Research Article

Asymptotic Law of the j th Records in the Bivariate Exponential Case

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We consider a sequence $(X_i, Y_i)_{1 \leq i \leq n}$ of independent and identically distributed random variables with joint cumulative distribution $H(x, y)$, which has exponential marginals $F(x)$ and $G(y)$ with parameter $\lambda = 1$. We also assume that $X_i(\omega) \neq Y_i(\omega)$, $\forall i \in \mathbb{N}$, and $\omega \in \Omega$. We denote $\{R_k^{(j)}\}_{k \geq 1}$ and $\{S_k^{(j)}\}_{k \geq 1}$ by the sequences of the j th records in the sequences $(X_i)_{1 \leq i \leq n}$, $(Y_i)_{1 \leq i \leq n}$, respectively. The main result of the paper is to prove the asymptotic independence of $\{R_k^{(j)}\}_{k \geq 1}$ and $\{S_k^{(j)}\}_{k \geq 1}$ using the property of stopping time of the j th record times and that of the exponential distribution.

1. Introduction

Let $(X_i)_{1 \leq i \leq n}$ be a sequence of independent and identically distributed (i.i.d) random variables (r.v.s.) from a distribution F . Let us consider the j th record time defined recurrently for $k = 2, 3, \dots$ and $j \geq 1$ as

$$L_1^{(j)} = j, \quad (1)$$

$$L_k^{(j)} = \min \left\{ m > L_{k-1}^{(j)}; X_{m-j+1, m} > X_{L_{k-1}^{(j)}-j+1, L_{k-1}^{(j)}} \right\},$$

and the k th ordinary j th record from F as

$$R_k^{(j)} = X_{L_k^{(j)}-j+1, L_k^{(j)}}; \quad R_1^{(j)} = X_{1, j}, \quad (2)$$

where $X_{i, m}$ denotes the i th order statistic of a sample of size m .

In 1976 Dziubdziela-Kopocinski [1] proved that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(R_n^{(j)} \leq a_n x + b_n \right) = N(-\log(-\log \Theta(x))), \quad (3)$$

where $\Theta(x)$ is one of the three well known limit laws of $X_{n, n} = \max_{1 \leq i \leq n} (X_i)$ (Frechet, Weibull, and Gumbell), $N(\cdot)$ is the standard normal distribution, and a_n, b_n are the constants of normalization.

Taking account of these three limit laws we get those of the j th record; namely,

(i) Type 1:

$$N(x), \quad x \in \mathbb{R}, \quad (4)$$

(ii) Type 2:

$$N_{1\alpha} = \begin{cases} N(\log x^\alpha), & x > 0, \alpha > 0, \\ 0 & x \leq 0, \end{cases} \quad (5)$$

(iii) Type 3:

$$N_{2\alpha} = \begin{cases} N(\log(-x)^{-\alpha}), & x < 0, \alpha > 0, \\ 1 & x \geq 0. \end{cases} \quad (6)$$

The authors have presented the expressions of the probability density function $f_n^{(j)}(x)$ and the distribution function $F_n^{(j)}(x)$ of the j th record as follows:

$$f_n^{(j)}(x) = \frac{j}{(n-1)!} [-j \log(1-F(x))]^{n-1} [1-F(x)]^{j-1} f(x), \quad n = 1, 2, \dots,$$

$$F_n^{(j)}(y) = \int_0^{-j \log(1-F(y))} e^{-u} \frac{u^{n-1}}{(n-1)!} du, \quad n = 1, 2, \dots \quad (7)$$

Contrary to that of records, the theory of limit law of the maximum has been extended to the bivariate case by the works of Finkelshtein [2], Geffroy [3], de Oliveira [4], Sibuya [5], Galombos [6], Marshall and Olkin [7], Kotz and Nadarajah [8], Coles [9], and Smith [10].

