Research Article

Nontrivial Solutions for Dirichlet Boundary Value Systems with the $(p_1, \dots, p_n)$ -Laplacian

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Using critical point theory due to Bonanno (2012), we prove the existence of at least one nontrivial solution for Dirichlet boundary value systems with the $(p_1, \dots, p_n)$ -Laplacian.

1. Introduction

In this note, we prove the existence of at least one nontrivial solution for Dirichlet boundary value systems with the $(p_1, \dots, p_n)$ -Laplacian as follows:

$$\begin{aligned} \Delta_{p_1} u_1 + \lambda F_{u_1}(x, u_1, \dots, u_n) &= 0, \quad \text{in } \Omega, \\ &\vdots \\ \Delta_{p_n} u_n + \lambda F_{u_n}(x, u_1, \dots, u_n) &= 0, \quad \text{in } \Omega, \\ u_1 = \dots = u_n &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

for $1 \leq i \leq n$, where $n \geq 1$ is an integer, where $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2} \nabla u_i)$ denotes the p_i -Laplacian differential operator and $p_i > 1$ for $i = 1, \dots, n$, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$), is a nonempty bounded open set with smooth boundary $\partial\Omega$, $\lambda > 0$, $F: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $F(\cdot, t_1, \dots, t_n)$ is measurable in Ω for all $(t_1, \dots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$ and for every $\varrho > 0$,

$$\sup_{|(t_1, \dots, t_n)| \leq \varrho} \sum_{i=1}^n |F_{t_i}(x, t_1, \dots, t_n)| \in L^1(\Omega), \quad (2)$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \leq i \leq n$.

Due to importance of second-order Dirichlet and Neumann problems in describing a large class of physic phenomena, many authors have studied the existence and multiplicity of solutions for such a problem; we refer the reader to [1–13] and references therein. Some authors also study the system case; see [14–20].

In [6], the authors, employing a three-critical-point theorem due to Bonanno [3], determined an exact open interval of the parameter λ for which system (1) in the case $n = 1$ admits non nontrivial weak solution.

The aim of this paper is to prove the existence of at least one nontrivial weak solution for (1) for appropriate values of the parameter λ belonging to a precise real interval, which extend the results in [7]. For basic notation and definitions and also for a thorough account on the subject, we refer the reader to [21].

2. Preliminaries and Basic Notation

First, we recall for the reader's convenience [22, Theorem 2.5] as given in [23, Theorem 2.1] (see also [3, Proposition 2.1]) which is our main tool to transfer the question of existence of at least one weak solution of (1) to the existence of a critical point of the Euler functional as follows.

For a given nonempty set X and two functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$, we define the following two functions as follows:

$$\begin{aligned}\beta(r_1, r_2) &= \inf_{v \in \Phi^{-1}([r_1, r_2])} \frac{\sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \\ \rho(r_1, r_2) &= \sup_{v \in \Phi^{-1}([r_1, r_2])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1},\end{aligned}\quad (3)$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 1 ([3, Theorem 5.1]). *Let X be a reflexive real Banach space and let $\Phi : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and let $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2). \quad (4)$$

Then, for each $\lambda \in]1/\rho(r_1, r_2), 1/\beta(r_1, r_2)[$, there is $u_{0,\lambda} \in \Phi^{-1}([r_1, r_2])$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u) \forall u \in \Phi^{-1}([r_1, r_2])$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let us introduce notation that will be used later. Let $Y := W_0^{1,p_i}(\Omega)$ be the Sobolev space with the usual norm given by $\|u_i\| = (\int_\Omega |\nabla u_i|^{p_i} dx)^{1/p_i}$, where $i = 1, \dots, n$. Let $X := W_0^{1,p_1}(\Omega) \times \dots \times W_0^{1,p_n}(\Omega)$ with the norm

$$\|u\| = \sum_{i=1}^n \left(\int_\Omega |\nabla u_i|^{p_i} dx \right)^{1/p_i} = \sum_{i=1}^n \|u_i\|, \quad (5)$$

where $u = (u_1, u_2, \dots, u_n) \in X$. Let

$$c := \max \left\{ \sup_{u_i \in W_0^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u_i|^{p_i}}{\|u_i\|^{p_i}}, 1 \leq i \leq n \right\}. \quad (6)$$

