Research Article

A Globally Convergent Parallel SSLE Algorithm for Inequality Constrained Optimization

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A new parallel variable distribution algorithm based on interior point SSLE algorithm is proposed for solving inequality constrained optimization problems under the condition that the constraints are block-separable by the technology of sequential system of linear equation. Each iteration of this algorithm only needs to solve three systems of linear equations with the same coefficient matrix to obtain the descent direction. Furthermore, under certain conditions, the global convergence is achieved.

1. Introduction

Consider the following inequality constrained optimization problems:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_j(x) \leq 0, \quad j \in I = \{1, \dots, m\}, \end{aligned} \quad (1)$$

where $f(x) : R^n \rightarrow R$, $g_j(x) : R^n \rightarrow R$ are continuously differentiable. We denote

$$\begin{aligned} X &= \{x \in R^n \mid g_j(x) \leq 0, \quad j \in I\}, \\ X^0 &= \{x \in R^n \mid g_j(x) < 0, \quad j \in I\}, \\ I(x) &= \{j \mid g_j(x) = 0, \quad j \in I\}. \end{aligned} \quad (2)$$

To solve the problem (1), there are two type methods with superlinear convergence: sequential quadratic programming (SQP) type algorithms (see [1–4], etc.) and SSLE (sequential system of linear equations) type algorithms (see [5–9], etc.). In general, since SQP algorithms are necessary to solve one or more quadratic programming subproblems in single iteration, the computation effort is very large.

SSLE algorithms were proposed to solve the problem (1), in which an iteration similar to the following linear system was considered:

$$\begin{aligned} Hd_0 + \nabla_x L(x, \lambda_0) &= 0, \\ \mu_j \nabla g_j(x)^T d_0 + \lambda_{0j} g_j(x) &= 0, \quad j = 1 \sim m, \end{aligned} \quad (3)$$

where $L(x, \lambda) = f(x) + \sum_{j=1}^m \lambda_j g_j(x)$ is Lagrangian function, H is an estimate of the Hessian of L , x is the current estimate of a solution x^* , d_0 is the search direction, and λ_0 is the next estimate of the Kuhn-Tucker multiplier vector associated with x^* . Obviously, it is simpler to solve system of linear equations than to solve the QP (quadratic programming) problem with inequality constraints.

In addition, parallel variable distribution (PVD) algorithm [10] is a method that distributes the variables among parallel processors. The problem is parted into many respective subproblems and each subproblem is arranged to a different processor in it. Each processor has the primary responsibility for updating its block of variables while allowing the remaining secondary variables to change in a restricted fashion along some easily computable directions. In 2002, Sagastizábal and Solodov [11] proposed two new variants of PVD for the constrained case. Without assuming convexity

of constraints, but assuming block-separable structure, they showed that PVD subproblems can be solved inexactly by solving their quadratic programming approximations. Han et al. [12] proposed an asynchronous PVT algorithm for solving large-scale linearly constrained convex minimization problems with the idea in 2009, which is based on the idea that a constrained optimization problem is equivalent to a differentiable unconstrained optimization problem by introducing the Fischer function. In 2011, Zheng et al. [13] gave a parallel SSLE algorithm, in which the PVD subproblems are solved inexactly by serial sequential linear equations, for solving large-scale constrained optimization with block-separable structure. Without assuming the convexity of constraints, the algorithm is proved to be globally convergent to a KKT point.