In this work, we assume that $X_n(\omega) \neq Y_n(\omega)$, $\forall n \in \mathbb{N}$ and $\omega \in \Omega$. Moreover, we will focus on the bivariate r.v's. $(R_n^{(j)}, S_n^{(j)})_{n \geq 1}$ issued from pairs of r.v's. $(X_n, Y_n)_{n \geq 1}$ with joint cumulative distribution $H(x, y)$ whose marginals $F(x)$ and $G(y)$ are exponential with parameter $\lambda = 1$ and prove that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left(R_n^{(j)} \leq a_n x + b_n, S_n^{(j)} \leq c_n y + d_n \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P} \left(R_n^{(j)} \leq a_n x + b_n \right) \lim_{n \rightarrow \infty} \mathbb{P} \left(S_n^{(j)} \leq c_n y + d_n \right) \quad (8) \\ &= N(x) N(y), \end{aligned}$$

where $N(\cdot)$ is the standard normal distribution, and $a_n = c_n = n/j$, $b_n = d_n = \sqrt{n/j^2}$.

2. Preliminaries

In this section we recall some relevant results for future use.

Theorem 1 (Deheuvels [11]). *Let $X_1, X_2, \dots, X_n$ be a sequence of i.i.d r.v's. with $\Phi(x)$ continues repartition function.*

Let $\{X_{k,n} : 1 \leq k \leq n\}$ and $\{R_k^{(j)} : k \geq 1\}$ be, respectively, the sequences of order statistics and j th records.

$\forall n \geq 2$ and for all $j \geq 1$ one has the following.

(i) *For $n \geq j \geq 1$ and $\Phi(s) < 1$, the sequences*

$$\{X_{k,n} : 1 \leq k \leq n\}, \quad \{R_k^{(j)} : k \geq 1\} \quad (9)$$

form a Markov chain with equal probability of transition. That means

$$\begin{aligned} \mathbb{P} \left(R_{k+1}^{(j)} \geq t \mid R_k^{(j)} = s \right) &= \mathbb{P} \left(X_{n-j+1,n} \geq t \mid X_{n-j,n} = s \right) \\ &= \left(\frac{1 - \Phi(t)}{1 - \Phi(s)} \right)^j. \end{aligned} \quad (10)$$

(ii) *The assertions $\{R_k^{(j)} \text{ and } R_{k+1}^{(j)} - R_k^{(j)} \text{ independent}\}$ and $\{X_{n-j,n} \text{ and } X_{n-j+1,n} - X_{n-j,n} \text{ independent}\}$ are equivalent.*

Corollary 2. *Let $(X_i)_{1 \leq i \leq n}$ be a sequence of i.i.d r.v's. with repartition $\Phi(x)$ continues and let, for $j \geq 1$ fixed, $(R_k^{(j)})_{k \geq 1}$ be the sequence of j th records from Φ .*

If $k \geq 1$ is a fixed integer, $R_k^{(j)}$ and $R_{k+1}^{(j)} - R_k^{(j)}$ are independent if and only if $\Phi(x) = 1 - e^{(-b(x-B))}$ with $x \geq B$ for B and b being positive finite constants.

Corollary 3. *For $\Phi(x) = 1 - e^{-x}$, $x \geq 0$ and $(R_k^{(j)})_{k \geq 1}$ being the sequence of j th records, then $\alpha_k^{(j)} = (R_k^{(j)} - R_{k-1}^{(j)})_{1 \leq k \leq n}$ is a sequence of i.i.d r.v's.s with repartition function $\Psi(x) = 1 - e^{-jx}$, $x \geq 0$.*

Corollary 4. *For $\Phi(x) = 1 - e^{-x}$, $x \geq 0$, and for all $m \leq k-1$, $R_k^{(j)} - R_{k-1}^{(j)}$ is independent of R_m .*

Let $(U_i)_{1 \leq i \leq n}$ be the sequence of i.i.d r.v's. uniformly distributed on $[0, 1]$; then one has the following.

