Research Article

Hermite-Hadamard and Simpson-Like Type Inequalities for Differentiable Harmonically Convex Functions

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A new identity for differentiable functions is derived. A consequence of the identity is that the author establishes some new general inequalities containing all of the Hermite-Hadamard and Simpson-like types for functions whose derivatives in absolute value at certain power are harmonically convex. Some applications to special means of real numbers are also given.

1. Introduction

Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

\[
\frac{a + b}{2} \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping \( f \). Both inequalities hold in the reversed direction if \( f \) is concave.

The following inequality is well known in the literature as Simpson inequality:

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^{4}.
\]

For some results which generalize, improve, and extend the Hermite-Hadamard and Simpson inequalities, one refers the reader to the recent papers (see [1–8]).

In [9], the author introduced the concept of harmonically convex functions and established some results connected with the right-hand side of new inequalities similar to inequality (1) for these classes of functions. Some applications to special means of positive real numbers were also given.

**Definition 2.** Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically convex, if

\[
f\left( \frac{xy}{tx + (1 - t)y} \right) \leq tf(y) + (1 - t)f(x)
\]

for all \( x, y \in I \) and \( t \in [0, 1] \). If inequality in (3) is reversed, then \( f \) is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 3.** Let \( f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold

\[
f\left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

The above inequalities are sharp.
Some results connected with the right part of (4) were given in [9] as follows.

**Theorem 4.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \([a, b] \) for \( q \geq 1 \), then

\[
\frac{f(a) + f(b)}{2} = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{ab(b-a)}{2} \lambda_1 \left[ \lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q \right]^{1/q},
\]

where

\[
\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]

\[
\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]

\[
\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right)
\]

\[= \lambda_1 - \lambda_2.
\]

**Theorem 5.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \([a, b] \) for \( q > 1 \), \( 1/p + 1/q = 1 \), then

\[
\frac{f(a) + f(b)}{2} = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q \right)^{1/q},
\]

where

\[
\mu_1 = \frac{a^{2-2q} + b^{2-2q} [(b-a) (1-2q)] - \lambda_1}{2(b-a)^2 (1-q) (1-2q)},
\]

\[
\mu_2 = \frac{b^{2-2q} - a^{2-2q} [(b-a) (1-2q)] + \lambda_3}{2(b-a)^2 (1-q) (1-2q)}.
\]

In this paper, one gives some general integral inequalities connected with the left and right parts of (4); as a result of this, one obtains some new midpoint, trapezoid, and Simpson-like type inequalities for differentiable harmonically convex functions.

### 2. Main Results

In order to prove our main results we need the following lemma.

**Lemma 6.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \). If \( f' \in L[a,b] \) then for \( \lambda \in [0, 1] \) one has the equality

\[
(1 - \lambda) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right)
\]

\[= \frac{ab(b-a)}{2} \left[ \int_0^{1/2} \frac{2-\lambda-2t}{A_1^2} f' \left( \frac{ab}{A_1} \right) dt + \int_{1/2}^1 \frac{2-\lambda-2t}{A_1^2} f' \left( \frac{ab}{A_1} \right) dt \right],
\]

where \( A_1 = tb + (1-t)a \).

**Proof.** It suffices to note that

\[
I_1 = ab(b-a) \int_0^{1/2} \frac{2-\lambda-2t}{A_1^2} f' \left( \frac{ab}{A_1} \right) dt
\]

\[= (2t-\lambda) f \left( \frac{ab}{A_1} \right) - \int_0^{1/2} f' \left( \frac{ab}{A_1} \right) dt
\]

\[= (1-\lambda) f \left( \frac{2ab}{a+b} \right) + \lambda f (b) - \int_0^{1/2} f' \left( \frac{ab}{A_1} \right) dt.
\]

Set \( x = \frac{ab}{A_1} \), and \( dx = (-\lambda b - \lambda a) dt \), which gives

\[
I_1 = (1-\lambda) f \left( \frac{2ab}{a+b} \right) + \lambda f (b) - \frac{2ab}{b-a} \int_{2ab/(a+b)}^{b} \frac{f(x)}{x^2} dx.
\]

Similarly, we can show that

\[
I_2 = ab(b-a) \int_{1/2}^1 \frac{2-\lambda-2t}{A_1^2} f' \left( \frac{ab}{A_1} \right) dt
\]

\[= \lambda f (a) + (1-\lambda) f \left( \frac{2ab}{a+b} \right)
\]

\[= \frac{2ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx.
\]