Since $p_i > N$ for $1 \leq i \leq n$, one has $c < +\infty$. In addition, it is known [24, formula (6b)] that

$$\begin{aligned}\sup_{u_i \in W_0^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u_i|^{p_i}}{\|u_i\|^{p_i}} &\leq \frac{N^{-1/p_i}}{\sqrt{\pi}} \left[\Gamma \left(1 + \frac{N}{2} \right) \right]^{1/N} \left(\frac{p_i - 1}{p_i - N} \right)^{1-1/p_i} \\ &\times [m(\Omega)]^{1/N-1/p_i}\end{aligned}\quad (7)$$

for $1 \leq i \leq n$, where Γ denotes the Gamma function and $m(\Omega)$ is the Lebesgue measure of the set Ω and equality occurs when Ω is a ball.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $0 < r_1 < r_2$ such that $S(x^0, r_1) \subset S(x^0, r_2) \subset \Omega$, where $S(x, r)$ stands for the open ball in $\mathbb{R}^N$ of radius r centered at x . Set

$$\begin{aligned}k_i &:= k_i(N, p_i, r_1, r_2) \\ &= \frac{1}{r_2 - r_1} \left((r_2^N - r_1^N) \frac{c\pi^{N/2}}{\Gamma(1 + N/2)} \right)^{1/p_i}, \quad 1 \leq i \leq n.\end{aligned}\quad (8)$$

We say that $u = (u_1, \dots, u_n)$ is a weak solution to (1) if $u = (u_1, \dots, u_n) \in X$ and

$$\int_\Omega \sum_{i=1}^n (|\nabla u_i|^{p_i-2} \nabla u_i \nabla v_i) dx = \lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1, \dots, u_n) v_i dx \quad (9)$$

for every $v = (v_1, \dots, v_n) \in X$. For $\gamma > 0$, we denote the set

$$K(\gamma) = \left\{ (t_1, \dots, t_n) \mid \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \frac{\gamma}{\prod_{i=1}^n p_i} \right\}. \quad (10)$$

Now, we prove the following lemma to get the main result.

Lemma 2. *Let $T : X \rightarrow X^*$ be the operator defined by*

$$\begin{aligned}T(u_1, \dots, u_n)(h_1, \dots, h_n) &= \int_\Omega \sum_{i=1}^n |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \cdot \nabla h_i(x) dx\end{aligned}\quad (11)$$

for every $(u_1, \dots, u_n), (h_1, \dots, h_n) \in X$. Then, T admits a continuous inverse on X^ .*

Proof. Since

$$\begin{aligned}T(u_1, \dots, u_n)(u_1, \dots, u_n) &= \int_\Omega \sum_{i=1}^n |\nabla u_i(x)|^{p_i} dx \\ &= \sum_{i=1}^n \|u_i\|_{p_i}^{p_i},\end{aligned}\quad (12)$$

T is coercive. For any $p > 1$, by (2.2) in [25], there exists $C_p > 0$ such that

$$\langle |x|^p x - |y|^p y, x - y \rangle \geq \begin{cases} C_p |x - y|^p, & \text{if } p \geq 2, \\ \frac{C_p |x - y|^2}{(|x| + |y|)^{2-p}}, & \text{if } 1 < p < 2 \end{cases} \quad (13)$$

for any $x, y \in \mathbb{R}^n$. Now, for any $u = (u_1, \dots, u_n) \in X$ and $v = (v_1, \dots, v_n) \in X$, we have

$$\begin{aligned}\langle T(u) - T(v), u - v \rangle &= \sum_{i=1}^n \int_\Omega (|\nabla u_i(x)|^{p_i-2} \nabla u_i(x) - |\nabla v_i(x)|^{p_i-2} \nabla v_i(x)) \\ &\quad \times (\nabla u_i(x) - \nabla v_i(x)) dx.\end{aligned}\quad (14)$$

Then, from (13), we see that, for $u, v \in X$ with $u \neq v$,

$$\begin{aligned}\langle T(u) - T(v), u - v \rangle &= \sum_{i \in I_1} C_{p_i} \int_\Omega |\nabla u_i(x) - \nabla v_i(x)|^{p_i} dx \\ &\quad + \sum_{i \in I_2} C_{p_i} \int_\Omega \frac{C_p |\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}} dx,\end{aligned}\quad (15)$$

where $I_1 = \{i \in \{1, \dots, n\} : p_i \geq 2\}$ and $I_2 = \{i \in \{1, \dots, n\} : 1 < p_i < 2\}$. Thus, T is strictly monotone. Clearly, T is also demicontinuous; so, by [26, Theorem 26.A(d)], the inverse operator T^{-1} of T exists.

