Research Article

Remarks on Homogeneous Al-Salam and Carlitz Polynomials

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Several multilinear generating functions of the homogeneous Al-Salam and Carlitz polynomials are derived from q -operator. In addition, two interesting relationships of product of this kind of polynomials are obtained.

1. Introduction

The Al-Salam and Carlitz polynomials

$$\phi_n^{(a)}(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k \quad (1)$$

have been studied by many researchers for a long time. The history of these polynomials may go back to Al-Salam and Carlitz in 1965. Since then, these polynomials have been studied by many mathematicians [1–14].

Recently, Cao [7] used Carlitz's q -operators to study the following homogeneous Al-Salam and Carlitz polynomials:

$$\phi_n^{(a)}(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k y^{n-k}, \quad y \neq 0 \quad (2)$$

and he gave some linear generating functions of them. In this paper, we will research these polynomials by some construction of q -operator. With this method, some new multilinear generating functions can be easily derived. Firstly, we give the following three results which originated from the results about $\phi_n^{(a)}(x)$ which appeared in [1, 2, 5–7, 9, 10].

Theorem 1. If $m, n \in \mathbb{N}$, $\max\{|xXz|, |xYz|, |yXz|, |yYz|\} < 1$, then

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_{m+n}^{(a)}(x, y) \phi_n^{(b)}(X, Y) \frac{z^n}{(q; q)_n} \\ &= \frac{y^m (q; q)_m (byXz, axXz; q)_{\infty}}{(yYz, yXz, xXz; q)_{\infty}} \\ & \times \sum_{k=0}^{\infty} \frac{(x/y)^k (a, yYz, yXz; q)_k}{(q, byXz, axXz; q)_k (q; q)_{m-k}} \\ & \times {}_3\Phi_2 \left(\begin{matrix} aq^k, & yXzq^k, & \frac{bX}{Y} \\ & byXzq^k, & axXzq^k \end{matrix}; q, xYz \right). \end{aligned} \quad (3)$$

Theorem 2. If $m, n \in \mathbb{N}$, $\max\{|xt|, |yt|, |Xz|, |Yz|, |yY|, |yXY|, |xY|, |xY| \} < 1$, then

$$\begin{aligned} & \sum_{m,n,k=0}^{\infty} \phi_{m+k}^{(a)}(x, y) \phi_{n+k}^{(b)}(X, Y) \frac{t^m z^n y^k}{(q; q)_m (q; q)_n (q; q)_k} \\ &= \frac{(bXz, axY; q)_{\infty}}{(Yz, Xz, yt, yY, xY; q)_{\infty}} \times \sum_{m=0}^{\infty} \frac{(yXY)^m (b, Yz; q)_m}{(bXz; q)_m} \end{aligned}$$

$$\begin{aligned} & \times \sum_{k=0}^{\infty} \frac{(x/y)^k (a, yt, yYq; q)_k}{(q, axYq; q)_k (q; q)_{m-k}} \\ & \times {}_2\Phi_1 \left(\begin{matrix} aq^k, & yYq^k \\ & axYq^k \end{matrix}; q, xt \right). \end{aligned} \quad (4)$$

Theorem 3 (cf. [7, Equation (1.9)]). If $k, m, n \in \mathbb{N}$, $\max\{|xYz|, |xXz|, |Xyz|, |Yyz|\} < 1$, then

$$\begin{aligned} & \sum_{k=0}^{\infty} \phi_{n+k}^{(a)}(x, y) \phi_{m+k}^{(b)}(X, Y) \frac{z^k}{(q; q)_k} \\ & = \frac{y^n Y^m (q; q)_n (q; q)_m (ybXz, xaXz; q)_{\infty}}{(yYz, yXz, xXz; q)_{\infty}} \\ & \times \sum_{k=0}^{\infty} \frac{(X/Y)^k (b, yYz; q)_k}{(q, ybXz; q)_k (q; q)_{m-k}} \\ & \times \sum_{j=0}^{\infty} \frac{(x/y)^j (a, yYzq^k, yXz; q)_j}{(q, ybXzq^k, xaXz; q)_j (q; q)_{n-j}} \\ & \times {}_3\Phi_2 \left(\begin{matrix} aq^j, & yXzq^j, & \frac{bX}{Y} \\ & byXzq^{k+j}, & axXzq^j \end{matrix}; q, xYzq^k \right). \end{aligned} \quad (5)$$

Now further using our method, we can deduce more results of multilinear generating functions.

