 33. *Let D be a self-map and let $V(x_0)$ be any branch of a BCI-algebra X. If, for any $x \in V(x_0)$, $D(x) = F(x_0)$, then D is a regular left (F,G)-derivation.*

Proof. Suppose that $x, y \in X$, $x \in V(x_0)$, and $y \in V(y_0)$; it is not necessary that $x_0 \neq y_0$. Then, $x * y \in V(x_0 * y_0)$. By using hypothesis, we get

$$D(x * y) = F(x_0 * y_0). \tag{58}$$

Also, by using hypothesis, we have

$$\begin{aligned} &(D(x) * F(y)) \wedge (D(y) * G(x)) \\ &\quad = (F(x_0) * F(y)) \wedge (F(y_0) * G(x)) \\ &\quad \leq F(x_0) * F(y) = F(x_0 * y) \\ &\quad = F(x_0 * y_0). \end{aligned} \tag{59}$$

From $x_0, y_0 \in L_p(X)$, we get $F(x_0 * y_0) \in L_p(X)$. By using (58) and (59), $D(x * y) = (D(x) * F(y)) \wedge (D(y) * G(x))$. So, D is a left (F,G)-derivation. It is clear that D is regular. □

Theorem 34. *A self-map D of a BCI-algebra X defined by $D(x) = 0 * (0 * F(x)) = F_x$, for all $x \in X$, is a left (F,G)-derivation.*

Proof. Suppose that $x, y \in X$. We have $D(x * y) = F_{x*y} = F_x * F_y$ and

$$\begin{aligned} &(D(x) * F(y)) \wedge (D(y) * G(x)) \\ &\quad = (F_x * F(y)) \wedge (F_y * G(x)) \leq F_x * F(y). \end{aligned} \tag{60}$$

So,

$$\begin{aligned}&(D(x)*F(y))\wedge(D(y)*G(x))*\left(F_x*F_y\right)\\&\quad\le(F_x*F(y))*\left(F_x*F_y\right)\le F_y*F(y)=0.\end{aligned}\tag{61}$$

Hence, $(D(x)*F(y))\wedge(D(y)*G(x))\le F_x*F_y$. Therefore, $(D(x)*F(y))\wedge(D(y)*G(x))=F_x*F_y=D(x*y)$, which implies that D is a left (F,G)-derivation. □

Theorem 35. *Let D be a left (F,G)-derivation of a commutative BCI-algebra X. Then, $x\le y$ implies that $D(x)$ and $D(y)$ belong to the same branch of X.*

Proof. Suppose that $x,y\in X$ and $x\le y$. Since X is commutative, $y*(y*x)=x$. Since D is a left (F,G)-derivation, we have

$$\begin{aligned}D(x)&=D(y*(y*x))\\&=(D(y)*F(y*x))\wedge(D(y*x)*G(y))\\&\le D(y)*F(y*x)=D(y)*(F(y)*F(x)).\end{aligned}\tag{62}$$

Since $x\le y$, $0=F(0)=F(x*y)=F(x)*F(y)$. So, $F(x)$ and $F(y)$ are contained in the same branch. Hence, $F(y)*F(x)\in X_+$. By (62), we have

$$\begin{aligned}D(x)*D(y)&\le(D(y)*(F(y)*F(x)))*D(y)\\&=(D(y)*D(y))*(F(y)*F(x))\\&=0*(F(y)*F(x))=0.\end{aligned}\tag{63}$$

Thus, $D(x)*D(y)=0$. So, $D(x)\le D(y)$. Hence, $D(x)$ and $D(y)$ are contained in the same branch of X. □

Theorem 36. *Let D be an (F,G)-derivation of a BCI-algebra X. Also, let $V(x_0)$ and $V(y_0)$ be two arbitrary branches of X such that $F(x_0)\in V(x_0)$ and, for all $y\in V(y_0)$, $F(y)\in V(y_0)$. Then, $D(x)=y_0$ implies $D(y)\in V(x_0)$.*