Now, we show that T satisfies property as follows:

$$(S_2) \text{ if } (u_{1_n}, \dots, u_{n_n}) \rightharpoonup (u_1, \dots, u_n) \text{ and } T(u_{1_n}, \dots, u_{n_n}) \rightarrow T(u_1, \dots, u_n), \text{ then } (u_{1_n}, \dots, u_{n_n}) \rightarrow (u_1, \dots, u_n).$$

Let us take a sequence $(u_{1_n}, \dots, u_{n_n}) \in X$ such that $(u_{1_n}, \dots, u_{n_n}) \rightharpoonup (u_1, \dots, u_n)$ in X and

$$T(u_{1_n}, \dots, u_{n_n}) \rightarrow T(u_1, \dots, u_n) \text{ in } X^*. \quad (16)$$

Then,

$$\langle T(u_{1_n}, \dots, u_{n_n}), (u_{1_n}, \dots, u_{n_n}) \rangle \rightarrow \langle T(u_1, \dots, u_n), (u_1, \dots, u_n) \rangle. \quad (17)$$

So,

$$\|u_{1_n}\|_{p_1}^{p_1} + \dots + \|u_{n_n}\|_{p_n}^{p_n} \rightarrow \|u_1\|_{p_1}^{p_1} + \dots + \|u_n\|_{p_n}^{p_n}. \quad (18)$$

According to the uniform convexity of X , $(u_{1_n}, \dots, u_{n_n}) \rightarrow (u_1, \dots, u_n)$ in X .

It suffices then to show the continuity of T^{-1} . Let $((f_{1_n})_n, \dots, (f_{n_n})_n)$ be a sequence of X^* such that $f_{i_n} \rightarrow f_i$ in $(W_0^{1,p_i}(\Omega))^*$. Let $(u_{1_n}, \dots, u_{n_n})$ and $(u_1, \dots, u_n)$ in X be such that

$$T_{p_i}^{-1}(f_{i_n}) = u_{i_n}, \quad T_{p_i}^{-1}(f_i) = u_i. \quad (19)$$

By the coercivity of T , one deduces that the sequence $(u_{1_n}, \dots, u_{n_n})$ is bounded in the reflexive space X . For a subsequence, we have $(u_{1_n}, \dots, u_{n_n}) \rightharpoonup (\widehat{u}_1, \dots, \widehat{u}_n)$ in X , which implies

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \langle T(u_{1_n}, \dots, u_{n_n}) - T(u_1, \dots, u_n), (u_{1_n}, \dots, u_{n_n}) \\ & \quad - (\widehat{u}_1, \dots, \widehat{u}_n) \rangle \\ &= \lim_{n \rightarrow +\infty} \langle (f_{1_n}, \dots, f_{n_n}) - (f_1, \dots, f_n), (u_{1_n}, \dots, u_{n_n}) \\ & \quad - (\widehat{u}_1, \dots, \widehat{u}_n) \rangle = 0. \end{aligned} \quad (20)$$

It follows from the property of (S_2) and the continuity of T that $(u_{1_n}, \dots, u_{n_n}) \rightarrow (\widehat{u}_1, \dots, \widehat{u}_n)$ in X and $T(u_{1_n}, \dots, u_{n_n}) \rightarrow T(\widehat{u}_1, \dots, \widehat{u}_n) = T(u_1, \dots, u_n)$ in X^* .