Thus,

\[
\frac{I_1 + I_2}{2} = (1 - \lambda) f \left( \frac{2ab}{a+b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right)
\]

\[- \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx
\]

which is required. \( \square \)

**Theorem 7.** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is
harmonically convex on $[a, b]$ for $q \geq 1$ and then one has the following inequality for $\lambda \in [0, 1]$:

$$
\left| (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right|
\leq \frac{ab(b - a)}{2}
$$

$$
\times \left\{ \left( \int_0^{1/2} \frac{|\lambda - 2t|}{A_1^2} \frac{dt}{A_1^2} \right)^{1-1/q} \times \left( \int_0^{1/2} \frac{f'(ab)}{A_1^2} \frac{dt}{A_1^2} \right)^{1/q}
+ \left( \int_{1/2}^1 \frac{|2 - \lambda - 2t|}{A_1^2} \frac{dt}{A_1^2} \right)^{1-1/q}
\times \left( \int_{1/2}^1 \frac{f'(ab)}{A_1^2} \frac{dt}{A_1^2} \right)^{1/q} \right\}.
$$

Hence, by harmonically convexity of $|f'|^q$ on $[a, b]$, we have

$$
\left| (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right|
\leq \frac{ab(b - a)}{2}
$$

$$
\times \left\{ \left( \int_0^{1/2} \frac{|\lambda - 2t|}{A_1^2} \frac{dt}{A_1^2} \right)^{1-1/q} \times \left( \int_0^{1/2} \frac{f'(ab)}{A_1^2} \frac{dt}{A_1^2} \right)^{1/q}
+ \left( \int_{1/2}^1 \frac{|2 - \lambda - 2t|}{A_1^2} \frac{dt}{A_1^2} \right)^{1-1/q}
\times \left( \int_{1/2}^1 \frac{f'(ab)}{A_1^2} \frac{dt}{A_1^2} \right)^{1/q} \right\}.
$$

\begin{align}
C_1(\lambda; u, \theta) &= \frac{1}{(\theta - u)^2} \\
&\times \left[ -4 + \frac{\lambda (\theta - u) + 2u (3u + \theta)}{u (u + \theta)} \\
&+ 2 \ln \left( \frac{2u (u + \theta)}{(2u + \lambda (\theta - u))^2} \right) \right],
\end{align}

\begin{align}
C_2(\lambda; u, \theta) &= \frac{1}{(\theta - u)^3} \\
&\times \left\{ \lambda (\theta - u) + 4u \ln \left( \frac{\lambda (\theta - u) + 2u^2}{2u (u + \theta)} \right) \\
&- \frac{\lambda (\theta - u) + 2u (5u + 3\theta)}{u + \theta} + 7u + \theta \right\},
\end{align}

\begin{align}
C_3(\lambda; u, \theta) &= C_1(\lambda; u, \theta) - C_2(\lambda; u, \theta), \quad u, \theta > 0.
\end{align}

**Proof.** Let $A_t = tb + (1 - t)a$. From Lemma 6 and using the Hölder inequality, we have

\begin{align}
&\left| (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right)
- \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right|
\leq \frac{ab(b - a)}{2}

&\times \left\{ \left( \int_0^{1/2} \frac{|\lambda - 2t|}{A_t^2} \frac{dt}{A_t^2} \right)^{1-1/q} \times \left( \int_0^{1/2} \frac{f'(ab)}{A_t^2} \frac{dt}{A_t^2} \right)^{1/q}
+ \left( \int_{1/2}^1 \frac{|2 - \lambda - 2t|}{A_t^2} \frac{dt}{A_t^2} \right)^{1-1/q}
\times \left( \int_{1/2}^1 \frac{f'(ab)}{A_t^2} \frac{dt}{A_t^2} \right)^{1/q} \right\}.
\end{align}
It is easily to check that
\[
\int_0^{1/2} \frac{\sqrt{\lambda - 2t}}{A^2_t} \, dt = C_1(\lambda; a, b)
\]
\[
= \frac{1}{(b - a)^2} \times \left[ -4 + \frac{[\lambda(b - a) + 2a]}{a(a + b)} + 2 \ln \left( \frac{2a(a + b)}{(2a + \lambda(b - a))^2} \right) \right],
\]
\[
\int_0^{1/2} \frac{\sqrt{\lambda - 2t} \cdot t}{A^2_t} \, dt = C_2(\lambda; a, b)
\]
\[
= \frac{1}{(b - a)^3} \times \left[ \frac{[\lambda(b - a) + 2a]}{a(a + b)} + \frac{5a(2a + b)}{a + b} + 7a + b \right],
\]
\[
\int_0^{1/2} \frac{\sqrt{\lambda - 2t} \cdot (1 - t)}{A^2_t} \, dt = C_3(\lambda; a, b) = C_1(\lambda; a, b) - C_2(\lambda; a, b).
\]
This concludes the proof. \[\square\]