Theorem 5 (Deheuvels [12]). *For all $A > 0$ and for all $P \geq 4$ a.s. for $n \rightarrow \infty$,*

$$\begin{aligned} U_{m,n} &\leq \frac{1}{n} \left\{ \log_2 n + (m+1) \log_3 n + \log_4 n + \dots + (1+A) \log_p n \right\}. \end{aligned} \quad (11)$$

Theorem 6 (Deheuvels [12]). *For all $A > 0$, and for all $p \geq 3$ a.s. for n large enough,*

$$U_{m,n} > \frac{1}{n \left\{ (\log n) (\log_2 n) \dots (\log_p n)^{1+A} \right\}^{1/m}}. \quad (12)$$

Corollary 7. *Let $X_1, X_2, \dots, X_n$ be a sequence of i.i.d r.v's. with repartition function $\Phi(x)$ and $\Gamma(t) = \inf\{x \in \mathbb{R} \mid \Phi(x) \geq t\}$. Then, a.s. for $n \rightarrow \infty$, $A > 0$, $p \geq 4$,*

$$\begin{aligned} & \Gamma \left\{ \frac{1}{n \left\{ (\log(n)) (\log_2(n)) \dots (\log_p(n))^{1+A} \right\}^{1/m}} \right\} \\ & \leq X_{m,n} \\ & \leq \Gamma \left\{ \frac{1}{n} \left\{ \log_2(n) + (m+1) \log_3(n) \right. \right. \\ & \quad \left. \left. + \dots + (1+A) \log_p(n) \right\} \right\}. \end{aligned} \quad (13)$$

Moreover, a.s., for $A = 0$, both members of these two inequalities are not satisfied infinitely many times.

Corollary 8. *If $\Phi(x) = 1 - e^{-x}$, $x > 0$, then $\Gamma(x) = -\log(1-x)$, $x < 1$.*

Theorem 9 (Barndorff-Nielsen [13]). *If*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \{X_{m,n} \leq \lambda_n\} = 0, \\ & \sum_n n^{m-1} \{ \Phi(\lambda_n) \}^{n-m+1} \{ 1 - \Phi(\lambda_{n+1}) \}^m < \infty, \end{aligned} \quad (14)$$

then $\lambda_n \ll X_{m,n}$ a.s. with $\{\lambda_n\}_{n \geq 1}$ an increasing sequence of numbers.

Lemma 10. *If $(X_i, Y_i)_{1 \leq i \leq n}$ is a sequence of i.i.d pairs of r.v's. with joint cumulative distribution function $\Phi(x, y)$, then*

$$\Phi(x, y) = \Phi_X(x) + \Phi_Y(y) + \mathbb{P}(X > x, Y > y) - 1. \quad (15)$$

Proof. The proof is obvious. $\square$

Theorem 11 (Marshall and Olkin [7]). *If*

$$\begin{aligned}\Phi_X(x) &= 1 - e^{-x}, \quad x > 0, \\ \Phi_Y(y) &= 1 - e^{-y}, \quad y > 0,\end{aligned}\quad (16)$$

then $\mathbb{P}(X > x, Y > y) = e^{-x-y-\lambda \max(x,y)}$, $\lambda > 0$.

Theorem 12 (Galombos [6]). *Let $(u_n)_{n \geq 1}$ and $(v_n)_{n \geq 1}$ be two sequences of r.v's.; suppose that there exists a distribution function $\Pi(x)$ such that at any point x of continuity of Π*

$$\mathbb{P}(u_n \leq x) \xrightarrow{n \rightarrow \infty} \Pi(x). \quad (17)$$

Suppose that for all $\epsilon > 0$

$$\mathbb{P}(|v_n| > \epsilon) \xrightarrow{n \rightarrow \infty} 0. \quad (18)$$

Then $\lim_{n \rightarrow \infty} \mathbb{P}(u_n + v_n < x) = \Pi(x)$ at all points of continuity of Π .

3. Main Result

Throughout the following, we consider $(X_i, Y_i)_{1 \leq i \leq n}$ as a sequence of i.i.d. r.v's. with joint cumulative distribution function $H(x, y)$ whose marginals are, respectively, $F(x) = 1 - e^{-x}$, $x > 0$ and $G(y) = 1 - e^{-y}$, $y > 0$.

3.1. Study of the Number of Coincidences. Let $\{n_k^{(j)}\}_{k \geq 1}$ and $\{m_k^{(j)}\}_{k \geq 1}$ be the sequences of j th record times associated, respectively, to X_i and Y_i , with $R_1^{(j)} = X_{1,j}$ and $S_1^{(j)} = Y_{1,j}$ dependent, suggesting that every time $n_k^{(j)} = m_{k'}^{(j)}$, for any k and k' , there is dependence of the j th corresponding records.