Moreover, since T is an injection, we conclude that $(u_{1_n}, \dots, u_{n_n}) = (\widehat{u}_1, \dots, \widehat{u}_n)$. $\square$

3. Results

For a given nonnegative constant θ and a positive constant τ with $\sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) \neq \theta / \prod_{i=1}^n p_i$, put

$$\begin{aligned} a_\tau(\theta) := & \left(\int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta)} F(x, t_1, \dots, t_n) dx \right. \\ & \left. - \int_{S(x^0, r_1)} F(x, \tau, \dots, \tau) dx \right) \\ & \times \left(\frac{\theta}{\prod_{i=1}^n p_i} - \sum_{i=1}^n \frac{(\tau k_i)^{p_i}}{p_i} \right)^{-1}, \end{aligned} \quad (21)$$

where $K(\theta) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n (|t_i|^{p_i} / p_i) \leq \theta / \prod_{i=1}^n p_i\}$ (see (10)).

We formulate our main result as follows.

Theorem 3. Assume that there exist a nonnegative constant θ_1 and two positive constants θ_2 and τ with $\sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) > \theta_1 / \prod_{i=1}^n p_i$ and $\sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) < \theta_2 / \prod_{i=1}^n p_i$ such that

$$(A1) \ F(x, t_1, \dots, t_n) \geq 0 \text{ for each } (x, t_1, \dots, t_n) \in (\overline{\Omega} \setminus S(x^0, r_1)) \times [0, \tau] \times \dots \times [0, \tau];$$

$$(A2) \ a_\tau(\theta_2) < a_\tau(\theta_1).$$

Then, for each $\lambda \in]1/a_\tau(\theta_1), 1/a_\tau(\theta_2)[$, system (1) admits at least one nontrivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that

$$\frac{\theta_1}{c \prod_{i=1}^n p_i} < \sum_{i=1}^n \frac{\|u_{0i}\|^{p_i}}{p_i} < \frac{\theta_2}{c \prod_{i=1}^n p_i}. \quad (22)$$

Proof. To apply Theorem 1 to our problem, arguing as in [7], we introduce the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$ as follows:

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|^{p_i}}{p_i}, \quad (23)$$

$$\Psi(u) = \int_{\Omega} F(x, u_1(x), \dots, u_n(x)) dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\langle \Phi'(u), v \rangle = \int_{\Omega} \sum_{i=1}^n |\nabla u_i|^{p_i-2} \nabla u_i \nabla v_i dx, \quad (24)$$

$$\Psi'(u)(v) = \int_{\Omega} \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \dots, v_n) \in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* , and Ψ is sequentially weakly upper semicontinuous (see Lemma 2). Furthermore, Ψ' :

$X \rightarrow X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X . For this, for fixed $(u_1, \dots, u_n) \in X$, let $(u_{1m}, \dots, u_{nm}) \rightarrow (u_1, \dots, u_n)$ weakly in X as $m \rightarrow +\infty$; then, we have that $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on Ω as $m \rightarrow +\infty$ (see [26]). Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$; so, for $1 \leq i \leq n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \rightarrow F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \rightarrow +\infty$ which follows $\Psi'(u_{1m}, \dots, u_{nm}) \rightarrow \Psi'(u_1, \dots, u_n)$ strongly as $m \rightarrow +\infty$. Thus, we proved that Ψ' is strongly continuous on X , which implies that Ψ' is a compact operator by Proposition 26.2 of [26]. Set $w(x) = (w_1(x), \dots, w_n(x))$ such that, for $1 \leq i \leq n$,

$$w_i(x) = \begin{cases} 0, & x \in \bar{\Omega} \setminus S(x^0, r_2), \\ \frac{\tau}{r_2 - r_1} \left(r_2 - \sqrt{\sum_{j=1}^N (x_j - x_j^0)^2} \right), & x \in S(x^0, r_2) \setminus S(x^0, r_1), \\ \tau, & x \in S(x^0, r_1), \end{cases} \quad (25)$$