**Corollary 8.** Under the assumptions of Theorem 7 with \(\lambda = 0\), one has
\[
\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}
\]
\[
\times \left\{ C_1^{-1/q}(0; a, b) \times \left[ C_2(0; a, b) \left| f'(a) \right|^q + C_3(0; a, b) \left| f'(b) \right|^q \right]^{1/q} + C_1^{-1/q}(0; b, a) \times \left[ C_3(0; b, a) \left| f'(a) \right|^q + C_2(0; b, a) \left| f'(b) \right|^q \right]^{1/q} \right\},
\]
where
\[
C_1(0; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \ln \left( \frac{u + \vartheta}{2u} \right) - \frac{\vartheta - u}{u + \vartheta} \right],
\]
\[
C_2(0; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{3(\vartheta + \vartheta)(\theta - u)}{u + \vartheta} + 4u \ln \left( \frac{2u}{u + \vartheta} \right) \right].
\]

**Corollary 9.** Under the assumptions of Theorem 7 with \(\lambda = 1\), one has
\[
\left| f\left(\frac{a}{b} + f\left(\frac{b}{a}\right) \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}
\]
\[
\times \left\{ C_1^{1-1/q}(1; a, b) \times \left[ C_2(1; a, b) \left| f'(a) \right|^q + C_3(1; a, b) \left| f'(b) \right|^q \right]^{1/q} + C_1^{1-1/q}(1; b, a) \times \left[ C_3(1; b, a) \left| f'(a) \right|^q + C_2(1; b, a) \left| f'(b) \right|^q \right]^{1/q} \right\},
\]
where
\[
C_1(1; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{2u + \vartheta}{u + \vartheta} \ln \left( \frac{u + \vartheta}{2u} \right) - \frac{u + 3\vartheta}{u + \vartheta} \right],
\]
\[
C_2(1; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{3u + \vartheta}{u + \vartheta} \ln \left( \frac{u + \vartheta}{2u} \right) - 2(\vartheta - u) \right],
\]
\[
C_3(1; u, \vartheta) = \frac{1}{(\vartheta - u)^2} \left[ \frac{u + \vartheta}{u + \vartheta} \ln \left( \frac{u + \vartheta}{2u} \right) - \frac{u + 3\vartheta}{u + \vartheta} \ln \left( \frac{u + \vartheta}{2u} \right) \right],
\]
and
\[
\left| \frac{1}{3} f\left(\frac{a}{b} + f\left(\frac{b}{a}\right) \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2}
\]
\[
\times \left\{ C_1^{1-1/q}\left(\frac{1}{3}; a, b\right) \times \left[ C_2\left(\frac{1}{3}; a, b\right) \left| f'(a) \right|^q + C_3\left(\frac{1}{3}; a, b\right) \left| f'(b) \right|^q \right]^{1/q} + C_1^{1-1/q}\left(\frac{1}{3}; b, a\right) \times \left[ C_3\left(\frac{1}{3}; b, a\right) \left| f'(a) \right|^q + C_2\left(\frac{1}{3}; b, a\right) \left| f'(b) \right|^q \right]^{1/q} \right\},
\]
for \(u, \vartheta > 0\).
where
\[ C_1 \left( \frac{1}{3}; u, \vartheta \right) = \frac{1}{(\vartheta - u)^2} \left[ \frac{(\vartheta - u)(\vartheta - 3u)}{3u(u + \vartheta)} + 2 \ln \left( \frac{18u(u + \vartheta)}{(5u + \vartheta)^2} \right) \right], \]
\[ C_2 \left( \frac{1}{3}; u, \vartheta \right) = \frac{1}{(\vartheta - u)^3} \left[ \frac{11u + \vartheta}{3} \ln \left( \frac{(5u + \vartheta)}{18u(u + \vartheta)} \right) + 4u(\vartheta - u) \right], \]
\[ C_3 \left( \frac{1}{3}; u, \vartheta \right) = \frac{1}{(\vartheta - u)^2} \left[ \frac{\vartheta^2 - 4u\vartheta - u^2}{3u(u + \vartheta)} + \frac{5u + 7\vartheta}{3(\vartheta - u)} \ln \left( \frac{18u(u + \vartheta)}{(5u + \vartheta)^2} \right) \right], \]
\[ u, \vartheta > 0. \]