In this paper, we use Zhu [8] as our main reference on SSLE-type PVD method for problem (1). Suppose that the problem (1) has the following block-separable structure:

$$\begin{aligned} x &= (x_1, x_2, \dots, x_p), \quad x_l \in R^{n_l}, \quad l = 1, 2, \dots, p, \\ \sum_{l=1}^p n_l &= n, \\ g(x) &= (g_1(x_1), g_2(x_2), \dots, g_p(x_p)), \\ g_l(x_l) : R^{n_l} &\longrightarrow R^{m_l}, \quad \sum_{l=1}^p m_l = m, \\ I &= (I_1, I_2, \dots, I_p) = \{1, 2, \dots, m\}, \\ I_l(x_l) &= \{l_j \mid g_{l_j}(x_l) = 0, \quad l_j \in I_l = \{1, 2, \dots, m_l\}\}. \end{aligned} \quad (4)$$

Then, the problem is distributed into p parallel subproblems which have been computed by the p parallel processors. In the algorithm, at each iteration, the search direction is obtained by solving three systems of linear equations with the same coefficient matrix, which guarantees that every iteration is feasible. Thereby, the computational effort of the proposed algorithm is reduced further. Furthermore, its global convergence is obtained under some suitable conditions.

The remaining part of this paper is organized as follows. In Section 2, a parallel SSLE algorithm is presented. Global convergence is established under some basic assumptions in Section 3. And concluding remarks are given in the last section.

2. Description of Algorithm

Now we state our algorithm as follows.

Algorithm

Step 0 (initialization). Given a starting point $x^0 \in X^0$. Choose parameters $\theta, \sigma \in (0, 1), \alpha \in (0, 1/2), 2 < \tau < \delta < 3, \bar{\mu} > 0$,

$0 < \mu_{l_j}^0 \leq \bar{\mu}$, and $(l_j \in I_l)$ and an initial symmetric positive definite matrix $H_l^0 \in R^{n_l \times n_l}$. Set $k = 0$.

Step 1 (parallelization). For each processor $l \in \{1, 2, \dots, p\}$, let

$$\begin{aligned} N_l^k &= (\nabla g_{l_j}(x_l^k), \quad l_j \in I_l), \\ G_l^k &= \text{diag}(g_{l_j}(x_l^k), \quad l_j \in I_l), \\ M_l^k &= \text{diag}(\mu_{l_j}^k, \quad l_j \in I_l). \end{aligned} \quad (5)$$

1.1 Computation of the Newton Direction. Solve the following system of linear equations:

$$\begin{pmatrix} H_l^k & N_l^k \\ M_l^k(N_l^k)^T & G_l^k \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = - \begin{pmatrix} \nabla_l f(x_l^k) \\ 0 \end{pmatrix}. \quad (6)$$

Let (d_l^{0k}, π_l^k) be the solution. If $d_l^{0k} = 0$, stop.

1.2 Computation of the Descent Direction. Solve the following system of linear equations:

$$\begin{pmatrix} H_l^k & N_l^k \\ M_l^k(N_l^k)^T & G_l^k \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = - \begin{pmatrix} \nabla_l f(x_l^k) \\ \|d_l^{0k}\|^\delta e_l \end{pmatrix}, \quad (7)$$

where $e_l = (1, \dots, 1)^T \in R^{m_l}$. Let (d_l^{1k}, π_l^k) be the solution.

1.3 Computation of the Main Search Direction. Establish a convex combination of d_l^{0k} and d_l^{1k} :

$$\begin{aligned} d_l^k &= (1 - \beta_l^k) d_l^{0k} + \beta_l^k d_l^{1k}, \\ \lambda_l^k &= (1 - \beta_l^k) \pi_l^k + \beta_l^k \pi_l^k, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \beta_l^k &= \max \left\{ \beta \in (0, 1] \mid (1 - \beta) \nabla_l f(x_l^k)^T d_l^{0k} \right. \\ &\quad \left. + \beta \nabla_l f(x_l^k)^T d_l^{1k} \leq \theta \nabla_l f(x_l^k)^T d_l^{0k} \right\}. \end{aligned} \quad (9)$$

1.4 Computation of the High-Order Corrected Direction. Set

$$L_l^k = \{l_j \in I_l \mid g_{l_j}(x_l^k) \geq -\lambda_{l_j}^k\}. \quad (10)$$