$r_1 = \theta_1/c \prod_{i=1}^n p_i$, and $r_2 = \theta_2/c \prod_{i=1}^n p_i$. It is easy to see that $w(x) = (w_1(x), \dots, w_n(x)) \in X$ and, in particular, one has, for $1 \leq i \leq n$,

$$\|w_i\|^{p_i} = (r_2^N - r_1^N) \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \left(\frac{\tau}{r_2 - r_1} \right)^{p_i} = \frac{(k_i \tau)^{p_i}}{c}, \quad (26)$$

so, from the definition of Φ , we have

$$\Phi(w) = \sum_{i=1}^n \frac{(k_i \tau)^{p_i}}{c}. \quad (27)$$

From the conditions $\sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) > \theta_1/\prod_{i=1}^n p_i$ and $\sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) < \theta_2/\prod_{i=1}^n p_i$, we obtain

$$r_1 < \Phi(w) < r_2. \quad (28)$$

Moreover, from (6), one has

$$\sup_{x \in \Omega} \sum_{i=1}^n |u_i(x)|^{p_i} \leq c \sum_{i=1}^n \|u_i\|^{p_i} \quad (29)$$

for each $u = (u_1, \dots, u_n) \in X$; so, from the definition of Φ , we observe that

$$\begin{aligned} \Phi^{-1}([-\infty, r_2]) &= \{(u_1, \dots, u_n) \in X; \Phi(u_1, \dots, u_n) < r_2\} \\ &= \left\{ (u_1, \dots, u_n) \in X; \sum_{i=1}^n \frac{\|u_i\|^{p_i}}{p_i} < r_2 \right\} \\ &\subseteq \left\{ (u_1, \dots, u_n) \in X; \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} < \frac{\theta_2}{\prod_{i=1}^n p_i} \quad \forall x \in \Omega \right\}, \end{aligned} \quad (30)$$

from which it follows

$$\begin{aligned} &\sup_{(u_1, \dots, u_n) \in \Phi^{-1}([-\infty, r_2])} \Psi(u) \\ &= \sup_{(u_1, \dots, u_n) \in \Phi^{-1}([-\infty, r_2])} \int_{\Omega} F(x, u_1(x), \dots, u_n(x)) dx \quad (31) \\ &\leq \int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta_2)} F(x, t_1, \dots, t_n) dx. \end{aligned}$$

Since, for $1 \leq i \leq n$, $0 \leq w_i(x) \leq \tau$ for each $x \in \Omega$, condition (A1) ensures that

$$\begin{aligned} &\int_{\bar{\Omega} \setminus S(x^0, r_2)} F(x, w_1, \dots, w_n) dx \\ &+ \int_{S(x^0, r_2) \setminus S(x^0, r_1)} F(x, w_1, \dots, w_n) dx \geq 0. \end{aligned} \quad (32)$$

So, one has

$$\begin{aligned} \beta(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}([-\infty, r_2])} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\leq \frac{\int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta_2)} F(x, t_1, \dots, t_n) dx - \Psi(w)}{r_2 - \Phi(w)} \\ &\leq a_{\tau}(\theta_2). \end{aligned} \quad (33)$$

On the other hand, by similar reasoning as before, one has

$$\begin{aligned} \rho(r_1, r_2) &\geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u)}{\Phi(w) - r_1} \\ &\geq \frac{\Psi(w) - \int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta_1)} F(x, t_1, \dots, t_n) dx}{\Phi(w) - r_1} \\ &\geq a_{\tau}(\theta_1). \end{aligned} \quad (34)$$

Hence, from Assumption (A2), one has $\beta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, from Theorem 1, taking into account that the weak solutions of the system (1) are exactly the solutions of the equation $\Phi'(u) - \lambda \Psi'(u) = 0$, we have the conclusion. $\square$

Now, we point out the following consequence of Theorem 3.

Theorem 4. Suppose that there exist two positive constants θ and τ with $\sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) < \theta/\prod_{i=1}^n p_i$ such that

$$(A3) \quad m(\Omega) \sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) \sup_{(x, t_1, \dots, t_n) \in \bar{\Omega} \times K(\theta)} F(x, t_1, \dots, t_n) < (\theta/\prod_{i=1}^n p_i) \int_{S(x^0, r_1)} F(x, \tau, \dots, \tau) dx;$$

$$(A4) \quad F(x, 0, \dots, 0) = 0 \text{ for every } x \in \Omega.$$

Then, for each

$$\lambda \in \left[\frac{\sum_{i=1}^n ((k_i \tau)^{p_i} / p_i)}{m(\Omega) \sup_{(x, t_1, \dots, t_n) \in \bar{\Omega} \times K(\theta)} F(x, t_1, \dots, t_n)}, \frac{\theta / \prod_{i=1}^n p_i}{\int_{S(x^0, r_1)} F(x, \tau, \dots, \tau) dx} \right], \quad (35)$$

system (1) admits at least one nontrivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\sum_{i=1}^n (\|u_{0i}\|^{p_i} / p_i) < \theta / c \prod_{i=1}^n p_i$.