Theorem 11. Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q > 1 \) and then one has the following inequality for \( \lambda \in [0, 1] \):
\[ \left| (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b - a)}{2} \times \left\{ \left( \int_0^{1/2} |\lambda - 2t|^p dt \right)^{1/p} \times \left( \int_0^{1/2} \frac{1}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right\}. \]

Using the harmonically convexity of \( |f'|^q \), we obtain
\[ \int_0^{1/2} \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \leq \int_0^{1/2} t \left| f'(a) \right|^q + (1 - t) \left| f'(b) \right|^q dt \]
\[ \frac{1}{2(1 - q)(1 - 2q)(b - a)^2} \times \left\{ \left[ \left( \frac{a + b}{2} \right)^{1-2q} \left[ b - 3a - q(b - a) \right] + a^{2-2q} \right] \left| f'(a) \right|^q \right. \]
\[ + \left. \left[ \left( \frac{a + b}{2} \right)^{1-2q} \left[ 3b - a - q(b - a) \right] + a^{2-2q} \right] \left| f'(b) \right|^q \right\}, \]

and \( 1/p + 1/q = 1. \)
\[
\int_0^{1/2} \frac{1}{A_t^{2q}} \left| f'(t) \left( \frac{ab}{A_t^q} \right) \right|^q dt \\
\leq \int_{1/2}^1 \frac{\left| f'(t) \right|^q + (1-t) \left| f'(b) \right|^q}{A_t^{2q}} dt \\
= \frac{1}{2(1-q)(1-2q)(b-a)^2} \\
\times \left\{ b^{1-2q} [b - 2a - 2q(b-a)] \\
+ \left( \frac{a+b}{2} \right)^{2-2q} \left[ 3a - b \right] + q + q(b-a) \right] \left| f'(a) \right|^q \\
+ \left( \frac{a+b}{2} \right)^{2-2q} \left[ a - 3b \right] + q + q(b-a) \right) \left| f'(b) \right|^q \\
\times \left| f'(b) \right|^q \right\}.
\] (28)

Further, we have
\[
\int_0^{1/2} |\lambda - 2t| dt = \int_{1/2}^1 |2 - \lambda - 2t| dt \\
= \frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{2(p+1)}.
\] (29)

A combination of (27)–(29) gives the required inequality (25).

**Corollary 12.** Under the assumptions of Theorem 11 with \( \lambda = 0 \), one has
\[
\left| f \left( \frac{2ab}{a+b} \right) - \frac{ab(b-a)}{2(p+1)^{1/p}} \right| \\
\leq \frac{ab(b-a)}{4(1-q)(1-2q)(b-a)^2} \\
\times \left\{ (C_5(q,a,b) \left| f'(a) \right|^q + C_6(q,a,b) \left| f'(b) \right|^q \right\}^{1/q} \\
+ \left( C_6(q,a,b) \left| f'(a) \right|^q + C_5(q,b,a) \left| f'(b) \right|^q \right) \right\}^{1/q}.\] (30)

**Corollary 13.** Under the assumptions of Theorem 11 with \( \lambda = 1 \), one has
\[
\left| f(a) + f(b) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \right| \\
\leq \frac{ab(b-a)}{4(1-q)(1-2q)(b-a)^2} \\
\times \left\{ (C_9(\lambda,q) \left| f'(a) \right|^q + C_{10}(\lambda,q) \left| f'(b) \right|^q \right\}^{1/q} \\
+ \left( C_9(\lambda,q) \left| f'(a) \right|^q + C_{10}(\lambda,q) \left| f'(b) \right|^q \right) \right\}^{1/q}.\] (31)
where
\[ C_7(a, b; p) = a^{1 - 2p} - \left(\frac{a + b}{2}\right)^{1 - 2p}, \]
\[ C_8(a, b; p) = \left(\frac{a + b}{2}\right)^{1 - 2p} - b^{1 - 2p}, \]
\[ C_9(\lambda, q) = \lambda^{q + 1} + (1 - \lambda)^{q + 1} (q + 1 + \lambda), \]
\[ C_{10}(\lambda, q) = \lambda^{q + 1} (4 + 2q - \lambda) + (1 - \lambda)^{q + 1} (3 + q - \lambda), \]
and \( 1/p + 1/q = 1 \).

