Research Article
On Generalized Derivations of BCI-Algebras and Their Properties

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We introduce the concept of \((f, g)\)-derivations of BCI-algebras and we investigate some fundamental properties and establish some results on \((f, g)\)-derivations. Also, we treat to generalization of right derivation and left derivation of BCI-algebras and consider some related properties.

1. Basic Facts about BCI-Algebras

In 1966, Iséki introduced the concept of BCI-algebra, which is a generalization of the BCK-algebra, as an algebraic counterpart of the BCI-logic [1]; also see [2–6]. In this section, we summarize some basic concepts which will be used throughout the paper. For more details, we refer to the references in [7–12]. Let us recall the definition.

A BCI-algebra \(X\) is an abstract algebra \((X, \ast, 0)\) of type \((2, 0)\), satisfying the following conditions, for all \(x, y, z \in X\):

\[
(BCI1) \; ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0;
(BCI2) \; (x \ast (x \ast y)) \ast y = 0;
(BCI3) \; x \ast x = 0;
(BCI4) \; x \ast y = 0 \; \text{and} \; y \ast x = 0 \; \text{implies} \; x = y.
\]

A nonempty subset \(S\) of a BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x \ast y \in S\), for all \(x, y \in S\). In any BCI-algebra \(X\), one can define a partial order “\(\leq\)" by putting \(x \leq y\) if and only if \(x \ast y = 0\). A BCI-algebra \(X\) satisfying \(0 \leq x\), for all \(x \in X\), is called a BCK-algebra. In any BCK/BCI-algebra, the following properties are valid, for all \(x, y, z \in X\):

\[
(1) \; x \ast 0 = x;
(2) \; (x \ast y) \ast z = (x \ast z) \ast y;
(3) \; x \leq y \implies x \ast z \leq y \ast z, z \ast y \leq z \ast x;
(4) \; (x \ast z) \ast (y \ast z) \leq x \ast y;
(5) \; x \ast (x \ast (x \ast y)) = x \ast y;
(6) \; 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y);
(7) \; x \ast 0 = 0 \implies \text{that} \; x = 0.
\]

Let \(X\) be a BCI-algebra; the set \(X_+ = \{x \in X : 0 \leq x\}\) is a subalgebra and is called the BCK-part of \(X\). A BCI-algebra \(X\) is called proper if \(X - X_+ \neq \emptyset\). If \(X_+ = \{0\}\), then \(X\) is called a p-semisimple BCI-algebra. In any BCI-algebra \(X\), the following properties are valid, for all \(x, y, z \in X\):

\[
(1) \; (x \ast z) \ast (y \ast z) = x \ast y,
(2) \; x \ast (0 \ast y) = y \ast (0 \ast x),
(3) \; x \ast y = x \ast z \implies y = z,
(4) \; y \ast z = z \ast x \implies y = z.
\]

For a BCI-algebra \(X\), the set \(G(X) = \{x \in X : 0 \ast x = x\}\) is called the BCI - G part of \(X\). Note that \(G(X) \cap X_+ = \{0\}\).

Let \(X\) be a p-semisimple BCI-algebra. We define addition "\(+\)" as \(x + y = x \ast (0 \ast y)\), for all \(x, y \in X\). Then, \((X, +)\) is an abelian group with identity 0 and \(x - y = x \ast y\). Conversely, let \((X, +)\) be an abelian group with identity 0 and \(x \ast y = x - y\). Then, \(X\) is a p-semisimple BCI-algebra and \(x + y = x \ast (0 \ast y)\), for all \(x, y \in X\) (see [2]).

Let \(X\) be a BCI-algebra; we define the binary operation \(\land\) as \(x \land y = y \ast (y \ast x)\), for all \(x, y \in X\). In particular, we denote \(a_0 := x \land 0 = 0 \ast (0 \ast x)\). An element \(a \in X\) is said to be an initial element (p-atom) of \(X\), if \(x \leq a\) implies \(x = a\). We
denote by $L_p(X)$ the set of all initial elements ($p$-atoms) of $X$, indeed $L_p(X) = \{ a \in X : x * a = 0 \Rightarrow x = a, \forall x \in X \}$, and we call it the center of $X$. Note that $L_p(X) = \{ x \in X : a_e = x \}$, which is the $p$-semisimple part of $X$ and $X$ is a $p$-semisimple BCI-algebra if and only if $L_p(X) = X$. Let $X$ be a BCI-algebra with $L_p(X)$ as its center and $a \in L_p(X)$. Then, the set $V(a) = \{ x \in X : a \leq x \}$ is called the branch of $X$ with respect to $a$.

For any BCI-algebra $X$ the following results are valid.

1. If $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$, for all $a, b \in L_p(X)$.
2. If $x \leq y$, then $x, y$ are contained in the same branch of $X$.
3. If $x, y \in V(a)$, for some $a \in L_p(X)$, then $x * y, y * x \in X$.
4. If $a, b \in L_p(X)$, then $a * y = a * b$, for all $y \in V(b)$.
5. $0 * x \in L_p(X)$, for all $x \in X$.
6. $a_e \in L_p(X)$, for all $x \in X$. Indeed, $0 * (0 * a_e) = a_e$, for all $x \in X$ which implies that $a_e * y \in L_p(X)$, for all $y \in X$.
7. $G(X) \subseteq L_p(X)$.
8. $x * (x * a) = a$ and $a * x \in L_p(X)$, for all $x \in X$ and $a \in L_p(X)$.

A self-map $f$ of a BCI-algebra $X$ (i.e., a mapping of $X$ into itself) is called an endomorphism of $X$ if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that $f(0) = 0$. A subset $I$ of a BCI-algebra $X$ is called an ideal of $X$, if it satisfies

1. $0 \in I$,
2. $x * y \in I$ and $y \in I$ imply that $x \in I$, for all $x, y \in X$.

Let $f$ be an endomorphism of a BCI-algebra $X$ and let $L_p(X)$ be its center. Then, according to [13], we have the following results:

1. $f(a) \in L_p(X)$, for all $a \in L_p(X)$,
2. $f_x * f_y \in L_p(X)$ and $f_{x * y} = f_x * f_y$, for all $x, y \in X$, where $f_x = 0 * (0 * f(x))$,
3. $f(a) = 0 * (0 * f(x))$, for all $x \in V(a)$.