Theorem 4. If $G \in \mathbb{Z}^+$, $a_1 = q^{-G}$, $|a_1 x_1 / y_1| < 1$, then

$$\begin{aligned} & \sum_{m_1, m_2, m_3=0}^{\infty} \phi_{m_1+m_2+m_3}^{(a_1)}(x_1, y_1) \phi_{m_1+m_3}^{(a_2)}(x_2, y_2) \\ & \times \phi_{m_1+m_2}^{(a_3)}(x_3, y_3) \frac{t_1^{m_1} t_2^{m_2} t_3^{m_3}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} \\ & = \frac{(a_1 x_1 / y_1, a_3 x_3 t_2 y_1, a_2 x_2 t_1 y_1 y_3; q)_{\infty}}{(x_1 / y_1, t_2 y_1 y_3, t_3 y_1 y_2, t_1 y_1 y_2 y_3, x_3 t_2 y_1, x_2 t_1 y_1 y_3; q)_{\infty}} \\ & \times \sum_{j_1=0}^{\infty} \frac{(a_3; q)_{j_1} (x_3 t_1 y_1 y_2)^{j_1} (q; q)_{j_1}}{(q; q)_{j_1}} \\ & \times \sum_{j_2=0}^{\infty} \frac{(x_2 / y_2)^{j_2} (a_2; q)_{j_2}}{(q; q)_{j_2} (q; q)_{j_1-j_2}} \sum_{j_3=0}^{\infty} \frac{(x_2 t_3 y_1)^{j_3} (a_2 q^{j_2}; q)_{j_3}}{(q; q)_{j_3}} \end{aligned}$$

$$\begin{aligned} & \times \sum_{j_4=0}^{\infty} \frac{(a_1, x_3 t_2 y_1, x_2 t_1 y_1 y_3; q)_{j_4} (-1)^{j_4} q^{-(j_4-1)j_4/2}}{(q, x_1 / (y_1 q^{j_4}); q)_{j_4}} \\ & \times \left(\frac{x_1 q^{j_1+j_3}}{y_1} \right)^{j_4} \frac{(t_3 y_1 y_2; q)_{j_2+j_4} (t_2 y_1 y_3; q)_{j_1+j_4}}{(a_3 x_3 t_2 y_1; q)_{j_1+j_4}} \\ & \times \frac{(t_1 y_1 y_2 y_3; q)_{j_2+j_3+j_4}}{(a_2 x_2 t_1 y_1 y_3; q)_{j_2+j_3+j_4}}, \end{aligned} \quad (6)$$

provided that $\max\{|t_i x_j y_l|, |t_1 x_i y_j y_l|, |t_1 y_1 y_2 y_3|\} < 1$, where $1 \leq i \neq j \neq l \leq 3$.

Theorem 5. If $G \in \mathbb{Z}^+$, $a_1 = q^{-G}$, then

$$\begin{aligned} & \sum_{m_1, m_2, m_3=0}^{\infty} \phi_{m_1+m_2+m_3}^{(a_1)}(x_1, y_1) \phi_{m_1+m_2}^{(a_2)}(x_2, y_2) \\ & \times \phi_{m_3}^{(a_3)}(x_3, y_3) \frac{t_1^{m_1} t_2^{m_2} t_3^{m_3}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} \\ & = \frac{(a_1 x_1 / y_1, a_3 x_3 t_3 y_1, a_2 x_2 t_2 y_1; q)_{\infty}}{(x_1 / y_1, t_3 y_1 y_3, t_3 y_1 x_3, t_1 y_1 y_2, t_2 y_1 y_2, t_2 x_2 y_1; q)_{\infty}} \\ & \times \sum_{j_1, j_2=0}^{\infty} \frac{(a_2; q)_{j_1} (t_1 x_2 y_1)^{j_1} (-1)^{j_2} q^{-(j_2-1)j_2/2}}{(q; q)_{j_1}} \\ & \times \frac{(a_1, t_3 y_3 y_1, t_3 x_3 y_1, t_2 x_2 y_1, t_1 y_2 y_1; q)_{j_2}}{(q, q x_1 / (y_1 q^{j_2}), a_3 t_2 x_3 y_1; q)_{j_2}} \\ & \times \frac{(t_2 y_2 y_1; q)_{j_1+j_2}}{(a_2 t_2 x_2 y_1; q)_{j_1+j_2}} \left(\frac{q^{j_1} x_1}{y_1} \right)^{j_2}, \end{aligned} \quad (7)$$