Lemma 13. *The number of coincidences, which means the number of times $n_k^{(j)}$ and $m_k^{(j)}$ belong, respectively, to the sets $\{m_1^{(j)}, m_2^{(j)}, \dots, m_k^{(j)}, \dots\}$ and $\{n_1^{(j)}, n_2^{(j)}, \dots, n_k^{(j)}, \dots\}$, is finite.*

Proof. Let us study the nature of the following series

$$\begin{aligned}\sum_n \mathbb{P}(A_n) \\ = \sum_n \mathbb{P}\{X_{n-j+1,n} > X_{n-j,n-1}; Y_{n-j+1,n} > Y_{n-j,n-1}\},\end{aligned}\quad (19)$$

where A_n is the event for which $n_k^{(j)} = m_{k'}^{(j)} = n$.

As $\{X_{n-j+1,n} > X_{n-j,n-1}; Y_{n-j+1,n} > Y_{n-j,n-1}\}$ is equivalent to $\{X_n > X_{n-j,n-1}; Y_n > Y_{n-j,n-1}\}$, studying the nature of the series $\sum_n \mathbb{P}(A_n)$ is equivalent to the study of the series $\sum_n \mathbb{P}(A'_n)$ with

$$A'_n = \{X_n > X_{n-j,n-1}; Y_n > Y_{n-j,n-1}\}. \quad (20)$$

Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. r.v's. with repartition $\Phi(x)$ and $(U_n)_{n \geq 1}$ the sequence of i.i.d. r.v's. uniformly distributed on $[0, 1]$ with

$$\Gamma(t) = \inf\{x \in \mathbb{R} \mid \Phi(x) \geq t\}. \quad (21)$$

$\{\Gamma(U_n)\}_{n \geq 1}$ and $(X_n)_{n \geq 1}$ are identically distributed.

An a.s. bounding of $X_{m,n}$ is equivalent to an a.s. bounding of $U_{m,n}$.

Using Theorems 5 and 6, and Corollary 7 of Deheuvels and Theorem 9 of Barndorff-Nielsen we obtain the a.s. bounding of $X_{n-j+1,n}$.

Choosing $\lambda_n = 1 - (1/n(\log n)^{(1+\epsilon)/(n-j+1)})$, $\epsilon > 0$, we obtain a lower bound of U_{n-j+1} .

Indeed

$$\begin{aligned}F_U(\lambda_n) &= 1 - \frac{1}{n(\log n)^{(1+\epsilon)/(n-j+1)}}, \\ \sum_n n^{m-1} (F_U(\lambda_n))^{n-m+1} (1 - F_U(\lambda_{n+1}))^m \\ &= \sum_n n^{n-j} \left(1 - \frac{1}{n(\log n)^{(1+\epsilon)/(n-j+2)}}\right)^j \\ &\quad \times \left(\frac{1}{(n+1)(\log(n+1))^{(1+\epsilon)/(n-j+2)}}\right)^{n-j+1}.\end{aligned}\quad (22)$$

Let S_n be the general term of the series in the right side of the above equality. It is clear that

$$S_n \leq V_n, \quad (23)$$

with

$$\begin{aligned}V_n &= \frac{n^{n-j}}{n^{n-j+1}} \frac{1}{(\log n)^{(1+\epsilon)((n-j+1)/(n-j+2))}} \\ &= \frac{1}{n} \frac{1}{(\log n)^{(1+\epsilon)((n-j+1)/(n-j+2))}}.\end{aligned}\quad (24)$$

For n large enough $V_n \sim W_n$ with $W_n = 1/n(\log n)^{1+\epsilon}$ is a general term of Bertrand convergent series.

Hence we get

$$\sum_n S_n \leq +\infty, \quad (25)$$

which implies that

$$U_{n-j+1,n} > 1 - \frac{1}{n(\log n)^{(1+\epsilon)/(n-j+1)}}. \quad (26)$$

Consequently, we have

$$X_{n-j+1,n} > \Gamma(\lambda_n). \quad (27)$$

Using Corollary 8, we get $\Gamma(\lambda_n) = -\log(1 - \lambda_n) = \log\{n(\log n)^{(1+\epsilon)/(n-j+1)}\}$.