A BCI-algebra $X$ is called commutative if $x \leq y$ implies $x = x \wedge y = y \wedge (y \wedge x)$. It is called branchwise commutative, if $x \wedge y = y \wedge x$, for all $x, y \in V(a)$ and all $a \in L_p(X)$. Note that a BCI-algebra $X$ is commutative if and only if it is branchwise commutative.

2. $(f, g)$-Derivations of BCI-Algebras

Recently greater interest has been developed in the derivation of BCI-algebras, introduced by Jun and Xin [14]. The notion was further explored in the form of $f$-derivations of BCI-algebras by Zhan and Liu [15]. We recall the following definition from [14]. Let $X$ be a BCI-algebra. A left-right derivation (briefly, $(l, r)$-derivation) of $X$ is a self-map $d$ of $X$ satisfying the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)), \quad \forall x, y \in X.$$  \hspace{1cm} (1)$$

If $d$ satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then $d$ is called a right-left derivation (briefly, $(r, l)$-derivation) of $X$. Moreover, if $d$ is both an $(l, r)$- and $(r, l)$-derivation, then it is called a derivation.

According to [15], a left-right $f$-derivation (briefly, $(l, r)$- $f$-derivation) of $X$ is a self-map $d$ of $X$ satisfying the identity

$$d(x * y) = (d(x) * f(y)) \wedge (f(x) * d(y)), \quad \forall x, y \in X,$$  \hspace{1cm} (2)$$

where $f$ is an endomorphism of $X$. If $d$ satisfies the identity $d(x * y) = (f(x) * d(y)) \wedge (d(x) * f(y))$, for all $x, y \in X$, then $d$ is called a right-left $f$-derivation (briefly, $(r, l)$- $f$-derivation) of $X$. Moreover, if $d$ is both an $(l, r)$- and $(r, l)$- $f$-derivation, then it is called an $f$-derivation.

Now, we introduce the notion of $(f, g)$-derivation of BCI-algebras.

Definition 1. Let $X$ be a BCI-algebra. A left-right $(f, g)$-derivation (briefly, $(l, r)$- $(f, g)$-derivation) of $X$ is a self-map $d$ of $X$ satisfying the identity

$$d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y)), \quad \forall x, y \in X,$$  \hspace{1cm} (3)$$

where $f, g$ are endomorphisms of $X$. If $d$ satisfies the identity $d(x * y) = (f(x) * d(y)) \wedge (d(x) * g(y))$, for all $x, y \in X$, then $d$ is called a right-left $(f, g)$-derivation (briefly, $(r, l)$- $(f, g)$-derivation) of $X$. Moreover, if $d$ is both an $(l, r)$- and $(r, l)$- $(f, g)$-derivation, it is called an $(f, g)$-derivation.

Example 2. Let $X = \{0, a, b\}$ and a binary operation $*$ is defined as follows:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, it forms a $p$-semisimple BCI-algebra (see [16]). Define maps $d : X \to X$ and $f, g : X \to X$ by $d(x) = 0$, for all $x \in X$, $f = 0$ and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \\ a & \text{if } x = b. \end{cases}$$  \hspace{1cm} (5)$$

Then, $f$ and $g$ are endomorphisms. It is easy to check that $d$ is both $(l, r)$- and $(r, l)$- $(f, g)$-derivation of $X$. So, $d$ is an $(f, g)$-derivation.

Now, we consider $h : X \to X$ by $h = I$. Then, $h$ is an endomorphism and $d$ is not an $(r, l)$- $(h, g)$-derivation, since $d(a * b) = 0$ but $(d(a) * h(b)) \wedge (g(a) * d(b)) = a \wedge b = a$. Also, $d$ is not an $(r, l)$- $(h, g)$-derivation, since $d(a * b) = 0$ but $(h(a) * d(b)) \wedge (d(a) * g(b)) = a \wedge b = a$. 


Example 3. Let $X = \{0, a, b, c, d, e\}$ be a BCI-algebra with the following Cayley table:

\[
\begin{array}{cccccc}
\ast & 0 & a & b & c & d \\
0 & 0 & b & b & b & b \\
a & a & 0 & b & b & b \\
b & b & b & 0 & 0 & 0 \\
c & c & b & a & 0 & 0 \\
d & d & b & a & a & 0 \\
e & e & b & a & a & 0 \\
\end{array}
\]

Define maps $d : X \to X$ and $f, g : X \to X$ by $g(x) = 0$, for all $x \in X$:

\[
d(x) = \begin{cases} 
  b & \text{if } x = 0, a \\
  0 & \text{otherwise}, 
\end{cases}
\]

\[
f(x) = \begin{cases} 
  0 & \text{if } x = 0, a \\
  b & \text{otherwise}. 
\end{cases}
\]

Then, $f$ and $g$ are endomorphisms. $d$ is both derivation and $f$-derivation of $X$ (see Example 2.2 of [15]). Also, it is easy to check that $d$ is both $(l, r)$- and $(r, l)$-$(f, g)$-derivation of $X$. So, $d$ is an $(f, g)$-derivation.

Example 4. Let both $X$ and $d : X \to X$ be as in Example 3. Define $f, g : X \to X$ by $f(0) = 0$ and $g(0) = 1$. Then, $f, g$ are endomorphisms. $d$ is not a $f$-derivation of $X$ (see Example 2.3 of [15]). Also, $d$ is not an $(l, r)$-$(f, g)$-derivation, since $d(b \ast c) = d(0) = b$ but $(d(b) \ast d(c)) \land (g(b) \ast d(c)) = 0 \land b = 0$. $d$ is not an $(r, l)$-$(f, g)$-derivation, since $d(b \ast c) = d(0) = b$ but $(f(b) \ast d(c)) \land (d(b) \ast g(c)) = 0 \land (0 \ast c) = 0 \land b = 0$.

Theorem 5. Let $f, g$ be endomorphisms of BCI-algebra $X$ and let $d$ be a self-map of $X$ defined by $d(x) = f_x$, for all $x \in X$. Then, $d$ is an $(l, r)$-$(f, g)$-derivation of $X$.