provided that $\max\{|y_1 y_2 t_1|, |y_1 y_2 t_2|, |y_1 y_3 t_3|, |x_3 y_1 t_3|, |x_2 y_1 t_1|, |x_1 y_1 t_2|, |a_1 x_1 / y_1|\} < 1$.

Theorem 6. If $m_i \in \mathbb{N}$, $\max\{|x_1|, |y_1 t_i|, |x_1 t_i|\} < 1$, $i = 1, 2, 3$, then

$$\begin{aligned} & \sum_{m_1, m_2, m_3=0}^{\infty} \phi_{m_1+m_2+m_3}^{(a)}(x_1, y_1) \\ & \times \frac{t_1^{m_1} t_2^{m_2} (-t_3)^{m_3} q^{m_3(m_3-1)/2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} \\ & = \frac{(y_1 t_3, ax_1 t_2; q)_{\infty}}{(y_1 t_1, y_1 t_2, x_1 t_2; q)_{\infty}} \\ & \times {}_3\Phi_2 \left(\begin{matrix} a, & y_1 t_2, & \frac{t_3}{t_1} \\ & y_1 t_3, & ax_1 t_2 \end{matrix}; q, x_1 t_1 \right). \end{aligned} \quad (8)$$

Theorem 7. If $m_i \in N, n, G \in Z^+, a = q^{-G}, \max\{|yt_i|, |ax/y|\} < 1, i = 1, 2, \dots, n$, then

$$\begin{aligned} & \sum_{m_1, \dots, m_n=0}^{\infty} \phi_{m_1+\dots+m_n}^{(a)}(x, y) \frac{t_1^{m_1} \dots t_n^{m_n}}{(q; q)_{m_1} \dots (q; q)_{m_n}} \\ &= \frac{(ax/y; q)_{\infty}}{(x/y, yt_1, yt_2, \dots, yt_n; q)_{\infty}} \\ & \times \sum_{j=0}^{\infty} \frac{(a, yt_1, yt_2, \dots, yt_n; q)_j (-1)^j q^{-(j-1)j/2}}{(q, x/(yq^j); q)_j} \left(\frac{x}{y}\right)^j. \end{aligned} \quad (9)$$

Theorem 8. If $m_i \in N, t_i \neq 0, \max\{|x|, |yt_i|, |xt_i|\} < 1, i = 1, 2, 3, 4$, then

$$\begin{aligned} & \sum_{m_1, m_2, m_3, m_4=0}^{\infty} \phi_{m_1+m_2+m_3+m_4}^{(a)}(x, y) \\ & \times \frac{(-t_1)^{m_1} t_2^{m_2} t_3^{m_3} (-t_4)^{m_4} q^{m_1(m_1-1)/2} q^{m_4(m_4-1)/2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4}} \\ &= \frac{(yt_1, yt_4; q)_{\infty}}{(yt_2, yt_3; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(a, t_1/t_2, yt_3; q)_j (xt_2)^j}{(q, yt_1, yt_4; q)_j} \\ & \times {}_2\Phi_1 \left(aq^j, \frac{t_4}{t_3}; q, xt_3 \right). \end{aligned} \quad (10)$$

Theorem 9. If $m_i \in N, \max\{|y_i yt_i|, |x_i yt_i|, |a_0/x_0|\} < 1, i = 0, 1, 2, 3$, then