Solve the following system of linear equations:

$$\begin{pmatrix} H_l^k & N_l^k \\ M_l^k(N_l^k)^T & G_l^k \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = - \begin{pmatrix} \nabla_l f(x_l^k) \\ \|d_l^{0k}\|^\tau e_l + M_l^k \tilde{g}_l^k \end{pmatrix}, \quad (11)$$

where

$$\begin{aligned} \tilde{g}_l^k &= (\tilde{g}_{l_j}^k, \quad l_j \in I_l), \\ \tilde{g}_{l_j}^k &= \begin{cases} g_{l_j}(x_l^k + d_l^k), & l_j \in L_l^k, \\ 0, & l_j \in I_l \setminus L_l^k. \end{cases} \end{aligned} \quad (12)$$

Let $(\tilde{d}_l^k + d_l^k, \tilde{\lambda}_l^k + \lambda_l^k)$ be the solution. If $\|\tilde{d}_l^k\| > \|d_l^k\|$, set $\tilde{d}_l^k = 0$.

Step 2 (synchronization). Let

$$d^k = (d_1^k, d_2^k, \dots, d_p^k), \quad \tilde{d}^k = (\tilde{d}_1^k, \tilde{d}_2^k, \dots, \tilde{d}_p^k),$$

$$\lambda^k = (\lambda_1^k, \lambda_2^k, \dots, \lambda_p^k), \quad J_k = \{j \in I \mid g_j(x^k) \geq \lambda_j^k\}. \quad (13)$$

2.1 The Line Search. Compute t_k , the first number t in the sequence $\{1, 1/2, 1/4, \dots\}$ satisfying

$$f(x^k + t d^k + t^2 \tilde{d}^k) \leq f(x^k) + \alpha t \nabla f(x^k)^T d^k, \quad (14)$$

$$g_j(x^k + t d^k + t^2 \tilde{d}^k) \leq g_j(x^k), \quad j \in J_k, \quad (15)$$

$$g_j(x^k + t d^k + t^2 \tilde{d}^k) < 0, \quad j \in I \setminus J_k. \quad (16)$$

Step 3 (update). Obtain H_l^{k+1} by updating the positive definite matrix H_l^k using some quasi-Newton formulas. Set

$$\mu_{l_j}^{k+1} = \min \left\{ \max \left\{ \pi_{l_j}^k, \|d_l^{0k}\| \right\}, \bar{\mu} \right\}, \quad l_j \in I_l, \quad (17)$$

and $x^{k+1} = x^k + t_k d^k + t_k^2 \tilde{d}^k$. Let $k = k + 1$. Go back to Step 1.

3. Global Convergence of Algorithm

We make the following general assumptions and let them hold throughout the paper.

(H3.1) For $l = 1, 2, \dots, p$, the sets $X_l = \{x_l \mid g_l(x_l) \leq 0\}$ and $X_l^0 = \{x_l \mid g_l(x_l) < 0\}$ are nonempty. The set $X_l \cap \{x_l \mid f(x_l) \leq f(x_l^0)\}$ is compact.

(H3.2) The functions f , g_j , and $j \in I$ are continuously differentiable.

(H3.3) For all $x_l \in X_l$, the set of vectors $\{\nabla g_{l_j}(x_l) : l_j \in I_l(x_l)\}$ is linearly independent.

(H3.4) There exist $a, b > 0$, such that $a\|d\|^2 \leq d^T H_l^k d \leq b\|d\|^2$, for all $k \in \mathbb{R}$ and $d \in \mathbb{R}^{n_l}$.

