Research Article

Essential Self-Adjointness of Anticommutative Operators

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The self-adjoint extensions of symmetric operators satisfying anticommutation relations are considered. It is proven that an anticommutative type of the Glimm-Jaffe-Nelson commutator theorem follows. Its application to an abstract Dirac operator is also considered.

1. Introduction and Main Theorem

In this paper, we consider the essential self-adjointness of anticommutative symmetric operators. Let H be a symmetric operator on a Hilbert space $\mathcal{H}$; that is, H satisfies $H \subset H^*$. It is said that H is self-adjoint if $H = H^*$ and H is essentially self-adjoint if its closure $\overline{H}$ is self-adjoint. We are interested in conditions under which a symmetric operator is essentially self-adjoint. The Glimm-Jaffe-Nelson commutator theorem (e.g., [1, Theorem 2.32], [2, Theorem X.36]) is one criterion for the essential self-adjointness of commutative symmetric operators. The commutator theorem shows that if a symmetric operator H and a self-adjoint operator X obey a commutation relation on a dense subspace $\mathcal{D}$, which is a core of X , then H is essentially self-adjoint on $\mathcal{D}$. Historically, Glimm and Jaffe [3] and Nelson [4] investigate the commutator theorem for quantum field models. Faris and Lavine [5] apply it to quantum mechanical models and Fröhlich [6] considers a generalization of the commutator theorem and proves that a multiple commutator formula follows. Here, we overview the commutator theorem.

Let H and X be linear operators on $\mathcal{H}$. Assume the following conditions.

(C.1) H is symmetric and X is self-adjoint.

(C.2) There exists $\delta_X > 0$ such that, for all $\Psi \in \mathcal{D}(X)$,

$$\delta_X (\Psi, \Psi) \leq |(\Psi, X\Psi)|. \quad (1)$$

(C.3) X has a core $\mathcal{D}_0$ satisfying $\mathcal{D}_0 \subset \mathcal{D}(H)$, and there exist constants $a \geq 0$ and $b \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$\|H\Psi\| \leq a \|X\Psi\| + b \|\Psi\|. \quad (2)$$

Theorem A (Glimm-Jaffe-Nelson commutator theorem). *Let H and X be operators satisfying (C.1)–(C.3). Suppose (i) or (ii) as follows.*

(i) *There exists a constant $c_1 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,*

$$|(H\Psi, X\Psi) - (X\Psi, H\Psi)| \leq c_1 |(\Psi, X\Psi)|. \quad (3)$$

(ii) *There exists a constant $c_2 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,*

$$c_2 |(\Psi, X\Psi)| \leq |(H\Psi, X\Psi) - (X\Psi, H\Psi)|. \quad (4)$$

Then, H is essentially self-adjoint on $\mathcal{D}_0$.

Remark 1. In the commutator theorem, condition (i) is usually supposed. It is also proven under condition (ii) in a similar way to Theorem 2.

The idea of the proof of the commutator theorem is as follows. Let X and Y be symmetric operators on a Hilbert space. Then, the real part and the imaginary part of the inner product $(X\Psi, Y\Psi)$ for $\Psi \in \mathcal{D}(XY) \cap \mathcal{D}(YX)$ are expressed by

$$\begin{aligned} \operatorname{Re} (X\Psi, Y\Psi) &= \frac{1}{2} (\Psi, \{X, Y\} \Psi), \\ \operatorname{Im} (X\Psi, Y\Psi) &= \frac{1}{2i} (\Psi, [X, Y] \Psi), \end{aligned} \quad (5)$$

respectively, where $\{X, Y\} = XY + YX$ and $[X, Y] = XY - YX$. In the proof of the commutator theorem, the imaginary part is estimated. In Theorem 2, we prove that an anticommutative symmetric operator is essentially self-adjoint on a dense subspace by estimating the real part.

Theorem 2. Assume (C.1)–(C.3). In addition, suppose that (I) or (II) holds.

(I) There exists a constant $d_1 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$|(H\Psi, X\Psi) + (X\Psi, H\Psi)| \leq d_1 |(\Psi, X\Psi)|. \quad (6)$$

(II) There exists a constant $d_2 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$d_2 |(\Psi, X\Psi)| \leq |(H\Psi, X\Psi) + (X\Psi, H\Psi)|. \quad (7)$$

Then, H is essentially self-adjoint on $\mathcal{D}_0$.