$$\begin{aligned} & \sum_{m, m_1, m_2, m_3=0}^{\infty} \phi_{m_1+m_2+m_3+s}^{(a)}(x, y) \phi_{m+r}^{(a)} \\ & \times (x_0, y_0) \phi_{m_1}^{(a_1)}(x_1, y_1) \phi_{m_2}^{(a_2)}(x_2, y_2) \\ & \times \phi_{m_3}^{(a_3)}(x_3, y_3) \\ & \times \frac{t^m t_1^{m_1} t_2^{m_2} t_3^{m_3}}{(q; q)_m (q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} \\ &= \left(y^s y_0^r (q; q)_r \left(\frac{ax}{y}, a_1 yx_1 t_1, a_2 yx_2 t_2, a_3 yx_3 t_3; q \right)_{\infty} \right) \\ & \times \left(\left(\frac{x}{y}, yy_1 t_1, yx_1 t_1, yy_2 t_2, yx_2 t_2, \right. \right. \\ & \left. \left. yy_3 t_3, yx_3 t_3, yy_0 t; q \right)_{\infty} \right)^{-1} \end{aligned}$$

$$\begin{aligned} & \times \sum_{j_1, j_2=0}^{\infty} \frac{(x_0/y_0)^{j_1} (yx_0 t)^{j_2} (a_0; q)_{j_1+j_2}}{(q; q)_{j_1} (q; q)_{j_2} (q; q)_{r-j_1}} \\ & \times \sum_{j_3=0}^{\infty} \left((a, yy_1 t_1, yx_1 t_1, yy_2 t_2, yx_2 t_2, yy_3 t_3, yx_3 t_3; q)_{j_3} \right. \\ & \quad \times (yy_0 t; q)_{j_1+j_3} \left. \right) \\ & \times \left(\left(\frac{qy}{x}, a_1 yx_1 t_1, a_2 yx_2 t_2, a_3 yx_3 t_3; q \right)_{j_3} \right)^{-1} \\ & \times q^{j_3(j_2+s+1)}, \end{aligned} \quad (11)$$

where $t_0 = t, m_0 = m$, and $F \in Z^+, a_0 = q^{-F}$.

Theorem 10. If $M_i \in Z^+, a_i = q^{-M_i}, x_i \neq 0$, and $y_i \neq 0$, then

$$\begin{aligned} & \sum_{m_1, m_2, m_3, m_4=0}^{\infty} \phi_{m_2+m_3+m_4}^{(a_1)}(x_1, y_1) \\ & \times \phi_{1+m_3+m_4}^{(a_2)}(x_2, y_2) \phi_{m_1+m_2+m_4}^{(a_3)} \\ & \times (x_3, y_3) \phi_{m_1+m_2+m_3}^{(a_4)}(x_4, y_4) \\ & \times \frac{(-t_1)^{m_1} t_2^{m_2} t_3^{m_3} (-t_4)^{m_4} q^{m_1(m_1-1)/2} q^{m_4(m_4-1)/2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4}} \\ &= \left(\frac{a_4 x_4}{y_4}, \frac{a_3 x_3}{y_3}, \frac{a_2 x_2}{y_2}, \frac{a_1 x_1}{y_1}, t_1 y_2 y_3 y_4, t_4 y_1 y_2 y_3; q \right)_{\infty} \\ & \times \left(\left(\frac{x_4}{y_4}, \frac{x_3}{y_3}, \frac{x_2}{y_2}, \frac{x_1}{y_1}, t_2 y_1 y_3 y_4, t_3 y_1 y_2 y_4; q \right)_{\infty} \right)^{-1} \\ & \times \sum_{k_1, k_2, k_3, k_4=0}^{\infty} \left((a_4; q)_{k_4} (a_3; q)_{k_3} (a_2; q)_{k_2} \right. \\ & \quad \times (a_1; q)_{k_1} q^{k_1+k_2+k_3+k_4} \left. \right) \\ & \times \left(\left(q, \frac{qy_4}{x_4}; q \right)_{k_4} \left(q, \frac{qy_3}{x_3}; q \right)_{k_3} \right. \\ & \quad \times \left(q, \frac{qy_2}{x_2}; q \right)_{k_2} \left(q, \frac{qy_1}{x_1}; q \right)_{k_1} \left. \right)^{-1} \\ & \times \frac{(t_1 y_2 y_3 y_4; q)_{k_2+k_3+k_4} (t_4 y_1 y_2 y_3; q)_{k_1+k_2+k_3}}{(t_2 y_1 y_3 y_4; q)_{k_1+k_3+k_4} (t_3 y_1 y_2 y_4; q)_{k_1+k_2+k_4}}, \end{aligned} \quad (12)$$

provided that $\max\{|a_i x_i/y_i|, |t_i y_j y_k y_l|\} < 1$, where $1 \leq i \neq j \neq k \neq l \leq 4$.