Lemma 1. For any $x_l \in X_l$, any positive-definite matrix $H_l \in \mathbb{R}^{n_l \times n_l}$, and nonnegative vector $\mu_l = (\mu_{l_j}, l_j \in I_l) \in \mathbb{R}^{m_l}$ such that $\mu_{l_j} > 0, l_j \in I_l(x_l)$, the matrix

$$F(x_l, H_l, \mu_l) = \begin{pmatrix} H_l & N_l \\ M_l N_l^T & G_l \end{pmatrix} \quad (18)$$

is nonsingular, where $N_l = (\nabla g_{l_j}(x_l), l_j \in I_l) \in \mathbb{R}^{n_l \times m_l}$, $M_l = \text{diag}(\mu_{l_j}, l_j \in I_l) \in \mathbb{R}^{m_l \times m_l}$, and $G_l = \text{diag}(g_{l_j}(x_l), l_j \in I_l) \in \mathbb{R}^{m_l \times m_l}$.

Proof. We need only to prove that $(d_l, \lambda_l) = (0, 0) \in \mathbb{R}^{n_l \times m_l}$ is the unique solution of the following linear equations:

$$F(x_l, H_l, \mu_l) \begin{pmatrix} d_l \\ \lambda_l \end{pmatrix} = 0, \quad \text{that is, } \begin{cases} H_l d_l + N_l \lambda_l = 0, \\ M_l N_l^T d_l + G_l \lambda_l = 0. \end{cases} \quad (19)$$

Now consider the cases $l_j \in I_l(x_l)$ and $l_j \notin I_l(x_l)$ separately.

For $l_j \in I_l(x_l)$, if $g_{l_j}(x_l) = 0$ and $\mu_{l_j} > 0$, it follows from (19) that

$$\nabla g_{l_j}(x_l)^T d_l = 0. \quad (20)$$

By (19), we have

$$\sum_{l_j \in I_l(x_l)} \lambda_{l_j} \nabla g_{l_j}(x_l) = 0. \quad (21)$$

Then from the assumption (H3.3), it shows that $\lambda_{l_j} = 0$ ($\forall l_j \in I_l(x_l)$).

For $l_j \notin I_l(x_l)$, if $\mu_{l_j} = 0$, it follows from the first equation of (19) that

$$\lambda_{l_j} = 0, \quad d_{l_j} = 0. \quad (22)$$

If $\mu_{l_j} > 0$, it follows from the second equation of (19) that

$$\nabla g_{l_j}(x_l)^T d_l = -\frac{\lambda_{l_j}}{\mu_{l_j}} g_{l_j}(x_l). \quad (23)$$

Hence, if $\mu_{l_j} \geq 0$, combine (22) and (23), and from the first equation of (19) and the assumption (H3.4), we get

$$\begin{aligned} 0 &\leq d_l^T H_l d_l = -\lambda_l^T N_l^T d_l \\ &= -\sum_{l_j \in I_l(x_l), \mu_{l_j} > 0} \lambda_{l_j} (\nabla g_{l_j}(x_l)^T d_l) \\ &= \sum_{l_j \in I_l(x_l), \mu_{l_j} > 0} \frac{(\lambda_{l_j})^2}{\mu_{l_j}} g_{l_j}(x_l) \leq 0. \end{aligned} \quad (24)$$

It shows that $d_{l_j} = 0$ and $\lambda_{l_j} = 0$ ($\forall l_j \notin I_l(x_l)$). $\square$

Lemma 2. For $l = 1, 2, \dots, p$,

- (1) if $d_l^{0k} = 0$, then x^k is a KKT point of (1);
- (2) if $d_l^{0k} \neq 0$, then d_l^k computed according to (8) is well defined and

$$\begin{aligned} \nabla_l f(x_l^k)^T d_l^{0k} &\leq -(d_l^{0k})^T H_l^k d_l^{0k} < 0, \\ \nabla_l f(x_l^k)^T d_l^k &\leq \theta \nabla_l f(x_l^k)^T d_l^{0k} < 0, \end{aligned} \quad (25)$$

$$\nabla g_{l_j}(x_l^k)^T d_l^k = -\frac{\lambda_{l_j}^k}{\mu_{l_j}^k} g_{l_j}(x_l^k) - \beta_k \|d_l^{0k}\|^\delta, \quad l_j \in I_l.$$

Proof. (1) It is obvious according to the definition of the KKT point of (1).