Proof. Let \( A_t = tb + (1 - t)a \). Using Lemma 6 and Holder’s integral inequality, we deduce
\[
\left| (1 - \lambda) f \left( \frac{2ab}{a + b} \right) + \lambda \left( \frac{f(a) + f(b)}{2} \right) \right|
\leq \frac{ab(b - a)}{2}
\times \left\{ \left( \int_0^{1/2} \frac{1}{A_t^{2p}} dt \right)^{1/p}
\times \left( \int_0^{1/2} |\lambda - 2t|^q \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}
+ \left( \int_{1/2}^1 \frac{1}{A_t^{2p}} dt \right)^{1/p}
\times \left( \int_{1/2}^1 |2 - \lambda - 2t|^q \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right\}. \tag{35}
\]

Using the harmonically convexity of \(|f'|^q| \), we obtain
\[
\int_0^{1/2} |\lambda - 2t|^q \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt
\leq \int_0^{1/2} |\lambda - 2t|^q \left[ t \left| f'(a) \right|^q + (1 - t) \left| f'(b) \right|^q \right] dt
= \frac{1}{4(q + 1)(q + 2)}
\times \left\{ \left[ \lambda^{q + 2} + (1 - \lambda)^{q + 1} (q + 1 + \lambda) \right] \left| f'(a) \right|^q
+ \left[ \lambda^{q + 1} (4 + 2q - \lambda) + (1 - \lambda)^{q + 1} (3 + q - \lambda) \right] \left| f'(b) \right|^q \right\}, \tag{36}
\]

Further, we have
\[
\int_0^{1/2} \frac{1}{A_t^{2p}} dt = \frac{1}{(b - a)(2p - 1)} \left[ a^{1 - 2p} - \left(\frac{a + b}{2}\right)^{1 - 2p} \right],
\int_{1/2}^1 \frac{1}{A_t^{2p}} dt = \frac{1}{(b - a)(2p - 1)} \left[ \left(\frac{a + b}{2}\right)^{1 - 2p} - b^{1 - 2p} \right]. \tag{37}
\]

A combination of (35)–(37) gives the required inequality (33). \( \square \)

Corollary 16. Under the assumptions of Theorem 15 with \( \lambda = 0 \), one has
\[
\left| f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b f(x) dx \right|
\leq \frac{ab(b - a)^{1 - 1/p}}{2(2p - 1)^{1/p}}
\times \left\{ C_7^{1/p}(a, b; p) \left[ (q + 1) \left| f'(a) \right|^q + (q + 3) \left| f'(b) \right|^q \right]^{1/q}
+ C_8^{1/p}(a, b; p)
\times \left[ (q + 3) \left| f'(a) \right|^q + (q + 1) \left| f'(b) \right|^q \right]^{1/q} \right\}. \tag{38}
\]

Corollary 17. Under the assumptions of Theorem 15 with \( \lambda = 1 \), one has
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b f(x) dx \right|
\leq \frac{ab(b - a)^{1 - 1/p}}{2(2p - 1)^{1/p}}.
\[
\begin{align*}
&\times \left[ \frac{1}{4(q+1)(q+2)} \right]^{1/q} \\
&\times \left\{ C_{7}^{1/p}(a,b,p) \left[ \sum_{q} \left[ f'(a)^q + f'(b)^q \right]^{1/q} \right] \\
&\quad + C_{8}^{1/p}(a,b,p) \left[ \sum_{q} \left[ f'(a)^q + f'(b)^q \right]^{1/q} \right] \right\}.
\end{align*}
\]

(39)

Corollary 18. Under the assumptions of Theorem 15 with \( \lambda = 1/3 \), one has

\[
\left\| \left( 1 - \lambda \right) H + \lambda A - \frac{G^2}{L} \right\| 
\leq \frac{ab(b-a)}{2} \left[ C_{1}(\lambda;a,b) + C_{1}(\lambda;b,a) \right],
\]

where \( C_{1}(\lambda;a,b) \) is defined as in Theorem 7.

\[
\begin{align*}
&\text{where } C_{9}(1/3,q) = \frac{1}{3q^2} (1 + 2q + 3q + 4) \quad \text{and} \\
&C_{10}(1/3,q) = \frac{1}{3q^2} (11 + 6q + 2q + 8q + 3q). 
\end{align*}
\]

(40)

3. Some Applications for Special Means

Let us recall the following special means of two nonnegative number \( a, b \) with \( b > a \).