Proof. Suppose that $x, y \in X$. Then, we have

\[
d(x \ast y) = f_x \ast f_y = 0 \ast (0 \ast f(x \ast y)) = 0 \ast (0 \ast (f(x) \ast f(y))) = 0 \ast (0 \ast f(x)) \ast (0 \ast f(y)) = 0 \ast (0 \ast f_x) \ast (0 \ast f_y) = 0 \ast (0 \ast (f_x \ast f_y)) = 0 \ast (0 \ast ((0 \ast f(y)) \ast f(x))) = 0 \ast (0 \ast (f(y) \ast x)) = 0 \ast f(y \ast x) = 0 \ast (f(y) \ast f(x)) = (0 \ast f(y)) \ast (0 \ast f(x)) = (0 \ast (0 \ast f(x))) \ast f(y) = f_x \ast f(y).
\]

Since $f_x \in L_p(X)$, $f_x \ast f(y) \in L_p(X)$ and $f_x \ast f(y) = (f_x \ast f(y)) \land (g(x) \ast f_y)$. Hence, by (8), we get

\[
d(x \ast y) = \begin{cases} 
  (f_x \ast f(y)) \land (g(x) \ast f_y) & \text{if } x = 0, a \\
  (f_x \ast f(y)) \land (g(x) \ast d(y)) & \text{otherwise}. 
\end{cases}
\]

So, $d$ is an $(l, r)$-$(f, g)$-derivation of $X$. 

Theorem 6. Let $d$ be a self-map of a BCI-algebra $X$. Then, the following properties hold.

1. If $d$ is an $(l, r)$-$(f, g)$-derivation of $X$, then $d(x) \ast d(y) = (d(x) \ast g(x)) \land (d(y) \ast f(y)) = d(x) \ast d(y)$, for all $x, y \in X$.

2. If $d$ is an $(r, l)$-$(f, g)$-derivation of $X$, then $d(x) = f(x) \land d(x)$, for all $x \in X$ if and only if $d(0) = 0$.

Proof. (1) Suppose that $x \in X$. Then, we have

\[
d(x) = d(x \ast 0) = (d(x) \ast f(0)) \land (g(x) \ast d(0)) = d(x) \land (g(x) \ast d(0)) = (g(x) \ast d(0)) \land (g(x) \ast d(0)) \land d(x) = (g(x) \ast d(0)) \land (g(x) \ast d(0)) \land d(x) \leq g(x) \ast (g(x) \ast d(x)) = d(x) \land g(x).
\]

It is clear that $d(x) \land g(x) \leq d(x)$. So, $d(x) = d(x) \land g(x)$.

(2) Suppose that $d$ is an $(r, l)$-$(f, g)$-derivation of $X$. If $d(x) = f(x) \land d(x)$, for all $x \in X$, then $d(0) = f(0) \land d(0) = 0 \land d(0) = 0$. Conversely, let $d(0) = 0$. Then, $d(x) = d(x \ast 0) = (f(x) \ast d(0)) \land (g(x) \ast d(0)) = f(x) \land d(x)$. 

Corollary 7. Let $d$ be an $(l, r)$-$(f, g)$-derivation of a BCI-algebra $X$. Then, $d(0) \leq L_p(X)$.

Proof. By Theorem 6(1), $d(0) = d(0) \land g(0) = 0 \ast (0 \ast d(0))$. So, $d(0) \leq L_p(X)$.

Theorem 8. Let $X$ be a $p$-semisimple BCI-algebra and let $d$ be an $(l, r)$-$(f, g)$-derivation of $X$. Then,

1. $d(a) = d(0) \ast (0 \ast f(a)) = d(0) + f(a)$, for all $a \in L_p(X)$;

2. $d(a) \in L_p(X)$, for all $a \in L_p(X)$;

3. $d(a + b) = d(a) + d(b) - d(0)$, for all $a, b \in L_p(X)$.

Proof. (1) Let $a \in L_p(X)$. Then, $f(a) \in L_p(X)$. Hence, by Corollary 7, we have

\[
d(a) = d(0 \ast (0 \ast a)) = (d(0) \ast f(0 \ast a)) \land (g(0) \ast d(0 \ast a)) = (d(0) \ast f(0 \ast a)) \land (0 \ast d(0 \ast a)) \]

\[
= (d(0) \ast f(0 \ast a)) \land (0 \ast d(0 \ast a)).
\]
\[= (0 \ast d (0 \ast a)) \ast ( (0 \ast d (0 \ast a)) \ast (d (0) \ast f (0 \ast a))) \]
\[= (0 \ast d (0 \ast a)) \ast ((0 \ast (d (0) \ast f (0 \ast a))) \ast d (0 \ast a)) \]
\[= 0 \ast ((d (0) \ast f (0 \ast a))) \]
\[= 0 \ast ((d (0) \ast (0 \ast f (0 \ast a)))) \]
\[= (0 \ast ((d (0) \ast 0 \ast f (0 \ast a)))) \]
\[= 0 \ast (0 \ast (d (0) \ast 0 \ast f (0 \ast a))) \]
\[= (0 \ast (0 \ast d (0) \ast 0 \ast f (0 \ast a))) \]
\[= (0 \ast (0 \ast d (0))) \ast (0 \ast f (0 \ast a)) \]
\[= d (0) \ast (0 \ast f (0 \ast a)) \]
\[= d (0) \ast f (0 \ast a) \]

(11)

The proof of (2) and (3) follows directly from (1).

Theorem 9. Let \( d \) be an \((r, l)\)-(\( f, g \))-derivation of BCI-algebra \( X \). Then,

1. \( d(a) \in G(X) \), for all \( a \in G(X) \);
2. \( d(a) \in L_p(X) \), for all \( a \in L_p(X) \);
3. \( d(a) = f(a) \ast d(0) = f(a) + d(0) \), for all \( a \in L_p(X) \);
4. \( d(a + b) = d(a) + d(b) - d(0) \), for all \( a, b \in L_p(X) \).

Proof. (1) Suppose that \( a \in G(X) \). Then, we have
\[
d(a) = d(0 \ast a)
\]
\[
= (f(0) \ast d(a)) \land (d(0) \ast g(a))
\]
(12)
\[
= (0 \ast d(a)) \land (d(0) \ast g(a)) = 0 \ast d(a).
\]
So, \( d(a) \in G(X) \).