(2) If $d_l^{0k} \neq 0$, from (6), we have

$$\begin{aligned} \nabla_l f(x_l^k)^T d_l^{0k} &= -(d_l^{0k})^T H_l^k d_l^{0k} - (N_k \pi^k)^T d_l^{0k} \\ &= -(d_l^{0k})^T H_l^k d_l^{0k} + \sum_{j \in I} \frac{(\pi_{l_j}^k)^2}{\mu_{l_j}^k} g_{l_j}(x^k) \\ &\leq -(d_l^{0k})^T H_l^k d_l^{0k} < 0, \\ \nabla g_{l_j}(x^k)^T d_l^{0k} &= -\frac{\pi_{l_j}^k}{\mu_{l_j}^k} g_{l_j}(x^k), \quad l_j \in I_l. \end{aligned} \quad (26)$$

Thereby, from (9), there exists some $\beta \in (0, 1]$, such that $\beta_k = \beta \in (0, 1]$; that is, d_l^k is well defined. In addition, from (7), it follows that

$$\nabla g_{l_j}(x_l^k)^T d_l^k = -\frac{\tilde{\pi}_{l_j}^k}{\mu_{l_j}^k} g_{l_j}(x_l^k) - \|d_l^{0k}\|^\delta, \quad l_j \in I_l. \quad (27)$$

Thus, from (8), it is clear to see that

$$\begin{aligned} \nabla g_{l_j}(x_l^k)^T d_l^k &= (1 - \beta_k) \nabla g_{l_j}(x_l^k)^T d_l^{0k} + \beta_k \nabla g_{l_j}(x_l^k)^T d_l^{1k} \\ &= -\frac{\lambda_{l_j}^k}{\mu_{l_j}^k} g_{l_j}(x_l^k) - \beta_k \|d_l^{0k}\|^\delta, \quad l_j \in I_l, \\ \nabla_l f(x_l^k)^T d_l^k &= (1 - \beta_k) \nabla_l f(x_l^k)^T d_l^{0k} + \beta_k \nabla_l f(x_l^k)^T d_l^{1k} \\ &\leq \theta \nabla_l f(x_l^k)^T d_l^{0k} < 0. \end{aligned} \quad (28)$$

The claim holds. $\square$

Lemma 3. *The line search in Step 2 of the algorithm is well defined; that is, there exists $t_k > 0$ such that (14)–(16) hold.*

Proof. Firstly, for (14), since f is continuously differentiable, we can see that

$$\begin{aligned} a_k &\triangleq f(x^k + td^k + t^2 \tilde{d}^k) - f(x^k) - \alpha t \nabla f(x^k)^T d^k \\ &= \nabla f(x^k)^T (td^k + t^2 \tilde{d}^k) - \alpha t \nabla f(x^k)^T d^k + o(t) \\ &= (1 - \alpha) t \nabla f(x^k)^T d^k + o(t). \end{aligned} \quad (29)$$

From (25), we have $\nabla_l f(x_l^k)^T d_l^k < 0$, ($l \in \{1, 2, \dots, p\}$). Then we can obtain

$$\nabla f(x^k)^T d^k = \sum_{l=1}^p \nabla_l f(x_l^k)^T d_l^k < 0. \quad (30)$$

Thus, for $\alpha \in (0, 1/2)$, there exists some $\bar{t} > 0$, such that $a_k \leq 0$, $\forall t \in [0, \bar{t}]$.