Since

$$\{X_n > X_{n-j,n-1}, Y_n > Y_{n-j,n-1}\} \quad (28)$$

implies

$$\{X_n > \Gamma(\lambda_n), Y_n > \Gamma(\lambda_n)\}, \quad (29)$$

it suffices to study the nature of the series $\sum_n \mathbb{P}(A_n'')$ to know that of $\sum_n \mathbb{P}(A_n')$, where A_n'' is given by

$$A_n'' = \left\{ X_n > \log \left(n(\log n)^{(1+\epsilon)/(n-j+1)} \right); \right. \\ \left. Y_n > \log \left(n(\log n)^{(1+\epsilon)/(n-j+1)} \right) \right\}. \quad (30)$$

Using Theorem 11 of Marshall and Olkin, we get

$$\begin{aligned} \mathbb{P}(X_n > \Gamma(\lambda_n), Y_n > \Gamma(\lambda_n)) \\ = \frac{1}{n^{2+\lambda} (\log n)^{(1+\epsilon)((2+\lambda)/(n-j+1))}} \\ = K_n \leq K'_n = \frac{1}{n^{2+\lambda}}, \end{aligned} \quad (31)$$

with $\sum_n K'_n < \infty$ (Riemann's series).

Then

$$\begin{aligned} \sum_n K'_n < \infty &\Rightarrow \sum_n K_n < \infty \Rightarrow \sum_n \mathbb{P}(A_n'') \\ &< \infty \Rightarrow \sum_n \mathbb{P}(A_n') < \infty \Rightarrow \sum_n \mathbb{P}(A_n) < \infty. \end{aligned} \quad (32)$$

By Borel-Cantelli's Lemma,

$$\sum_n \mathbb{P}(A_n) < \infty \Rightarrow \overline{\lim}_n A_n = 0, \quad (33)$$

which means that with probability one at most a finite number of A_n are realized simultaneously and consequently that the number of coincidences is finite *a.s.* $\square$

3.2. Some Record Series. Let us consider and give some properties of the following sequences:

$$\begin{aligned} R_k^{(j)} &= \sum_{i=1}^k \alpha_i^{(j)} \quad \text{with} \quad \alpha_i^{(j)} = R_{i+1}^{(j)} - R_i^{(j)} \\ S_k^{(j)} &= \sum_{i=1}^k \beta_i^{(j)} \quad \text{with} \quad \beta_i^{(j)} = S_{i+1}^{(j)} - S_i^{(j)} \\ R_k^{*(j)} &= \sum_{i=1}^n \alpha_i^{(j)} \gamma_i \\ \text{with} \quad \gamma_i &= \begin{cases} 1 & \text{if } n_i^{(j)} \notin \{m_1^{(j)}, \dots, m_k^{(j)}, \dots\}, \\ 0 & \text{otherwise} \end{cases} \\ S_k^{*(j)} &= \sum_{i=1}^n \beta_i^{(j)} \delta_i \\ \text{with} \quad \delta_i &= \begin{cases} 1 & \text{if } m_i^{(j)} \notin \{n_1^{(j)}, \dots, n_k^{(j)}, \dots\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (34)$$

Lemma 14. Consider the following

$$R_n^{(j)} - R_n^{*(j)} \xrightarrow{n \rightarrow \infty} C_1 < \infty \quad \text{a.s.} \quad (35)$$

Proof. Consider the following

$$\begin{aligned} R_n^{(j)} - R_n^{*(j)} &= \sum_{i=1}^n \alpha_i^{(j)} - \sum_{i=1}^n \alpha_i^{(j)} \gamma_i \\ &= \sum_{i=1}^n (1 - \gamma_i) \alpha_i^{(j)} \\ &= \sum_{i=1}^n \theta_i \alpha_i^{(j)}, \end{aligned} \quad (36)$$

with

$$\theta_i = \begin{cases} 1 & \text{if } n_i^{(j)} \in \{m_1^{(j)}, \dots, m_k^{(j)}, \dots\}, \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