(2) Suppose that \( a \in L_p(X) \). Then, we have
\[
d(a) = d(0 \ast (0 \ast a))
\]
\[
= (f(0) \ast d(0 \ast a)) \land (d(0) \ast g(0 \ast a))
\]
(13)
\[
= 0 \ast d(0 \ast a) \in L_p(X).
\]

(3) Suppose that \( a \in L_p(X) \). Then, we have
\[
d(a) = d(a \ast 0)
\]
\[
= (f(a) \ast d(0)) \land (d(a) \ast g(0))
\]
\[
= f(a) \ast d(0) = f(a) \ast (0 \ast d(0)) = f(a) \ast d(0).
\]
(14)

(4) The proof follows from (3).

Definition 10. An \((f, g)\)-derivation \( d \) of a BCI-algebra \( X \) is called regular if \( d(0) = 0 \). If \( d(0) \neq 0 \), then \( d \) is called irregular.

Theorem 11. Let \( X \) be a commutative BCI-algebra and let \( d \) be a regular \((r, l)\)-(\( f, g \))-derivation of \( X \). Then, both \( f(x) \) and \( d(x) \) belong to the same branch, for all \( x \in X \).

Proof. Suppose that \( x \in X \). Then, we have
\[
0 = d(0) = d(a \ast x) = (f(a) \ast d(x)) \land (d(a) \ast g(x))
\]
\[
= (f_x \ast d(x)) \land (d(a) \ast g(x)) = f_x \ast d(x).
\]
(15)

Hence, \( f_x \leq d(x) \) and so \( d(x) \in V(f_x) \). Also, we have \( f(x) \in V(f_x) \), since \( f_x \leq f(x) \). This completes the proof.

Theorem 12. Let \( d \) be a regular \((r, l)\)-(\( f, g \))-derivation of BCI-algebra \( X \). Then,

1. \( d(x) \leq f(x) \), for all \( x \in X \);
2. \( d(x) \ast f(y) \leq f(x) \ast f(y) \), for all \( x, y \in X \).

Proof. (1) By Theorem 6 (2), \( d(x) = f(x) \land d(x) \leq f(x) \), for all \( x \in X \).

(2) By part (1), \( d(x) \ast f(y) \leq f(x) \ast f(y) \), for all \( x, y \in X \).

Theorem 13. Every \((r, l)\)-(\( f, g \))-derivation \((l, r)\)-(\( f, g \))-derivation) of a BCK-algebra \( X \) is regular.

Proof. Suppose that \( X \) is a BCK-algebra and \( d \) is an \((r, l)\)-(\( f, g \))-derivation of \( X \). Then, for all \( x \in X \), we have
\[
d(0) = d(0 \ast x) = (f(0) \ast d(x)) \land (d(0) \ast g(x))
\]
\[
= (0 \ast d(x)) \land (d(0) \ast g(x)) = 0 \land (d(0) \ast g(x)) = 0
\]
(16)

So, \( d \) is regular.

Now, suppose that \( d \) is an \((l, r)\)-(\( f, g \))-derivation of \( X \). Then, \( d(0) = d(0 \ast x) = (d(0) \ast f(x)) \land (g(0) \ast d(x)) = (d(0) \ast f(x)) \land 0 = 0 \).

Theorem 14. Let \( f \) be an epimorphism and let \( g \) be an endomorphism of a BCI-algebra \( X \). Also, let \( d \) be an \((r, l)\)-(\( f, g \))-derivation of \( X \) and \( a \in X \) such that \( a \ast d(x) = 0 \) and \( f(a) \ast d(x) = 0 \), for all \( x \in X \). Then, \( d \) is regular. Moreover, \( X \) is a BCK-algebra.

Proof. For all \( x \in X \), we have
\[
0 = a \ast d(a \ast x) = a \ast ((f(a) \ast d(x)) \land (d(a) \ast g(x)))
\]
\[
= a \ast (0 \land (d(a) \ast g(x))) = a.
\]
(17)

Hence, \( d(0) = d(0 \ast x) = (f(0) \ast d(0)) \land 0 = d(0) \ast 0 = 0 \ast 0 = 0 \).

By Theorem 12(1), \( 0 \ast d(x) \leq 0 \ast d(x) = 0 \). So, \( (0 \ast f(x)) \ast 0 = 0 \), for all \( x \in X \). Then, \( 0 \ast f(x) = 0 \), for all \( x \in X \). Therefore, \( f(X) \) is a BCK-algebra. This implies that \( X \) is a BCK-algebra, since \( f \) is an epimorphism.

Theorem 15. Let \( f \) be an epimorphism and let \( g \) be an endomorphism of a BCI-algebra \( X \). Also, let \( d \) be an \((f, g)\)-derivation of \( X \) and \( a \in X \) such that \( d(x) \ast a = 0 \) and \( d(x) \ast f(a) = 0 \), for all \( x \in X \). Then, \( d \) is regular. Moreover, \( X \) is a BCK-algebra.
Proof. For all \(x \in X\), we have
\[
0 = d(x \ast a) \ast a = ((d(x) \ast f(a)) \land (g(x) \ast d(a))) \ast a
= (0 \land (g(x) \ast d(a))) \ast a = 0 \ast a.
\] (18)

So,
\[
d(0) = d(0 \ast a) = (d(0) \ast f(a)) \land (g(0) \ast d(a))
= 0 \land (0 \ast d(a)) = 0.
\] (19)

By Theorem 12(1), we get
\[
0 \ast f(x) \leq 0 \ast d(x) = (d(x) \ast a) \ast d(x)
= (d(x) \ast d(x)) \ast a = 0 \ast a = 0.
\] (20)

Thus, \((f(x)) \ast 0 = 0\), for all \(x \in X\). So, \(0 \ast f(x) = 0\), for all \(x \in X\). Hence, \(f(X)\) is a \(BCK\)-algebra. This implies that \(X\) is a \(BCK\)-algebra, since \(f\) is an epimorphism. \(\square\)

Theorem 16. Let \(X\) be a \(p\)-semisimple \(BCI\)-algebra, and let \(d\) and \(d'\) be \((l,r)\)-(f,g)-derivations (resp., \((r,l)\)-(f,g)-derivations) of \(X\). Also, let \(f \ast f = f\). Then, \(d \ast d'\) is also an \((l,r)\)-(f,g)-derivation (resp., \((r,l)\)-(f,g)-derivation) of \(X\).