Proof. The conclusion follows from Theorem 3, by taking $\theta_1 = 0$ and $\theta_2 = \theta$. Indeed, owing to our assumptions, one has

$$\begin{aligned} a_\tau(\theta_2) &< \left(1 - \frac{\prod_{i=1}^n p_i \sum_{i=1}^n ((k_i \tau)^{p_i} / p_i)}{\theta} \right) \\ &\quad \times \int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta)} F(x, t_1, \dots, t_n) dx \\ &\quad \times \left(\frac{\theta}{\prod_{i=1}^n p_i} - \sum_{i=1}^n \frac{(\tau k_i)^{p_i}}{p_i} \right)^{-1} \\ &= \left(1 - \frac{\prod_{i=1}^n p_i \sum_{i=1}^n ((k_i \tau)^{p_i} / p_i)}{\theta} \right) \\ &\quad \times \int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta)} F(x, t_1, \dots, t_n) dx \\ &\quad \times \left(\frac{\theta}{\prod_{i=1}^n p_i} - \sum_{i=1}^n \frac{(\tau k_i)^{p_i}}{p_i} \right)^{-1} \\ &= \int_{\Omega} \sup_{(t_1, \dots, t_n) \in K(\theta)} F(x, t_1, \dots, t_n) dx \\ &\quad \times \left(\frac{\theta}{\prod_{i=1}^n p_i} \right)^{-1}. \end{aligned} \quad (36)$$

On the other hand, taking Assumption (A4) into account, one has

$$\frac{\int_{S(x^0, r_1)} F(x, \tau, \dots, \tau) dx}{\sum_{i=1}^n ((k_i \tau)^{p_i} / c)} = a_\tau(\theta_1). \quad (37)$$

Moreover, since

$$\sup_{x \in \Omega} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq c \sum_{i=1}^n \frac{\|u_i\|^{p_i}}{p_i} \quad (38)$$

for each $u = (u_1, \dots, u_n) \in X$, an easy computation ensures that $\sum_{i=1}^n (\|u_i\|_\infty^{p_i} < \theta)$ whenever $\Phi(u) < r_2$. Now, owing to Assumption (A3), it is sufficient to invoke Theorem 3 for concluding the proof. $\square$

Let f be a continuous function in $\bar{\Omega}$ and let g_i , for $1 \leq i \leq n$, be a function in C^1 and

$$F(x, u_1(x), \dots, u_n(x)) = f(x) \prod_{i=1}^n g_i(u_i(x)) \quad (39)$$

for each $(x, u_1, \dots, u_n) \in \bar{\Omega} \times \mathbb{R}^n$.

Theorem 5. Assume that Assumption (A1) in Theorem 3 holds. Suppose that there exist two positive constants θ and τ with $\sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) < \theta / \prod_{i=1}^n p_i$ such that

- (B1) $f(x) \geq 0$ for each $x \in (\bar{\Omega} \setminus S(x^0, r_1))$ and $g_i(t_i) \geq 0$ for $t_i \in [0, \tau]$, $1 \leq i \leq n$;
- (B2) $m(\Omega) \sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) \max_{(x, t_1, \dots, t_n) \in \bar{\Omega} \times K} f(x) \prod_{i=1}^n g_i(t_i(x)) < \theta \prod_{i=1}^n (g_i(\tau) / p_i) \int_{S(x^0, r_1)} f(x) dx$;
- (B3) $g_i(0) = 0$ for every $x \in \Omega$ for $1 \leq i \leq n$.