Polynomials (2) evidently reduce to the Rogers-Szegő polynomials (cf. [9])

$$\phi_n^{(0)}(x) = H_n(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad (13)$$

when $a = 0$ and $y = 1$. And when $y = 1$ they reduce to the common Al-Salam and Carlitz polynomials (1) (cf. [9]).

So now we take some special cases for checking.

Let $a = b = 0$, $x = Y = 1$ in (3); we have the following.

Corollary 11 (cf. [6, Theorem 1.1] or [9, Equation (4.1)]). *If $m, n \in N$, then*

$$\begin{aligned} & \sum_{n=0}^{\infty} H_{m+n}(y) H_n(X) \frac{z^n}{(q; q)_n} \\ &= \frac{y^m (yXz^2; q)_{\infty}}{(z, Xz, yz, yX; q)_{\infty}} \\ & \times {}_3\Phi_1 \left(\begin{matrix} q^{-m}, & yz, & yXz \\ & yXz^2 & \end{matrix}; q, \frac{q^m}{y} \right), \end{aligned} \quad (14)$$

provided that $\max\{|z|, |yz|, |Xz|, |yXz|\} < 1$.

Remark 12 (from [6, Equation (3.1)]). We know that Corollary 11 is equivalent to Theorem 1.1 given in [6]. And if we take $m = 0$ and $y = Y = 1$, (3) turns to [9, Equation (1.2)].

Taking $a = b = 0$, $x = Y = 1$ in (4), we have the following.

Corollary 13 (cf. [6, Theorem 1.2] or [9, Equation (1.3)]). *If $m, n \in N$, then*

$$\begin{aligned} & \sum_{m,n,k=0}^{\infty} H_{m+k}(y) H_{n+k}(X) \frac{t^m z^n \gamma^k}{(q; q)_m (q; q)_n (q; q)_k} \\ &= \frac{(yt\gamma, yXz\gamma; q)_{\infty}}{(t, z, Xz, yt, \gamma, y\gamma, yX\gamma; q)_{\infty}} \\ & \times {}_3\Phi_2 \left(\begin{matrix} z, & yt, & y\gamma \\ & yt\gamma, & yXz\gamma \end{matrix}; q, X\gamma \right), \end{aligned} \quad (15)$$

provided that $\max\{|t|, |z|, |\gamma|, |yt|, |y\gamma|, |Xz|, |X\gamma|, |yX\gamma|\} < 1$.

Remark 14. Using Hall's transformation [15, Equation (3.2.7)]

$$\begin{aligned} & {}_3\Phi_2 \left(\begin{matrix} a, & b, & c \\ d, & e \end{matrix}; q, \frac{de}{(abc)} \right) \\ &= \frac{(e/a, de/bc; q)_{\infty}}{(e, de/abc; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} a, & \frac{d}{b}, & \frac{d}{c} \\ d, & \frac{de}{(bc)} \end{matrix}; q, \frac{e}{a} \right), \end{aligned} \quad (16)$$

we find that Corollary 13 is equivalent to Theorem 1.2 given in [6]. And if we take $y = Y = 1$, then, with simplifying, (4) turns to [9, Equation (3.5)].

Taking $a = b = 0$ and $x = X = 1$ in (5), we have the following.

Corollary 15 (cf. [6, Theorem 1.3] or [9, Equation (1.4)]). *If $m, n, k \in N$, then*

$$\begin{aligned} & \sum_{k=0}^{\infty} H_{n+k}(y) H_{m+k}(Y) \frac{z^k}{(q; q)_k} \\ &= \frac{(yYz^2; q)_{\infty}}{(z, yz, Yz, yYz; q)_{\infty}} \\ & \times \sum_{k=0}^m \sum_{j=0}^n \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \frac{Y^{m-k} y^{n-j} (yYz; q)_{k+j}}{(yYz^2; q)_{k+j}} \\ & \times (Yz; q)_k (yz; q)_j, \end{aligned} \quad (17)$$

provided that $\max\{|z|, |yz|, |Yz|, |yYz|\} < 1$.