Proof. Suppose that \(d\) and \(d'\) are \((l,r)\)-(f,g)-derivations of \(X\). Then, for all \(x, y \in X\),
\[
(d \ast d')(x \ast y)
= d((d'(x) \ast f(y)) \land (g(x) \ast d'(y)))
= d(d'(x) \ast f(y))
= (d(d'(x)) \ast f(f(y))) \land (g(d'(x)) \ast d(f(y)))
= d((d(d')(x) \ast f(y)) \land (g(x) \ast (d \ast d')(y)))
= ((d \ast d')(x) \ast f(y)) \land ((d \ast d')(y)) \land (g(x) \ast (d \ast d')(y)).
\] (21)

So, \(d \ast d'\) is an \((l,r)\)-(f,g)-derivation of \(X\). Now, let \(d, d'\) be \((r,l)\)-(f,g)-derivations of \(X\). Then, for all \(x, y \in X\), we have
\[
(d \ast d')(x \ast y)
= d((d'(x) \ast f(y)) \land (g(x) \ast d'(y)))
= d(d'(x) \ast f(y))
= (f(x) \ast d'(y)) \land (d(f(x)) \ast g(d'(y)))
= f(x) \ast d \ast d'(y)
= f(x) \ast (d \ast d')(y) \land ((d \ast d')(x) \ast g(y)).
\] (22)

Theorem 17. Let \(X\) be a \(p\)-semisimple \(BCI\)-algebra, and let \(d\) and \(d'\) be, respectively, \((r,l)\)-(f,g)-derivation and \((l,r)\)-(f,g)-derivation of \(X\) such that \(f \circ d = d \circ f\), \(d' \circ f = f \circ d'\). Then, \(d \circ d' = d' \circ d\).

Proof. For all \(x, y \in X\), we have
\[
(d \circ d')(x \ast y)
= d((d'(x) \ast f(y)) \land (g(x) \ast d'(y)))
= d(d'(x) \ast f(y))
= (f(d'(x)) \ast d(f(y))) \land (d(d'(x)) \ast g(f(y)))
= f(d'(x)) \ast d(f(y)) = f \circ d'(x) \ast d \circ f(y).
\] (23)

Also, for all \(x, y \in X\),
\[
(d' \circ d)(x \ast y)
= d'(f(x) \ast d(y)) \land (d(x) \ast g(y)))
= d'(f(x) \ast d(y))
= (f(d'(x)) \ast f(d(y))) \land (g(f(x) \ast d'(d(y)))
= (f(d'(x)) \ast f(d(y)) = d'(f(x)) \ast d \circ f(y).
\] (24)

By using (23) and (24), we obtain \(d \circ d'(x \ast y) = d' \circ d(x \ast y)\), for all \(x, y \in X\). By putting \(y = 0\), we get \(d \circ d'(x) = d'(d(x))\), for all \(x \in X\). \(\square\)

Definition 18 (see [16]). Let \(X\) be a \(BCI\)-algebra and let \(d, d'\) be two self-maps of \(X\). We define \(d \ast d' : X \to X\) by \((d \ast d')(x) = d(x) \ast d'(x)\), for all \(x \in X\).

Theorem 19. Let \(X\) be a \(p\)-semisimple \(BCI\)-algebra and let \(d, d'\) be \((f, g)\)-derivations of \(X\). Then,
\[
(1) \ (f \circ d') \ast (d \circ f) = (d \circ f) \ast (f \circ d'),
(2) \ (d \ast d') \ast (f \circ f) = (f \circ f) \ast (d \ast d').
\]

Proof. (1) Since \(d\) is an \((r,l)\)-(f,g)-derivation and \(d'\) is an \((l,r)\)-(f,g)-derivation of \(X\), for all \(x, y \in X\), we have
\[
(d \ast d')(x \ast y)
= d((d'(x) \ast f(y)) \land (g(x) \ast d'(y)))
= d(d'(x) \ast f(y))
= (f(d'(x)) \ast d(f(y))) \land (d(d'(x)) \ast g(f(y)))
= f(d'(x)) \ast d \circ f(y).
\] (25)
Also, since \(d\) is an \((l, r)-(f, g)\)-derivation and \(d'\) is an \((r, l)-(f, g)\)-derivation of \(X\), for all \(x, y \in X\), we have
\[
(d \circ d')(x \ast y) = d((f(x) \ast d'(y)) \ast (d'(x) \ast g(y)))
\]
\[
= d(f(x) \ast d'(y))
\]
\[
= (d(f(x)) \ast f((d'(y)))) \ast (g(d'(x)) \ast d(f(y)))
\]
\[
= d \circ d'(x) \ast f \circ f(y).
\]
(26)

From (25) and (26), we get \(f \circ d(x) \ast d \circ f(y) = d \circ f(x) \ast f \circ d'(y)\), for all \(x, y \in X\). By putting \(x = y\), \((f \circ d(x) \ast d \circ f(x)) = (d \circ f \ast f \circ d'(x))\), for all \(x \in X\).

(2) Since \(d\) and \(d'\) are \((l, r)-(f, g)\)-derivations, then for all \(x, y \in X\), we have
\[
(d \circ d')(x \ast y) = d((f(x) \ast d'(y)) \ast (d'(x) \ast g(y)))
\]
\[
= d(f(x) \ast d'(y))
\]
\[
= (f(f(x))) \ast d(d'(y)) \ast (d'(x) \ast g(d'(y)))
\]
\[
= f \circ f(x) \ast d \circ d'(y).
\]
(27)

On the other hand, since \(d\) and \(d'\) are \((r, l)-(f, g)\)-derivations, then for all \(x, y \in X\), we have
\[
(d \circ d')(x \ast y) = d((f(x) \ast d'(y)) \ast (d'(x) \ast g(y)))
\]
\[
= d(f(x) \ast d'(y))
\]
\[
= (f(f(x))) \ast d(d'(y)) \ast (d'(x) \ast g(d'(y)))
\]
\[
= f \circ f(x) \ast d \circ d'(y).
\]
(28)

By (27) and (28), for all \(x, y \in X\),
\[
d \circ d'(x) \ast f \circ f(y) = f \circ f(x) \ast d \circ d'(y).
\]
(29)

By putting \(x = y\), we get \((d \circ d' \ast f \circ f)(x) = (f \circ f \ast d \circ d')(x)\), for all \(x \in X\).