Hence

$$R_n^{(j)} - R_n^{*(j)} = \sum_{i \in I} \alpha_i^{(j)}, \quad (38)$$

where I is the set of i for which there are coincidences; as this set is finite $\sum_{i \in I} \alpha_i^{(j)} < \infty$. By the same method we have

$$S_n^{(j)} - S_n^{*(j)} \xrightarrow{n \rightarrow \infty} C_2 < \infty \quad \text{a.s.} \quad (39)$$

$\square$

Lemma 15. Consider

$$\frac{(R_n^{*(j)} - (n/j))}{\sqrt{n/j^2}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} N(0, 1). \quad (40)$$

Proof. Consider the following

$$\begin{aligned} \frac{R_n^{*(j)} - (n/j)}{\sqrt{n/j^2}} &= \frac{R_n^{*(j)} - R_n^{(j)} + R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \\ &= \frac{R_n^{*(j)} - R_n^{(j)}}{\sqrt{n/j^2}} + \frac{R_n^{(j)} - (n/j)}{\sqrt{n/j^2}}. \end{aligned} \quad (41)$$

Using Corollary 3 and Theorem 12 of Galombos and the central limit theorem we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{R_n^{*(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right\} \\ = \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{R_n^{*(j)} - R_n^{(j)} + R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right\} \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{R_n^{*(j)} - R_n^{(j)}}{\sqrt{n/j^2}} + \frac{R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right\} \\
&= \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du \\
&= N(x).
\end{aligned} \tag{42}$$

By the same method, we have

$$\frac{S_n^{*(j)} - (n/j)}{\sqrt{n/j^2}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} N(0, 1). \tag{43}$$

□

Now, let us consider the two following sequences:

$$R_n^{** (j)} = \sum_{k=1}^{N(n)} \alpha_k^{(j)} \gamma_k, \quad S_n^{** (j)} = \sum_{k=1}^{M(n)} \beta_k^{(j)} \delta_k \tag{44}$$

with

$$\begin{aligned}
N(n) &= \inf \left\{ m \geq 1 \mid \sum_{k=1}^m \gamma_k = n \right\}, \\
M(n) &= \inf \left\{ m \geq 1 \mid \sum_{k=1}^m \delta_k = n \right\}.
\end{aligned} \tag{45}$$

Lemma 16. Consider the following

$$R_n^{** (j)} - R_n^{*(j)} \xrightarrow[n \rightarrow \infty]{} k_1 < \infty \quad \text{a.s.} \tag{46}$$

Proof. Since the number of coincidences is finite, there exists n_0 finite, such that, for all $n > n_0$, $\gamma_n = 1$;

$$\begin{aligned}
&R_n^{** (j)} - R_n^{*(j)} \\
&= \sum_{k=1}^{N(n)} \alpha_k^{(j)} \gamma_k - \sum_{k=1}^n \alpha_k^{(j)} \gamma_k \\
&= \sum_{k=1}^{n_0} \alpha_k^{(j)} \gamma_k + \sum_{k=n_0+1}^{N(n)} \alpha_k^{(j)} \gamma_k \\
&\quad - \sum_{k=1}^{n_0} \alpha_k^{(j)} \gamma_k - \sum_{k=n_0+1}^n \alpha_k^{(j)} \gamma_k \\
&= \sum_{k=n_0+1}^{N(n)} \alpha_k^{(j)} \gamma_k - \sum_{k=n_0+1}^n \alpha_k^{(j)} \gamma_k \\
&= \sum_{k=n_0+1}^{N(n)} \alpha_k^{(j)} - \sum_{k=n_0+1}^n \alpha_k^{(j)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=n_0+1}^n \alpha_k^{(j)} + \sum_{k=n+1}^{N(n)} \alpha_k^{(j)} - \sum_{k=n_0+1}^n \alpha_k^{(j)} \\
&= \sum_{k=n+1}^{N(n)} \alpha_k^{(j)}.
\end{aligned} \tag{47}$$

$R_n^{** (j)} - R_n^{*(j)}$ is the sum of $(N(n) - n)$ i.i.d. r.v's.