Proof of Theorem 2. We show that, for some $z \in \mathbf{C} \setminus \mathbf{R}$, $\dim \ker((H|_{\mathcal{D}_0})^* + z^\sharp) = 0$ where $z^\sharp = z, z^*$. Let $\Psi \in \mathcal{D}((H|_{\mathcal{D}_0})^*)$ and let $\Xi = X^{-1}\Psi$. Since $((H|_{\mathcal{D}_0})^*)^* = \overline{H|_{\mathcal{D}_0}}$, we have

$$\begin{aligned} & \operatorname{Re} \left(\Xi, \left((H|_{\mathcal{D}_0})^* + z^\sharp \right) \Psi \right) \\ &= \frac{1}{2} \left(\left(\overline{H|_{\mathcal{D}_0}} \Xi, X\Xi \right) + \left(X\Xi, \overline{H|_{\mathcal{D}_0}} \Xi \right) \right) \\ &+ \operatorname{Re} z (\Xi, X\Xi). \end{aligned} \quad (8)$$

First, we assume that (I) holds. Let $z \in \mathbf{C} \setminus \mathbf{R}$ satisfy $|\operatorname{Re} z| > d_1/2$. Since $\mathcal{D}_0$ is a core of X , it follows from (C.3) and (I) that $\mathcal{D}(X) \subset \mathcal{D}(\overline{H|_{\mathcal{D}_0}})$ and for all $\Phi \in \mathcal{D}(X)$,

$$\left| \left(\overline{H|_{\mathcal{D}_0}} \Phi, X\Phi \right) + \left(X\Phi, \overline{H|_{\mathcal{D}_0}} \Phi \right) \right| \leq d_1 |(\Phi, X\Phi)|. \quad (9)$$

By (8) and (9), we have

$$\begin{aligned} \left| \operatorname{Re} \left(\Xi, \left((H|_{\mathcal{D}_0})^* + z^\sharp \right) \Psi \right) \right| &\geq \left(|\operatorname{Re} z| - \frac{d_1}{2} \right) |\Xi, X\Xi| \\ &\geq \delta_X \left(|\operatorname{Re} z| - \frac{d_1}{2} \right) (\Xi, \Xi). \end{aligned} \quad (10)$$

Since $\Psi \in \ker((H|_{\mathcal{D}_0})^* + z^\sharp)$, we have $\Xi = X^{-1}\Psi = 0$ from (10). Then, we have $\Psi = 0$. Next, we suppose that (II) follows. Let $z \in \mathbf{C} \setminus \mathbf{R}$ satisfy $|\operatorname{Re} z| < d_2/2$. Since $\mathcal{D}_0$ is a core of X , it also follows from (C.3) and (II) that $\mathcal{D}(X) \subset \mathcal{D}(\overline{H|_{\mathcal{D}_0}})$ and for all $\Phi \in \mathcal{D}(X)$,

$$d_2 |(\Phi, X\Phi)| \leq \left| \left(\overline{H|_{\mathcal{D}_0}} \Phi, X\Phi \right) + \left(X\Psi, \overline{H|_{\mathcal{D}_0}} \Phi \right) \right|. \quad (11)$$

Then from (8) and (11), we have

$$\begin{aligned} \left| \operatorname{Re} \left(\Xi, \left((H|_{\mathcal{D}_0})^* + z^\sharp \right) \Psi \right) \right| &\geq \left(\frac{d_2}{2} - |\operatorname{Re} z| \right) |\Xi, X\Xi| \\ &\geq \delta_X \left(\frac{d_2}{2} - |\operatorname{Re} z| \right) (\Xi, \Xi). \end{aligned} \quad (12)$$

Since $\Psi \in \ker((H|_{\mathcal{D}_0})^* + z^\sharp)$ and $\Xi = X^{-1}\Psi$, we have $\Psi = 0$ from (12). Thus, the proof is obtained. $\square$

2. Application of Theorem 2

We apply Theorem 2 to abstract Dirac operator theory [1, 7]. Let $\mathcal{H}$ be a Hilbert space. Let H and τ be self-adjoint operators on $\mathcal{H}$. Assume that τ is bounded, $\tau^2 = I$ and, $\tau\mathcal{D}(H) \subset \mathcal{D}(H)$. Then H is called an abstract Dirac operator on $\mathcal{H}$ with unitary involution τ . We construct an abstract Dirac operator by weakly commuting operators. Let X and Y be densely defined linear operators on a Hilbert space. The weak commutator of X and Y is defined for $\Phi \in \mathcal{D}(X^*) \cap \mathcal{D}(Y^*)$ and for $\Psi \in \mathcal{D}(X) \cap \mathcal{D}(Y)$ by

$$[X, Y]^0(\Phi, \Psi) = (X^*\Phi, Y\Psi) = (Y^*\Phi, X\Psi). \quad (13)$$

Let $\{P_j\}_{j=1}^N$, $N \in \mathbf{N}$, be self-adjoint operators on a Hilbert space $\mathcal{H}$. Set $\mathcal{D}_0 = \cap_{j=1}^N \mathcal{D}(P_j)$. Assume that $\{P_j\}_{j=1}^N$ satisfy the following condition.