Theorem 20. Let \(X\) be a commutative \(p\)-semisimple BCI-algebra and let \(d, d'\) be \((f, g)\)-derivations of \(X\). Then, \(d = d'\) if \(d(x)\) and \(d'(x)\) lie in the same branch.

Proof. For all \(x, y \in X\), we have
\[
(d \wedge d')(x \ast y) = d((x \ast y) \wedge d'(x \ast y)) = d(x \ast y).
\]
(30)

On the other hand,
\[
(d \wedge d')(x \ast y) = d((x \ast y) \wedge d'(x \ast y))
\]
\[
= d'(x \ast y) \wedge d(x \ast y) = d'(x \ast y).
\]
(31)

By (30) and (31), \(d(x \ast y) = d'(x \ast y)\), for all \(x, y \in X\). By putting \(y = 0\), we obtain \(d(x) = d'(x)\), for all \(x \in X\).

We denote by \(\text{Der}(X)\) the set of all \((f, g)\)-derivations on \(X\).

Definition 21 (see [16]). Let \(d, d' \in \text{Der}(X)\). The binary operation \(\wedge\) on \(\text{Der}(X)\) is defined as \((d \wedge d')(x) = d(x) \wedge d'(x)\), for all \(x \in X\).

Theorem 22. Let \(X\) be a \(p\)-semisimple BCI-algebra and let \(d, d'\) be \((l, r)-(f, g)\)-derivations (resp., \((r, l)-(f, g)\)-derivations) of \(X\). Then, \(d \wedge d'\) is also an \((l, r)-(f, g)\)-derivation (resp., \((r, l)-(f, g)\)-derivation) of \(X\).

Proof. Suppose that \(d, d'\) are \((l, r)-(f, g)\)-derivations of \(X\) and \(x, y \in X\). Then, we have
\[
(d \wedge d')(x \ast y) = d((x \ast y) \wedge d'(x \ast y))
\]
\[
= d(x \ast y) \wedge d'(x \ast y) = d(x \ast y)
\]
(32)

By (31), \(d(x \ast y) = d'(x \ast y)\), for all \(x, y \in X\). By putting \(y = 0\), we obtain \(d(x) = d'(x)\), for all \(x \in X\).

So, \(d \wedge d'\) is an \((l, r)-(f, g)\)-derivation of \(X\).

Suppose that \(d, d'\) are \((r, l)-(f, g)\)-derivations of \(X\) and \(x, y \in X\). Then,
\[
(d \wedge d')(x \ast y) = d((x \ast y) \wedge d'(x \ast y))
\]
\[
= d(x \ast y) \wedge d'(x \ast y) = d(x \ast y)
\]
(33)

So, \(d \wedge d'\) is an \((l, r)-(f, g)\)-derivation of \(X\).

Theorem 23. Let \(X\) be a \(p\)-semisimple BCI-algebra. Then, \(\text{Der}(X), \wedge\) is a semigroup.

Proof. By Theorem 22, \(d \wedge d' \in \text{Der}(X)\). Suppose that \(d, d', d'' \in \text{Der}(X)\). Then,
\[
(d \wedge (d' \wedge d''))(x \ast y) = (d(x \ast y) \wedge (d' \wedge d''))(x \ast y)
\]
\[
= d(x \ast y).
\]
Also, we have
\[
((d \land d') \land d''') (x * y) = (d \land d') (x * y)
\]
= \(d (x * y) \land d' (x * y)\) \hspace{1cm} (35)
\[
= d (x * y).
\]
Therefore, \(d \land (d' \land d''') = (d \land d') \land d''\). This completes the proof. □

At the end of this section, we classify \((f, g)\)-derivations on BCK-algebras of order 3.

There is only one BCK-algebra of order 2. It is called \(B_{2-1}\) and it is as follows:

\[
\begin{array}{c|cc}
* & 0 & a \\
\hline
0 & 0 & 0 \\
a & a & 0 \\
\end{array}
\] (36)

There are only two endomorphisms on \(B_{2-1}\). They are as \(d_1 = 0\) and \(d_2 = 1\). We have Table 1, and so

\[
\begin{array}{c|ccc}
\land & d_1 & d_2 \\
\hline
d_1 & d_1 & d_1 \\
d_2 & d_2 & d_1 \\
\end{array}
\] (37)

There are only three BCK-algebras of order 3. In the following, we classify \((f, g)\)-derivations on them.

Let \(B_{3-1} = \{0, a, b\}\). Consider the following table:

\[
\begin{array}{c|ccc}
* & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & a & 0 & 0 \\
b & b & a & 0 \\
\end{array}
\] (38)

There are only two endomorphisms on \(B_{3-1}\). They are as \(d_1 = 0\) and \(d_2 = 1\). Set

\[
d_3 = \begin{cases} 0 & \text{if } x = 0, a \\ a & \text{if } x = b; \end{cases}
\] (39)
then we have Table 2, and so

\[
\begin{array}{c|cccc}
\land & d_1 & d_2 & d_3 \\
\hline
d_1 & d_1 & d_1 & d_1 \\
d_2 & d_2 & d_1 & d_1 \\
d_3 & d_3 & d_1 & d_1 \\
\end{array}
\] (40)

Let \(B_{3-2} = \{0, a, b\}\). Consider the following table:

\[
\begin{array}{c|ccc}
* & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & a & 0 & 0 \\
b & b & b & 0 \\
\end{array}
\] (41)

There are only four endomorphisms on \(B_{3-2}\). They are as \(d_1 = 0, d_2 = 1,\)

\[
d_3 = \begin{cases} 0 & \text{if } x = 0, a \\ a & \text{if } x = b; \end{cases}
\] (42)

\[
d_4 = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b. \end{cases}
\] (43)
then we have Table 3, and so

\[
\begin{array}{c|cccc}
\land & d_1 & d_2 & d_3 & d_4 \\
\hline
d_1 & d_1 & d_1 & d_1 & d_1 \\
d_2 & d_2 & d_1 & d_1 & d_1 \\
d_3 & d_3 & d_1 & d_1 & d_1 \\
d_4 & d_4 & d_1 & d_1 & d_1 \\
\end{array}
\] (44)

Let \(B_{3-3} = \{0, a, b\}\). Consider the following table:

\[
\begin{array}{c|ccc}
* & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & a & 0 & 0 \\
b & b & b & 0 \\
\end{array}
\] (45)