Between 1 and n_0 we have K coincidences and r noncoincidences, so we must have $(n - r)$ noncoincidences between $(n_0 + 1)$ and $N(n)$; hence

$$N(n) = K + r + n - r = K + n. \tag{48}$$

This implies that $N(n) - n = K$ with K finite, so $R_n^{** (j)} - R_n^{*(j)}$ is a finite sum of r.v's., which proves our result.

By the same method, we have

$$S_n^{** (j)} - S_n^{*(j)} \xrightarrow[n \rightarrow \infty]{} k_2 < \infty \quad \text{a.s.} \tag{49}$$

□

Lemma 17. $R_n^{** (j)}$ and $S_n^{** (j)}$ are individually the sum of n i.i.d exponential r.v's. with parameter j so by the central limit theorem one gets

$$\begin{aligned}
&\frac{R_n^{** (j)} - (n/j)}{\sqrt{n/j^2}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} N(0, 1), \\
&\frac{S_n^{** (j)} - (n/j)}{\sqrt{n/j^2}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} N(0, 1).
\end{aligned} \tag{50}$$

3.3. Study of the j th Record Times

Definition 18. Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathcal{F}_t)_{t \in I}$ be an increasing family of sub- σ -algebra of $\mathcal{F}$, defined on an interval I of $\mathbb{R}$ or $\mathbb{N}$.

According to $(\mathcal{F}_t)_{t \in I}$ a r.v. T defined on $(\Omega, \mathcal{F})$ with values in I is called a stopping time if

$$\{\omega : T(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall t \in I. \tag{51}$$

Interpretation. The notion of stopping time corresponds to the notion of stopping some process at a random time T . Indeed, the stopping time is a function of ω . Also the definition (51) expresses that all the contingencies that lead to the previous stopping time are an event of $\mathcal{F}_t$, that means an event definable by the behavior of the phenomenon before time t .

Lemma 19. The sequence of j th record times $\{n_k^{(j)}\}_{k \geq 1}$ is a sequence of stopping time.

Proof. Considering the sub- σ -algebra $a_n = \sigma(X_1, Y_1, \dots, X_n, Y_n)$ and $a_t = \sigma(X_1, Y_1, \dots, X_t, Y_t)$ the increasing family of sub- σ -algebra of a_n , $n_k^{(j)}$ is a stopping time if and only if

$$\{n_k^{(j)} = t\} \in a_t \quad \forall t \in \mathbb{N}. \tag{52}$$

But $n_k^{(j)} = t$ is equivalent to $X_{n_k^{(j)} - j + 1, n_k^{(j)}} = X_{t - j + 1, t} \mathbb{1}_{\{n_k^{(j)} = t\}}$.

$X_{t-j+1,t}$ is $\sigma(X_{1,t}, \dots, X_{t-j+1,t})$ measurable and included in $\sigma(X_1, \dots, X_t)$ because the ordered statistic is exhaustive.

But $\sigma(X_1, \dots, X_t)$ is included in $\sigma(X_1, Y_1, \dots, X_t, Y_t)$ which is equal to a_t .

Consequently

$$\{n_k^{(j)} = t\} \in a_t \quad \forall t \in I \subseteq \overline{\mathbb{N}}, \quad (53)$$

which implies that $n_k^{(j)}$ is a stopping time.

By the same method $m_k^{(j)}$ is also a stopping time. $\square$

3.4. Asymptotic Independence of $R_n^{(j)}$ and $S_n^{(j)}$. In this section we will prove the asymptotic independence of $\{R_n^{(j)}\}_{n \geq 1}$ and $\{S_n^{(j)}\}_{n \geq 1}$ sequences of records issued from $(X_i)_{1 \leq i \leq n}$ and $(Y_i)_{1 \leq i \leq n}$, respectively.

Proposition 20. Let $(R_n^{(j)}, S_n^{(j)})_{n \geq 1}$ be a bivariate r.v.s. of records issued from pairs of r.v.s. $(X_n, Y_n)_{n \geq 1}$ with joint cumulative distribution $H(x, y)$ whose marginals $F(x)$ and $G(y)$ are exponential with parameter $\lambda = 1$; then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq x, \frac{S_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right) \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{S_n^{(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right) \\ &= N(x) N(y), \end{aligned} \quad (54)$$

where $N(\cdot)$ is the standard normal distribution.