On the other hand, from (25), if $l_j \in I_l$, ($l \in \{1, 2, \dots, p\}$), we have

$$\nabla g_{l_j}(x_l^k)^T d_l^k = -\frac{\lambda_{l_j}^k}{\mu_{l_j}^k} g_{l_j}(x_l^k) - \beta_k \|d_l^{0k}\|^\delta, \quad l_j \in I_l. \quad (31)$$

So, for all $j \in \{1, 2, \dots, m\}$,

$$\begin{aligned} \nabla g_j(x^k)^T d^k &= \sum_{l=1}^p \sum_{j=1}^{m_l} \nabla g_{l_j}(x_l^k)^T d_l^k \\ &= -\frac{\lambda_j^k}{\mu_j^k} g_j(x^k) - \beta_k \|d^{0k}\|^\delta, \quad j \in I. \end{aligned} \quad (32)$$

Then, from (13), we obtain, for $j \in J_k$, that

$$\begin{aligned} b_j^k &\triangleq g_j(x^k + td^k + t^2 \tilde{d}^k) - g_j(x^k) \\ &= t \nabla g_j(x^k)^T d^k + o(t) \\ &= -\frac{\lambda_j^k}{\mu_j^k} t g_j(x^k) - t \beta_k \|d_0^k\|^\delta + o(t) \\ &\leq -\frac{1}{\mu_j^k} t g_j^2(x^k) - t \beta_k \|d_0^k\|^\delta + o(t). \end{aligned} \quad (33)$$

So, there exists some $\bar{t}_j > 0$, $j \in J_k$, such that $b_j^k \leq 0$, $\forall t \in [0, \bar{t}_j]$; that is, (15) holds.

Thirdly, for (16), since g is continuous and $g_j(x^k) < 0$, there exists some $\bar{t}_j > 0$, $j \notin J_k$, such that

$$g_j(x^k + td^k + t^2 \tilde{d}^k) \leq \frac{1}{2} g_j(x^k) < 0. \quad (34)$$

Let $t_k = (1/2)^i = \min\{\bar{t}, \bar{t}_j, j \in I\}$; then the conclusion holds. $\square$

According to (H3.1), (H3.2), and (H3.4), we might assume that there exists a subsequence K as well, such that

$$\begin{aligned} x^k &\longrightarrow x^*, \quad H_l^k \longrightarrow H_l^*, \quad d_l^{0k} \longrightarrow d_l^{0*}, \\ \pi_l^k &\longrightarrow \pi_l^*, \quad \mu_l^k \longrightarrow \mu_l^*, \\ &k \in K. \end{aligned} \quad (35)$$

In order to obtain the global convergence of the algorithm, we assume the following condition.

(H3.5) The number of stationary points of (1) is finite.

Theorem 4. *The algorithm in Section 2 either stops at the KKT point x^k of (1) in finite iteration or generates an infinite sequence $\{x^k\}$ whose all accumulation points are KKT points of (1).*

Proof. The first statement is obvious, the only stopping point being in Step 1.1. Firstly, we show that

$$\{x^k\}_{k \in K} \rightarrow x^*, \quad d^{0k} \rightarrow 0, \quad k \in K, \quad (36)$$

where $x^* = (x_1^*, x_2^*, \dots, x_p^*)$, $d^{0k} = (d_1^{0k}, d_2^{0k}, \dots, d_p^{0k})$.

Since $\{f(x^k)\}$ is monotonically decreasing, the facts $\{x^k\}_{k \in K} \rightarrow x^*$ and continuity of f imply that

$$f(x^k) \rightarrow f(x^*), \quad k \rightarrow \infty. \quad (37)$$

For $l_j \in I_l$, ($l \in \{1, 2, \dots, p\}$), suppose by contradiction that $d_l^{0*} \neq 0$. Then, from (17), we have

$$\mu_{l_j}^k \rightarrow \mu_{l_j}^* > 0, \quad l_j \in I_l, \quad k \in K. \quad (38)$$