There are only seven endomorphisms on \(B_{3-3}\). They are as \(d_1 = 0, d_2 = 1,\)

\[
d_3 = \begin{cases} 0 & \text{if } x = 0, a \\ a & \text{if } x = b; \end{cases}
\] (46)

\[
d_4 = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b. \end{cases}
\] (47)

\[
d_5 = \begin{cases} 0 & \text{if } x = 0, a \\ a & \text{if } x = a. \end{cases}
\] (48)

\[
d_6 = \begin{cases} 0 & \text{if } x = 0, b \\ b & \text{if } x = a. \end{cases}
\] (49)

\[
d_7 = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = b \\ b & \text{if } x = a. \end{cases}
\] (50)
3. \((F, G)\)-Derivations of BCI-Algebras

Let \(X\) be a BCI-algebra. A right derivation of \(X\) is a self-map \(D\) of \(X\) satisfying the identity

\[
D(x \ast y) = (x \ast D(y)) \wedge (y \ast D(x)), \quad \forall x, y \in X. \quad (48)
\]

If \(D\) satisfies the identity \(D(x \ast y) = (D(x) \ast F(y)) \wedge (D(y) \ast F(x)), \) for all \(x, y \in X, \) then \(D\) is called a left derivation of \(X.\)

Let \(X\) be a BCI-algebra. A right \(F\)-derivation of \(X\) is a self-map \(D\) of \(X\) satisfying the identity

\[
D(x \ast y) = (F(x) \ast D(y)) \wedge (F(y) \ast D(x)), \quad \forall x, y \in X, \quad (49)
\]

where \(F\) is an endomorphism of \(X.\) If \(D\) satisfies the identity \(D(x \ast y) = (D(x) \ast F(y)) \wedge (D(y) \ast F(x)), \) for all \(x, y \in X, \) then \(D\) is called a left \(F\)-derivation of \(X.\) Moreover, if \(D\) is both right and left \(F\)-derivation, then \(D\) is called an \(F\)-derivation of \(X;\) see [13].

The notion of left \((F, G)\)-derivation of BCI-algebras is introduced in [17]. In this section, we introduce the notion of right \((F, G)\)-derivation and give some examples and propositions to explain the theory of left \((F, G)\)-derivation and right \((F, G)\)-derivation in BCI-algebras.

**Definition 24.** Let \(X\) be a BCI-algebra. A right \((F, G)\)-derivation of \(X\) is a self-map \(D\) of \(X\) satisfying the identity

\[
D(x \ast y) = (F(x) \ast D(y)) \wedge (G(y) \ast D(x)), \quad \forall x, y \in X, \quad (50)
\]

where \(F\) and \(G\) are endomorphisms of \(X.\) If \(D\) satisfies the identity \(D(x \ast y) = (D(x) \ast F(y)) \wedge (D(y) \ast G(x)), \) for all \(x, y \in X, \) then \(D\) is called a left \((F, G)\)-derivation of \(X.\) Moreover, if \(D\) is both right and left \((F, G)\)-derivation, then it is said that \(D\) is an \((F, G)\)-derivation of \(X.\)

**Example 25.** Let \(X, d, f, \) and \(g\) be as in Example 3. Define \(D = d, F = f, \) and \(G = g.\) It is easy to see that \(D\) is both right \((F, G)\)-derivation and left \((F, G)\)-derivation. So, \(D\) is an \((F, G)\)-derivation.

**Example 26.** Let \(X\) and \(d\) be as in Example 3. Define \(D = d, F = 0, \) and \(G = 1.\) \(D\) is not a right \((F, G)\)-derivation, since \(D(b \ast c) = b \) but \((F(b) \ast D(c)) \wedge (G(c) \ast D(b)) = 0 \wedge c = 0.\) Also, \(D\) is not a left \((F, G)\)-derivation, since \(D(b \ast c) = b \) but \((D(b) \ast F(c)) \wedge (D(c) \ast G(b)) = 0 \wedge b = 0.\)

**Example 27.** Let \(X = \{0, a, b\}\) and the binary operation \(*\) is defined as follows:

\[
\begin{array}{ccc}
\ast & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & 0 & b
\end{array}
\]

Then, \((X, \ast)\) is a commutative BCK-algebra. Define maps \(D, F, G : X \rightarrow X\) by \(D(x) = 0, \) for all \(x \in X, F = I\) and \(G(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \\ a & \text{if } x = b. \end{cases}\)

Then, \(F\) and \(G\) are endomorphisms. It is easy to check that \(D\) is a left \((F, G)\)-derivation. \(D\) is not a right \((F, G)\)-derivation, since \(D(b \ast a) = 0 \) but \((F(b) \ast D(a)) \wedge (G(a) \ast D(b)) = b \ast 0 = b.\)

**Definition 28.** An \((F, G)\)-derivation \(D\) of a BCI-algebra \(X\) is called regular if \(D(0) = 0.\) If \(D(0) \neq 0,\) then \(D\) is called irregular.

**Theorem 29.** Every right \((F, G)\)-derivation of a BCK-algebra is regular.

**Proof.** Suppose that \(X\) is a BCK-algebra and \(D\) is a right \((F, G)\)-derivation of \(X.\) Then,

\[
D(0) = D(0 \ast 0) = (F(0) \ast D(0)) \wedge (G(0) \ast D(0)) = 0. \quad (53)
\]

So, \(D\) is regular.

**Theorem 30.** Let \(D\) be a right \((F, G)\)-derivation of a BCI-algebra \(X.\) Then, \(D(x) \in G(X), \) for all \(x \in G(X).\)

**Proof.** Suppose that \(x \in G(X).\) Then,

\[
D(x) = D(0 \ast x) = (F(0) \ast D(x)) \wedge (G(x) \ast D(0)) \leq F(0) \ast D(x) = 0 \ast D(x) \in L_p(X). \quad (54)
\]

Thus, \(D(x) = 0 \ast D(x)\). So, \(D(x) \in G(X).\)
Theorem 31. Let $D$ be a right $(F, G)$-derivation of a BCI-algebra $X$. Then, the following hold:

1. $D(x) \in L_p(X)$, for all $x \in X$;
2. $F(y) \ast (F(y) \ast D(x)) = D(x)$, for all $x, y \in X$;
3. $D(x) \ast F(y) = 0 \ast (F(y) \ast D(x))$, for all $x, y \in X$;
4. $D(x) \ast F(y) \in L_p(X)$, for all $x, y \in X$.