Then, for each

$$\lambda \in \left[\frac{\sum_{i=1}^n ((k_i \tau)^{p_i} / p_i)}{m(\Omega) \max_{(x, t_1, \dots, t_n) \in \bar{\Omega} \times K} f(x) \prod_{i=1}^n g_i(t_i(x))}, \frac{\theta / \prod_{i=1}^n p_i}{\prod_{i=1}^n g_i(\tau) \int_{S(x^0, r_1)} f(x) dx} \right], \quad (40)$$

system

$$\begin{aligned} \Delta_{p_1} u_1 + \lambda f(x) g_1'(u_1) \prod_{i=1, i \neq 1}^n g_i(u_i) &= 0, \quad \text{in } \Omega, \\ &\vdots \\ \Delta_{p_n} u_n + \lambda f(x) g_n'(u_n) \prod_{i=1, i \neq n}^n g_i(u_i) &= 0, \quad \text{in } \Omega, \\ u_1 = \dots = u_n &= 0, \quad \text{on } \partial\Omega \end{aligned} \quad (41)$$

admits at least one nontrivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\sum_{i=1}^n (\|u_{0i}\|^{p_i} / p_i) < \theta / c \prod_{i=1}^n p_i$.

The following result gives the existence of at least one nontrivial weak solution in X to the problem (1) in the autonomous case. For a given nonnegative constant θ and a positive constant τ with $\sum_{i=1}^n ((\tau k_i)^{p_i} / p_i) \neq \theta / \prod_{i=1}^n p_i$, put

$$\begin{aligned} b_\tau(\theta) &:= \left(m(\Omega) \sup_{(t_1, \dots, t_n) \in K(\theta)} H(t_1, \dots, t_n) \right. \\ &\quad \left. - \frac{r_1^N \pi^{N/2}}{\Gamma(1 + N/2) H(\tau, \dots, \tau)} \right) \\ &\quad \times \left(\frac{\theta}{\prod_{i=1}^n p_i} - \sum_{i=1}^n \frac{(\tau k_i)^{p_i}}{p_i} \right)^{-1}, \end{aligned} \quad (42)$$

where $K(\theta) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n (|t_i|^{p_i}/p_i) \leq \theta / \prod_{i=1}^n p_i\}$ (see (10)). We have the following result as a direct consequence of Theorem 5.

Theorem 6. Let $H : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function and assume that there exist a nonnegative constant θ_1 and two positive constants θ_2 and τ with $\sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) > \theta_1 / \prod_{i=1}^n p_i$ and $\sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) < \theta_2 / \prod_{i=1}^n p_i$ such that

$$(C1) \ H(t_1, \dots, t_n) \geq 0 \text{ for each } (t_1, \dots, t_n) \in [0, \tau] \times \dots \times [0, \tau];$$

$$(C2) \ m(\Omega) \sum_{i=1}^n ((\tau k_i)^{p_i}/p_i) \max_{(t_1, \dots, t_n) \in K} H(t_1, \dots, t_n) < (\theta_1^N \pi^{N/2} / \Gamma(1 + N/2) \prod_{i=1}^n p_i) H(\tau, \dots, \tau).$$

Then, for each $\lambda \in]1/b_\tau(\theta_1), 1/b_\tau(\theta_2)[$, the problem

$$\begin{aligned} \Delta_{p_1} u_1 + \lambda H_{u_1}(u_1, \dots, u_n) &= 0, \quad \text{in } \Omega, \\ &\vdots \\ \Delta_{p_n} u_n + \lambda H_{u_n}(u_1, \dots, u_n) &= 0, \quad \text{in } \Omega, \\ u_1 &= \dots = u_n = 0, \quad \text{on } \partial\Omega \end{aligned} \quad (43)$$

admits at least one nontrivial weak solution $u_0 \in X$ such that $\theta_1/c \prod_{i=1}^n p_i < \sum_{i=1}^n (\|u_{0i}\|^{p_i}/p_i) < \theta_2/c \prod_{i=1}^n p_i$.