Proof. Since there exists an n_0 from which there is no coincidences, throughout the following, we will consider the case where we observe, first, a record $R_k^{(j)}$ for the X_i .

$a_{n_k^{(j)}} = \{A \mid A \cap n_k^{(j)} = t \in a_t \quad \forall t \in I \subseteq \overline{\mathbb{N}}\}$ being the algebra of events prior $n_k^{(j)}$, all that has led to the appearance of $R_k^{(j)}$ that depends only on what has been observed before $n_k^{(j)}$.

Let $n_k^{(j)} = M$. Given $X_{1,M}, \dots, X_{M-j+1,M}, Y_{1,M}, \dots, Y_{M-j+1,M}$ and using Corollary 4, the first record $S_i^{(j)}$ with $n_i^{(j)} > M$ is such that $S_i^{(j)} - S_{i-1}^{(j)}$ is independent of $Y_{1,M}, \dots, Y_{M-j+1,M}$ and fortiori of $X_{1,M}, \dots, X_{M-j+1,M}$.

Thus, $\alpha_i^{(j)} = R_i^{(j)} - R_{i-1}^{(j)}, n_i^{(j)}$ and $\beta_i^{(j)} = S_i^{(j)} - S_{i-1}^{(j)}, m_i^{(j)} \geq M$ are independent from all what has been observed before $n_k^{(j)} = M$, including $R_k^{(j)}$, which means it is independent of a_M .

We conclude that

- (i) for all i such that $n_i^{(j)} \geq n_k^{(j)}$, $R_i^{(j)}$ is asymptotically independent of $S_{i-1}^{(j)}$ with $m_{i-1}^{(j)} < n_k^{(j)}$,
- (ii) for all t such that $m_t^{(j)} \geq n_k^{(j)}$, $S_t^{(j)}$ is asymptotically independent of $R_{i-1}^{(j)}$ with $n_{i-1}^{(j)} < n_k^{(j)}$.

This sequential reasoning will be taken each time we have either a record for the X_i or a record for the Y_i .

Thus, we prove that $R_n^{(j)}$ and $S_n^{(j)}$ are asymptotically independent and consequently $R_n^{(j)}$ and $S_n^{(j)}$ will be as well.

Indeed, if we denote, for the sake of simplicity, $\lim_{n \rightarrow \infty} \mathbb{P}((R_n^{(j)} - (n/j))/\sqrt{n/j^2} \leq x, (S_n^{(j)} - (n/j))/\sqrt{n/j^2} \leq y)$ by α , we get

$$\alpha = \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{(j)} - R_n^{*(j)} + R_n^{*(j)} - R_n^{***(j)} + R_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq x, \frac{S_n^{(j)} - S_n^{*(j)} + S_n^{*(j)} - S_n^{***(j)} + S_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right). \quad (55)$$

Using (35), (39), and Theorem 12

$$\alpha = \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{*(j)} - R_n^{***(j)} + R_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq x, \frac{S_n^{*(j)} - S_n^{***(j)} + S_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right). \quad (56)$$

Using (46), (49), and Theorem 12

$$\alpha = \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq x, \frac{S_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right). \quad (57)$$

Now using (50) we obtain

$$\begin{aligned} \alpha &= \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{R_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq x \right) \\ &\quad \times \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{S_n^{***(j)} - (n/j)}{\sqrt{n/j^2}} \leq y \right) \\ &= N(x) N(y), \end{aligned} \quad (58)$$

which proves our result. $\square$

4. Conclusion

In this paper we proved the asymptotic independence of the j th records $R_k^{(j)}$ and $S_k^{(j)}$ in the exponential case ($\lambda = 1$), from sequences $(X_i)_{1 \leq i \leq n}$ and $(Y_i)_{1 \leq i \leq n}$, respectively. In this case we know that $X_{n,n} = \max_{1 \leq i \leq n} (X_i)$ and $Y_{n,n} = \max_{1 \leq i \leq n} (Y_i)$ are the type Gumbell, respectively. It would be interesting to see if we can obtain the same result when $X_{n,n}$ and $Y_{n,n}$ are Fréchet or Weibull type.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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