Proof. (1) For all $x \in X$, we have

$$D(x) = D(x \ast 0) = (F(x) \ast D(0)) \wedge (G(0) \ast D(x))$$

$$= (0 \ast D(x)) \ast ((0 \ast D(x)) \ast (F(x) \ast D(0))) \leq 0 \ast (0 \ast (F(x) \ast D(0))).$$

So, $D(x) \in L_p(X)$.

(2), (3), and (4) are clear.

Theorem 32. Let $X$ be a $p$-semisimple BCI-algebra. Then

1. If $D$ is an $(l, r)$-$(F, G)$-derivation of $X$, then $D$ is a left $(F, G)$-derivation of $X$;
2. If $D$ is an $(r, l)$-$(F, G)$-derivation of $X$, then $D$ is a right $(F, G)$-derivation of $X$.

Proof. (1) Suppose that $x, y \in X$ and $D$ is an $(l, r)$-$(F, G)$-derivation. Then,

$$D(x \ast y) = (D(x) \ast F(y)) \wedge (G(x) \ast D(y)) \leq D(x) \ast F(y).$$

(56)

So, $D(x \ast y) = D(x) \ast F(y) = (D(x) \ast F(y)) \wedge (D(y) \ast G(x))$, which implies that $D$ is a left $(F, G)$-derivation of $X$.

(2) Suppose that $x, y \in X$ and $D$ is an $(r, l)$-$(F, G)$-derivation. Then,

$$D(x \ast y) = (F(x) \ast D(y)) \wedge (D(x) \ast G(y))$$

$$\leq F(x) \ast D(y).$$

So, $D(x \ast y) = F(x) \ast D(y) = (F(x) \ast D(y)) \wedge (G(y) \ast D(x))$, which implies that $D$ is a right $(F, G)$-derivation of $X$.

Theorem 33. Let $D$ be a self-map and let $V(x_0)$ be any branch of a BCI-algebra $X$. If, for any $x \in V(x_0)$, $D(x) = F(x_0)$, then $D$ is a regular left $(F, G)$-derivation.

Proof. Suppose that $x, y \in X, x \in V(x_0)$, and $y \in V(y_0)$; it is not necessary that $x_0 \neq y_0$. Then, $x \ast y \in V(x_0 \ast y_0)$. By using hypothesis, we get

$$D(x \ast y) = F(x_0 \ast y_0).$$

(58)

Also, by using hypothesis, we have

$$(D(x) \ast F(y)) \wedge (D(y) \ast G(x))$$

$$= (F(x_0) \ast F(y)) \wedge (F(y_0) \ast G(x))$$

$$\leq F(x_0) \ast F(y) = F(x_0 \ast y)$$

$$= F(x_0 \ast y_0).$$

(59)

From $x_0, y_0 \in L_p(X)$, we get $F(x_0 \ast y_0) \in L_p(X)$. By using (58) and (59), $D(x \ast y) = (D(x) \ast F(y)) \wedge (D(y) \ast G(x))$. So, $D$ is a left $(F, G)$-derivation. It is clear that $D$ is regular.

Theorem 34. A self-map $D$ of a BCI-algebra $X$ defined by $D(x) = 0 \ast (0 \ast F(x)) = F_{x'}$ for all $x \in X$, is a left $(F, G)$-derivation.

Proof. Suppose that $x, y \in X$. We have $D(x \ast y) = F_{x \ast y} = F_x \ast F_y$ and

$$D(x \ast F(y)) \wedge (D(y) \ast G(x))$$

$$= (F_x \ast F(y)) \wedge (F_y \ast G(x)) \leq F_x \ast F(y).$$

(60)
So,
\[(D(x) \ast F(y)) \land (D(y) \ast G(x)) \ast (F_x \ast F_y)\]
(61)

Hence, \((D(x) \ast F(y)) \land (D(y) \ast G(x)) \leq F_x \ast F_y = 0\).

**Theorem 35.** Let \(D\) be a left \((F, G)\)-derivation of a commutative BCI-algebra \(X\). Then, \(x \leq y\) implies that \((D(x) \ast D(y))\) belong to the same branch of \(X\).

**Proof.** Suppose that \(x, y \in X\) and \(x \leq y\). Since \(X\) is commutative, \(y \ast (y \ast x) = x\). Since \(D\) is a left \((F, G)\)-derivation, we have
\[D(x) = D(y \ast (y \ast x)) = (D(y) \ast F(y \ast x)) \land (D(x \ast x) \ast G(y))\] (62)
\[
\leq D(y) \ast F(y \ast x) = D(y) \ast (F(y) \ast F(x)).
\]
Since \(x \leq y\), \(0 = F(0) = F(x \ast y) = F(x) \ast F(y)\). So, \(F(x)\) and \(F(y)\) are contained in the same branch. Hence, \(F(y) \ast F(x) \in X\). By (62), we have
\[D(x) \ast D(y) \leq (D(y) \ast (F(y) \ast F(x))) \ast D(y)\]
\[= (D(y) \ast D(y)) \ast (F(y) \ast F(x)) = 0 \ast (F(y) \ast F(x)) = 0.\]

Thus, \(D(x) \ast D(y) = 0\). So, \(D(x) \leq D(y)\). Hence, \(D(x)\) and \(D(y)\) are contained in the same branch of \(X\).

**Theorem 36.** Let \(D\) be an \((F, G)\)-derivation of a BCI-algebra \(X\). Also, let \(V(x_0)\) and \(V(y_0)\) be two arbitrary branches of \(X\) such that \(F(x_0) \in V(x_0)\) and, for all \(y \in V(y_0)\), \(F(y) \in V(y_0)\). Then, \(D(x) = y_0\) implies \(D(y) \in V(x_0)\).

**Proof.** Suppose that \(x \in V(x_0)\), \(y \in V(y_0)\), and \(D(x) = y_0\). Since \(D\) is a left \((F, G)\)-derivation, we have
\[D(x \ast y) = (D(x) \ast F(y)) \land (D(y) \ast G(x))\]
\[\leq D(x) \ast F(y) = y_0 \ast F(y).\]

On the other hand, \(y_0 \ast F(y) = 0\), since \(F(y) \in V(y_0)\).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**

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