Proof. Note that $\int_{S(x^0, r_1)} H(\tau, \dots, \tau) dx = (r_1^N \pi^{N/2} / \Gamma(1 + N/2)) H(\tau, \dots, \tau)$, which together with (B2) implies

$$\begin{aligned} m(\Omega) \sum_{i=1}^n \frac{(\tau k_i)^{p_i}}{p_i} \max_{(t_1, \dots, t_n) \in K} H(t_1, \dots, t_n) \\ < \frac{\theta}{\prod_{i=1}^n p_i} \int_{S(x^0, r_1)} H(\tau, \dots, \tau) dx. \end{aligned} \quad (44)$$

Therefore, we have the conclusion by Theorem 5. $\square$

As an example, we point out the following special case of our main result.

Theorem 7. Let $h : \Omega \rightarrow \mathbb{R}$ be a positive and essentially bounded function and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative continuous function such that $\lim_{t \rightarrow 0^+} (g(t)/t^{p-1}) = +\infty$. Then, for each $\lambda \in]0, (1/pc \int_{\Omega} h(x) dx) \sup_{\theta > 0} (\theta^p / \int_0^\theta g(\xi) d\xi)[$, the problem

$$\begin{aligned} \Delta_p u + \lambda h(x) g(u) &= 0, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega \end{aligned} \quad (45)$$

admits at least one nontrivial weak solution in Y .

Proof. For fixed λ as in the conclusion, there exists positive constant θ such that

$$\lambda < \frac{1}{pc \int_{\Omega} h(x) dx} \sup_{\theta > 0} \frac{\theta^p}{\int_0^\theta g(\xi) d\xi}. \quad (46)$$

Moreover, $\lim_{t \rightarrow 0^+} (g(t)/t^{p-1}) = +\infty$ implies $\lim_{t \rightarrow 0^+} (\int_0^t g(\xi) d\xi / t^p) = +\infty$. Therefore, a positive constant τ satisfying $(k\tau)^p < \theta$ can be chosen such that

$$\frac{k^p}{\lambda pc \int_{B(x^0, r_1)} h(x) dx} < \frac{\int_0^\tau g(\xi) d\xi}{\tau^p}. \quad (47)$$

Hence, arguing as in the proof of Theorem 4, the conclusion follows from Theorem 6 with $\theta_1 = 0$, $\theta_2 = \theta$, and $f(t) = g(t)$ for every $t \in \mathbb{R}$. $\square$

We remark that if $h(x) = 1$ for every $x \in \Omega$, then Theorem 7 gives [7, Theorem 1.1].

Remark 8. For fixed ρ , put $\lambda_\rho := (1/pc \int_{\Omega} h(x) dx) \sup_{\theta > 0} (\theta^p / \int_0^\theta g(\xi) d\xi)$. The result of Theorem 7 for every $\lambda \in]0, \lambda_\rho[$ holds with $|u_0(x)| < \rho$ for all $x \in \Omega$, where u_0 is the ensured nontrivial weak solution in Y (see [7, Corollary 3.1]).

We close this paper by presenting the following example to illustrate our results.

Example 9. Consider the problem

$$\begin{aligned} -\Delta_3 u &= \lambda \left(1 + e^{-u^+} (u^+)^2 (3 - u^+) \right), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (48)$$

where $u^+ = \max\{u, 0\}$, $N = 2$, and $\Omega = \{(x, y) : x^2 + y^2 \leq 4\}$. Let

$$g(t) = 1 + e^{-t^+} (t^+)^2 (3 - t^+) \quad (49)$$

for all $t \in \mathbb{R}$, where $t^+ = \max\{t, 0\}$. It is clear that $\lim_{t \rightarrow 0^+} (g(t)/t) = +\infty$. Note that $p = 3$ and $N = 2$; we have $c = \sqrt[3]{2/\pi}$, and so $k = \sqrt[6]{54\pi^2}$. Hence, taking Remark 8 into account, since $\sup_{\theta \in]0, 1[} (\theta^2 / \int_0^\theta g(\xi) d\xi) = \sup_{\theta \in]0, 1[} (\theta^2 / (\theta + e^{-\theta} \theta^3)) = e/(1 + e)$, by applying Theorem 7, for every $\lambda \in]0, (\sqrt[3]{\pi}/3 \sqrt[3]{2})(e/(1 + e))]$, the problem (48) has at least one nontrivial classical solution $u_0 \in Y$ such that $\|u_0\|_\infty < 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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