(1) The arithmetic mean

\[
A = A(a,b) := \frac{a + b}{2}.
\]

(2) The geometric mean

\[
G = G(a,b) := \sqrt{ab}.
\]

(3) The harmonic mean

\[
H = H(a,b) := \frac{2ab}{a + b}.
\]

(4) The logarithmic mean

\[
L = L(a,b) := \frac{b - a}{\ln b - \ln a}.
\]

(5) The \( p \)-logarithmic mean

\[
L_p = L_p(a,b) := \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{1/p}, \quad p \in \mathbb{R} \setminus \{-1,0\}.
\]

(6) The identric mean

\[
I = I(a,b) := \frac{1}{e} \left( \frac{b^q}{a^q} \right)^{1/(b-a)}.
\]

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

\[
H \leq G \leq L \leq I \leq A.
\]

(48)

It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

Proposition 19. Let \( 0 < a < b \) and \( \lambda \in [0,1] \). Then one has the following inequality:

\[
\left\| \left( 1 - \lambda \right) H + \lambda A - \frac{G^2}{L} \right\| 
\leq \frac{ab(b-a)}{2} \left[ C_{1}(\lambda;a,b) + C_{1}(\lambda;b,a) \right],
\]

where \( C_{1} \) is defined as in Theorem 7.

\[
\begin{align*}
&\text{Proof. The assertion follows from inequality (14) in Theorem 7, for } f : (0,\infty) \to \mathbb{R}, \ f(x) = x. \quad \square \\
&\text{Proposition 20. Let } 0 < a < b \text{ and } \lambda \in [0,1]. \text{ Then one has the following inequality:}
\end{align*}
\]

\[
\left\| \left( 1 - \lambda \right) H + \lambda A - \frac{G^2}{L} \right\| 
\leq \frac{ab(b-a)}{2} \left[ C_{1}(\lambda;a,b) + C_{1}(\lambda;b,a) \right],
\]

where \( q > 1, 1/p + 1/q = 1, \) and \( C_{4}, C_{5}, \) and \( C_{6} \) are defined as in Theorem 11.

\[
\begin{align*}
&\text{Proof. The assertion follows from inequality (25) in Theorem 11, for } f : (0,\infty) \to \mathbb{R}, \ f(x) = x. \quad \square
\end{align*}
\]
Proposition 21. Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:
\[
(1 - \lambda) H + \lambda A - \frac{G^2}{L} \leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[ \frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{2(q+1)} \right]^{1/q} \times \left[ C_7^{1/p}(a,b,p) + C_8^{1/p}(a,b,p) \right],
\]
where $q > 1$, $1/p + 1/q = 1$, and $C_7$ and $C_8$ are defined as in Theorem 15.

Proof. The assertion follows from inequality (33) in Theorem 15, for $f : (0, \infty) \rightarrow \mathbb{R}, f(x) = x$.

Proposition 22. Let $0 < a < b$, $\lambda \in [0,1]$ and $q \geq 1$. Then one has the following inequality:
\[
\left[ (1 - \lambda) H^2 + \lambda A \left( a^2, b^2 \right) - G^2 \right]
\leq ab(b-a) \times \left[ C_1^{1-1/q}(\lambda,a,b) \left\{ C_2(\lambda,a,b)a^q + C_3(\lambda,a,b)b^q \right\}^{1/q} + C_1^{1-1/q}(\lambda,b,a) \times \left[ C_3(\lambda,b,a)a^q + C_3(\lambda,b,a)b^q \right]^{1/q} \right],
\]
where $C_1, C_2,$ and $C_3$ are defined as in Theorem 7.

Proof. The assertion follows from inequality (14) in Theorem 7, for $f : (0, \infty) \rightarrow \mathbb{R}, f(x) = x^2$.

Proposition 23. Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:
\[
\left[ (1 - \lambda) H^2 + \lambda A \left( a^{n+2}, b^{n+2} \right) - G^2 \right]
\leq \frac{ab(b-a)}{2} \times \left[ C_4^{1/p}(\lambda,p) \left\{ C_5(\lambda,a,b)a^{n+1} + C_5(\lambda,a,b)b^{n+1} \right\}^{1/q} \right],
\]
where $q > 1$, $1/p + 1/q = 1$, and $C_4, C_5,$ and $C_6$ are defined as in Theorem 11.