(S.1) $\mathcal{D}_0$ is dense in $\mathcal{H}$. For all $\Phi, \Psi \in \mathcal{D}_0$, $[P_j, P_l]^0(\Phi, \Psi) = 0$, $j, l = 1, \dots, N$.

Let M be a bounded self-adjoint operator $\mathcal{H}$ satisfying the following condition.

(S.2) For all $\Phi, \Psi \in \mathcal{D}_0$, $[M, P_j]^0(\Phi, \Psi) = 0$, $j = 1, \dots, N$.

Let $\mathcal{H}$ be a Hilbert space. Let $\{\Gamma_j\}_{j=1}^N$ and B be bounded self-adjoint operators on $\mathcal{H}$ satisfying the following anticommutation relations:

(S.3) (i) $\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}$, $j, k = 1, \dots, N$, (ii) $\{\Gamma_j, B\} = 1$, $j = 1, \dots, N$, (iii) $B^2 = I$.

Then, the next assertion holds.

Theorem 3. Let $\mathcal{H}_D = \mathcal{H} \otimes \mathcal{H}$. Assume (S.1)–(S.3). Then,

$$H_D = \sum_{j=1}^N \Gamma_j \otimes P_j + B \otimes M \quad (14)$$

is self-adjoint on $\mathcal{D}(H_D) = \mathcal{H} \otimes \mathcal{D}_0$.

Remark 4. In the case where $\{P_j\}_{j=1}^N$ strongly commute, Theorem 3 has been proven ([8, Theorem 4.3], [9, Lemma 6.7]) by strongly anticommuting methods [10, 11].

It is seen that $(I \otimes B)^2 = I$ and $(I \otimes B)\mathcal{D}(H_D) \subset \mathcal{D}(H_D)$. Then, from Theorem 3, H_D is an abstract Dirac operator on $\mathcal{H}_D$ with the unitary involution $I \otimes B$.

To prove Theorem 3, we show some lemmas.

Lemma 5. Let $\{C_j\}_{j=1}^N$, $N \in \mathbf{N}$, be closed operators on a Hilbert space on $\mathcal{X}$. Suppose that $\cap_{j=1}^N \mathcal{D}(C_j)$ is dense in $\mathcal{X}$ and for $j \neq l$, $(C_j\Psi, C_l\Psi) + (C_l\Psi, C_j\Psi) = 0$, $\Psi \in \cap_{j=1}^N \mathcal{D}(C_j)$. Then $C = \sum_{j=1}^N C_j$ is closed.

Proof. We see that $(C\Psi, C\Psi) = \sum_{j=1}^N \|C_j\Psi\|^2 \geq (1/N) (\sum_{j=1}^N \|C_j\Psi\|)^2$. Then, $\sum_{j=1}^N \|C_j\Psi\| \leq \sqrt{N}\|C\Psi\|$. Then from a closedness criterion (e.g., [1, Theorem B1], [12, Proposition 1]), C is closed. $\square$

From an argument of quadratic forms, there exists a self-adjoint operator L on $\mathcal{H}$ such that $L \geq 1$, $\mathcal{D}(\sqrt{L}) = \mathcal{D}_0$ and for all $\Phi, \Psi \in \mathcal{D}(\sqrt{L})$,

$$(\sqrt{L}\Phi, \sqrt{L}\Psi) = \sum_{j=1}^N (P_j\Phi, P_j\Psi) + (\Phi, \Psi). \quad (15)$$

Lemma 6. Assume (S.1). Then, for all $\Phi, \Psi \in \mathcal{D}(\sqrt{L})$,

$$[\sqrt{L}, P_j]^0 (\Phi, \Psi) = 0, \quad j = 1, \dots, N. \quad (16)$$

Proof. Since L is positive and self-adjoint, it follows that $\sqrt{L}\Xi = \int_0^\infty (1/\sqrt{\lambda})(L + \lambda)^{-1}L\Xi, \Xi \in \mathcal{D}(L)$. Then, for all $\Phi, \Psi \in \mathcal{D}(L)$,