Hence, it is easy to prove that (d_l^{0*}, π_l^*) is the unique solution of the following linear system:

$$\begin{aligned} H_l^* d_l + \nabla_l f(x_l^*) + N_l^* \pi_l &= 0, \\ \mu_{l_j}^* \nabla g_{l_j}(x_l^*)^T d_l + \pi_{l_j} g_{l_j}(x_l^*) &= 0, \quad l_j \in I_l, \end{aligned} \quad (39)$$

where $N_l^* = (\nabla g_{l_j}(x_l^*), l_j \in I_l)$. Then we can obtain

$$\begin{aligned} H^* d + \nabla f(x^*) + N^* \pi &= 0, \\ \mu_j^* \nabla g_j(x^*)^T d + \pi_j g_j(x^*) &= 0, \quad j \in I. \end{aligned} \quad (40)$$

Thereby,

$$\nabla f(x^*)^T d^{0*} < 0. \quad (41)$$

Similar to (8), we define d^* , and by imitating the proof of Lemma 2, it follows that

$$\begin{aligned} \nabla f(x^*)^T d^* &< 0, \\ \nabla g_j(x^*)^T d^* &\leq -\frac{\lambda_j^*}{\mu_j^*} g_j(x^*) - \beta^* \|d_0^*\|^\delta, \quad j \in I. \end{aligned} \quad (42)$$

From (41), (42), and the proof of Lemma 3, we can conclude that the step-size t_k obtained by the linear search in Step 2.2 is bounded away from zero on K ; that is,

$$t_k \geq t_* = \inf \{t_k, k \in K\} > 0, \quad k \in K. \quad (43)$$

So, from (14), (37), and (42), we get

$$\begin{aligned} 0 &= \lim_{k \in K} (f(x^{k+1}) - f(x^k)) \leq \lim_{k \in K} \alpha t_k \nabla f(x^k)^T d^k \\ &\leq \frac{1}{2} \alpha t_* \nabla f(x^*)^T d^* < 0. \end{aligned} \quad (44)$$

It is a contradiction, which shows that $d^{0k} \rightarrow 0$, $k \in K$.

Furthermore, from (40), we have

$$\nabla f(x^*) + N_* \pi^* = 0, \quad \pi_j^* g_j(x^*) = 0, \quad j \in I. \quad (45)$$

If $g_j(x^*) < 0$, $\forall j \in I$, then $\pi^* = 0$, $\nabla f(x^*) = 0$, and it is obvious that x^* is a KKT point of (1).

Without loss of generality, we suppose that there exists some $j_0 \in I$, such that $g_{j_0}(x^*) = 0$. If $\pi_{j_0}^* \geq 0$, then it is easy to see that x^* is a KKT point of (1). Suppose that $\pi_{j_0}^* < 0$. Since there are only finitely many choices for sets $J_k \subseteq I$, we might assume, for $k \in K$, k large enough, that $J_k \equiv J$ as well, where J is a constant set. Obviously, $j_0 \in J \neq \emptyset$. From condition (H3.5), it holds that $x^k \rightarrow x^*$, $k \rightarrow \infty$. Thereby, it holds that

$$\begin{aligned} \lambda_{j_0}^k &\leq g_{j_0}(x^k), \quad \lambda_{j_0}^k \rightarrow \lambda_{j_0}^* < 0, \\ g_{j_0}(x^k) &\rightarrow g_{j_0}(x^*) = 0. \end{aligned} \quad (46)$$

While, from (15), there exists some k_0 , such that, for $k \geq k_0$,

$$g_{j_0}(x^k) \leq g_{j_0}(x^{k-1}) \leq \dots \leq g_{j_0}(x^{k_0+1}) \leq g_{j_0}(x^{k_0}) < 0, \quad (47)$$

it is in contradiction to (46), which shows that x^* is a KKT point of (1). $\square$

4. Concluding Remarks

In this paper, combined with the idea of parallel variable distribution, we proposed a new interior point SSLE algorithm for solving constrained optimization problems. The presented algorithm is a special structure of the objective function or constraints with a special structure. Under some mild conditions, the theoretical analysis shows that the convergence of this algorithm can be obtained.

It is noted that there are still some problems worthy of further discussion such as study of the parallel algorithm with inequality and equality constraints.

Conflict of Interests

The authors have declared that there is no conflict of interests.

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