Proof. The assertion follows from inequality (25) in Theorem 11, for $f : (0, \infty) \rightarrow \mathbb{R}, f(x) = x^2$.

Proposition 24. Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:
\[
\left[ (1 - \lambda) H^2 + \lambda A \left( a^2, b^2 \right) - G^2 \right]
\leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[ \frac{1}{4(q+1)(q+2)} \right]^{1/q} \times \left[ C_7^{1/p}(a,b,p) \left\{ C_8(\lambda,a,b)a^q + C_9(\lambda,b,b) \left\{ C_8(\lambda,a,b)b^q \right\}^{1/q} \right\} + C_8^{1/p}(a,b,p) \left\{ C_9(\lambda,a,b)a^q + C_9(\lambda,b,b) \left\{ C_8(\lambda,a,b)b^q \right\}^{1/q} \right\} \right],
\]
where $q > 1$, $1/p + 1/q = 1$, and $C_7, C_8, C_9$, and $C_{10}$ are defined as in Theorem 15.

Proof. The assertion follows from inequality (33) in Theorem 15, for $f : (0, \infty) \rightarrow \mathbb{R}, f(x) = x^2$.

Proposition 25. Let $0 < a < b$, $n \in (-\infty, \infty) \setminus \{0\}$, $\lambda \in [0,1]$, and $q \geq 1$. Then one has the following inequality:
\[
\left[ (1 - \lambda) H^{n+2} + \lambda A \left( a^{n+2}, b^{n+2} \right) - G^2 \right] \cdot L_n^n
\leq \frac{ab(b-a)(n+2)}{2} \times \left[ C_1^{1-1/q}(\lambda,a,b) \left\{ C_2(\lambda,a,b)a^{n+1} + C_3(\lambda,a,b)b^{n+1} \right\}^{1/q} \right],
\]
where $C_1, C_2,$ and $C_3$ are defined as in Theorem 7.

Proof. The assertion follows from inequality (14) in Theorem 7, for $f : (0, \infty) \rightarrow \mathbb{R}, f(x) = x^{n+2}, n \in (-\infty, \infty) \setminus \{0\}$.

Proposition 26. Let $0 < a < b$, $\lambda \in [0,1]$, and $n \in (-\infty, \infty) \setminus \{0\}$. Then one has the following inequality:
\[
\left[ (1 - \lambda) H^{n+2} + \lambda A \left( a^{n+2}, b^{n+2} \right) - G^2 \right] \cdot L_n^n
\leq \frac{ab(b-a)}{4} \times \left[ C_4^{1/p}(\lambda,p) \left\{ C_5(\lambda,a,b)a^{n+1} + C_5(\lambda,a,b)b^{n+1} \right\}^{1/q} \right],
\]
where $C_4, C_5,$ and $C_6$ are defined as in Theorem 11.
where $q > 1$, $1/p + 1/q = 1$, and $C_5$ and $C_6$ are defined as in Theorem 11.

**Proof.** The assertion follows from inequality (25) in Theorem 11, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. \qed 

**Proposition 27.** Let $0 < a < b$, $\lambda \in [0, 1]$, and $n \in (-1, \infty) \setminus \{0\}$. Then one has the following inequality:

\[
\left| (1 - \lambda) H^{n+2} + \lambda A \left( a^{n+2}, b^{n+2} \right) - G^2 \cdot L^a_n \right| \leq \frac{ab(n+2)(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[ \frac{1}{4(q+1)(q+2)} \right]^{1/q} 
\]

\[
\times \left\{ C_{1/2}^{1/p} (a, b; p) \right. 
\times \left[ C_5 (\lambda, q) a^{(n+1)q} + C_7 (\lambda, q) b^{(n+1)q} \right]^{1/q} 
+ C_8^{1/p} (a, b; p) 
\times \left[ C_9 (\lambda, q) b^{(n+1)q} + C_{10} (\lambda, q) a^{(n+1)q} \right]^{1/q} \right\},
\]

where $q > 1$, $1/p + 1/q = 1$, and $C_7$, $C_8$, $C_9$, and $C_{10}$ are defined as in Theorem 15.

**Proof.** The assertion follows from inequality (33) in Theorem 15, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. \qed 

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


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