$$\begin{aligned} & [\sqrt{L}, P_j]^0 (\Phi, \Psi) \\ &= \int_0^\infty \frac{1}{\sqrt{\lambda}} [(L + \lambda)^{-1}L, P_j]^0 (\Phi, \Psi) d\lambda \\ &= \int_0^\infty \sqrt{\lambda} [P_j, L]^0 ((L + \lambda)^{-1}\Phi, (L + \lambda)^{-1}\Psi) d\lambda \\ &= \sum_{l=1}^N \int_0^\infty \sqrt{\lambda} \\ &\quad \times \left\{ [P_j, P_l]^0 ((L + \lambda)^{-1}\Phi, P_l(L + \lambda)^{-1}\Psi) \right. \\ &\quad \left. + [P_j, P_l]^0 (P_l(L + \lambda)^{-1}\Phi, (L + \lambda)^{-1}\Psi) \right\} d\lambda. \end{aligned} \quad (17)$$

By (S.1) and (17), we have $[\sqrt{L}, P_j]^0 (\Phi, \Psi) = 0$ for all $\Phi, \Psi \in \mathcal{D}(\sqrt{L})$. Note that $\mathcal{D}(L)$ is a core of $\sqrt{L}$, since L is self-adjoint. In addition, for all $\Psi \in \mathcal{D}(\sqrt{L})$, $\|P_j\Psi\| \leq \|\sqrt{L}\Psi\|$, $j = 1, \dots, N$. Hence, it follows that $[\sqrt{L}, P_j]^0 (\Phi, \Psi) = 0$ for all $\Phi, \Psi \in \mathcal{D}(\sqrt{L})$. Thus, the proof is obtained. $\square$

Proof of Theorem 3. Since $B \otimes M$ is bounded, it is enough to show that $H = \sum_{j=1}^N \Gamma_j \otimes P_j$ is self-adjoint. Let $X = B \otimes \sqrt{L}$. We show that H and X satisfy (C.1)–(C.3) and (I) in Theorem 2. Since H is symmetric and X is self-adjoint, (C.1) is satisfied. Since $\sigma(B) = \{\pm 1\}$ and $\sqrt{L} \geq 1$, we see that, for all $\Psi \in \mathcal{D}(H)$,

$$|(\Psi, X\Psi)| = (\Psi, (I \otimes \sqrt{L})\Psi) \geq (\Psi, \Psi). \quad (18)$$

Then, (C.2) is satisfied. Since $\mathcal{D}_0 = \mathcal{D}(\sqrt{L})$, it follows that $\mathcal{D}(H) = \mathcal{D}(X)$. Then, by (S.3), we see that, for all $\Psi \in \mathcal{D}(H)$,

$$\begin{aligned} \|H\Psi\|^2 &= \sum_{j=1}^N ((I \otimes P_j)\Psi, (I \otimes P_j)\Psi) \\ &\leq \|(I \otimes \sqrt{L})\Psi\|^2 = \|X\Psi\|^2. \end{aligned} \quad (19)$$

Then, $\|H\Psi\| \leq \|X\Psi\|$ for all $\Psi \in \mathcal{D}(H)$, and hence (C.3) is satisfied. By Lemma 6, it is seen that, for all $\Psi \in \mathcal{D}(H)$,

$$\begin{aligned} & (H\Psi, X\Psi) + (X\Psi, H\Psi) \\ &= \sum_{j=1}^N (((\Gamma_j \otimes P_j)\Psi, (B \otimes \sqrt{L})\Psi) \\ &\quad + ((B \otimes \sqrt{L})\Psi, (\Gamma_j \otimes P_j)\Psi)) \\ &= \sum_{j=1}^N ((I \otimes \sqrt{L})\Psi, (\{\Gamma_j, B\} \otimes P_j)\Psi). \end{aligned} \quad (20)$$

Then by (S.3) and (20), we have $(H\Psi, X\Psi) + (X\Psi, H\Psi) = 0$. Then, from (18), it follows that $(H\Psi, X\Psi) + (X\Psi, H\Psi) \leq |(\Psi, X\Psi)|$ for all $\Psi \in \mathcal{D}(H)$. Then (I) is satisfied, and hence $\bar{H}$ is self-adjoint from Theorem 2. In addition, by (S.1) and (S.3), we see that, for $j \neq l$,

$$\begin{aligned} & ((\Gamma_j \otimes P_j)\Psi, (\Gamma_l \otimes P_l)\Psi) + ((\Gamma_l \otimes P_l)\Psi, (\Gamma_j \otimes P_j)\Psi) \\ &= ((I \otimes P_l)\Psi, (\{\Gamma_j, \Gamma_l\} \otimes P_j)\Psi) = 0. \end{aligned} \quad (21)$$

Then, from Lemma 5, $\bar{H} = H$, and hence the proof is obtained. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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