Proof. Suppose that $x\in V(x_0)$, $y\in V(y_0)$, and $D(x)=y_0$. Since D is a left (F,G)-derivation, we have

$$\begin{aligned}D(x*y)&=(D(x)*F(y))\wedge(D(y)*G(x))\\&\le D(x)*F(y)=y_0*F(y).\end{aligned}\tag{64}$$

On the other hand, $y_0*F(y)=0$, since $F(y)\in V(y_0)$. Therefore, $D(x*y)=0$. Since D is a right (F,G)-derivation, $0=D(x*y)=(F(x)*D(y))\wedge(G(y)*D(x))\le F(x)*D(y)$. Hence, $0*F(x)\le(F(x)*D(y))*F(x)=0*D(y)$. So, $0*(0*D(y))\le 0*(0*F(x))=F(x_0)$. Since $x_0\in L_p(X)$, $F(x_0)\in L_p(X)$. Therefore, $F(x_0)=0*(0*D(y))\le D(y)$. Hence, $F(x_0)$ and $D(y)$ belong to the same branch of X. Then, $D(y)\in V(x_0)$, since $F(x_0)\in V(x_0)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] K. Iséki, "An algebra related with a propositional calculus," *Proceedings of the Japan Academy*, vol. 42, pp. 26–29, 1966.

[2] M. Daoji, "BCI-algebras and abelian groups," *Mathematica Japonica*, vol. 32, pp. 693–696, 1987.

[3] W. A. Dudek, "On BCI-algebras with conditions," *Mathematica Japonica*, vol. 31, no. 1, pp. 25–29, 1986.

[4] Q. P. Hu and K. Iseki, "On BCI-algebras satisfying $(x*y)*z=x*(y*z)$," *Mathematics Seminar Notes*, vol. 8, no. 3, pp. 553–555, 1980.

[5] F. Nisar and S. A. Bhatti, "A note on BCI-algebras of order five," *Punjab University Journal of Mathematics*, vol. 38, pp. 15–37, 2006.

[6] L. Tiande and X. Changchang, "p-radical in *BCI*-algebras," *Mathematica Japonica*, vol. 30, no. 4, pp. 511–517, 1985.

[7] S. A. Bhatti, M. A. Chaudhry, and B. Ahmad, "On classification of BCI-algebras," *Mathematica Japonica*, vol. 34, no. 6, pp. 865–876, 1989.

[8] S. A. Bhatti and M. A. Chaudhry, "Ideals in BCI-algebras," *International Journal of Mathematical Education in Science and Technology*, vol. 21, no. 4, pp. 637–643, 1990.

[9] M. A. Chaudhry, "Branchwise commutative BCI-algebras," *Mathematica Japonica*, vol. 37, no. 1, pp. 163–170, 1992.

[10] K. Iseki and S. Tanaka, "An introduction to the theory of BCK-algebras," *Mathematica Japonica*, vol. 23, no. 1, pp. 1–26, 1978.

[11] K. Iseki, "On BCI-algebras," *Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125–130, 1980.

[12] J. Meng and X. L. Xin, "Commutative BCI-algebras," *Mathematica Japonica*, vol. 37, no. 3, pp. 569–572, 1992.

[13] F. Nisar, "Characterization of F-derivations of BCI-algebras," *East African Medical Journal*, vol. 25, pp. 69–87, 2009.

[14] Y. B. Jun and X. L. Xin, "On derivations of BCI-algebras," *Information Sciences*, vol. 159, no. 3-4, pp. 167–176, 2004.

[15] J. Zhan and Y. L. Liu, "On f-derivations of BCI-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 11, pp. 1675–1684, 2005.

[16] M. A. Javed and M. Aslam, "A note on f-derivations of BCI-algebras," *Communications of the Korean Mathematical Society*, vol. 24, no. 3, pp. 321–331, 2009.

[17] G. Muhiuddin and A. M. Al-Roqi, "On (α,β)-derivations in BCI-algebras," *Discrete Dynamics in Nature and Society*, pp. 1–12, 2012.