The rest of this paper is organized as follows. In Section 2, we will give some notations and lemmas. In Section 3, we give the proofs of theorems. Section 4 describes the relationship between $\Psi_n(a, b; x_1, y_1; z)$ (cf. [4, Equation (2.2)]) and $\phi_n^{(a)}(x, y)$. In addition, an interesting relationship between the polynomial multiplications is given.

2. Notations and Some Lemmas

In this paper, we apply the standard notations that follow from [15] and assume that $0 < |q| < 1$; the q -shifted factorial and its compact factorials are defined, respectively, by

$$\begin{aligned} (a; q)_0 &= 1; \\ (a; q)_n &= \prod_{k=0}^{n-1} (1 - aq^k), \\ (a; q)_{\infty} &= \prod_{k=0}^{\infty} (1 - aq^k) \end{aligned} \quad (18)$$

and $(a_1, \dots, a_m; q)_n = (a_1; q)_n \dots (a_m; q)_n$, where $n = 0, 1, \dots, \infty$. And we use N to denote the set of nonnegative integers and Z^+ to denote the set of positive integers.

For any complex number α , we have

$$(a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(aq^{\alpha}; q)_{\infty}}. \quad (19)$$

The q -binomial coefficient and the q -binomial theorem are given by

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & \sum_{m=0}^{\infty} \frac{(a; q)_m}{(q; q)_m} y^m &= \frac{(ay; q)_{\infty}}{(y; q)_{\infty}}, \end{aligned} \quad (20)$$

respectively.

The basic hypergeometric series ${}_r\Phi_s$ is defined by

$$\begin{aligned} & {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \\ & \quad \times [(-1)^n q^{n(n-1)/2}]^{1+s-r} x^n. \end{aligned} \quad (21)$$

The q -derivative operator D_q and q -shifted operator η (cf. [1, 6, 7, 9, 10, 16–23]), acting on the variable x , are defined by

$$\begin{aligned} D_q \{f(x)\} &= \frac{f(x) - f(xq)}{x}, \\ \eta \{f(x)\} &= f(xq). \end{aligned} \quad (22)$$

The q -Leibnitz rule for the product of two functions (cf. [17–25]) is given by

$$D_q^n \{f(x)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{f(x)\} D_q^{n-k} \{g(xq^k)\}. \quad (23)$$

From definition, we can easily obtain $D_q = x^{-1}(1 - \eta)$; by mathematical induction, we have the following proposition.

Proposition 16 (cf. [20, Equation (6)]). *If $f(x)$ is any analytical function, then*

$$D_q^n \{f(x)\} = x^{-n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} (q\eta)^k \{f(x)\}. \quad (24)$$

In [19, 20], we had established the following q -operator structure:

$${}_1\Phi_0 \left(\begin{matrix} b \\ - \end{matrix}; q, cD_q \right) = \sum_{n=0}^{\infty} \frac{(b; q)_n (cD_q)^n}{(q; q)_n}. \quad (25)$$

For convenient, we use $F(a; xD_q)_y$ to denote the operator ${}_1\Phi_0 \left(\begin{matrix} a \\ - \end{matrix}; q, xD_q \right)$ acting on variable y . From this definition we can easily get the following Lemma.

Lemma 17. *One has*

$$\phi_n^{(a)}(x, y) = F(a; xD_q)_y \{y^n\}. \quad (26)$$

Proof. For

$$D_q^{(k)} \{y^n\} = \frac{(q; q)_n}{(q; q)_{n-k}} y^{n-k}, \quad (27)$$

we have

$$\begin{aligned} & F(a; xD_q)_y \{y^n\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k} \frac{(q; q)_n}{(q; q)_{n-k}} y^{n-k} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k y^{n-k}, \end{aligned} \quad (28)$$

which completes the proof. $\square$

It is the reason why we employ this operator to study the properties of homogeneous Al-Salam and Carlitz polynomials.

From Proposition 16, we may easily obtain the following identity.

Lemma 18. *If $G \in \mathbb{Z}^+$, $a = q^{-G}$, $|ax/y| < 1$ and $f(x)$ is any analytical function, then*

$$\begin{aligned} & F(a; xD_q)_y \{f(y)\} \\ &= \frac{(ax/y; q)_{\infty}}{(x/y; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k (a; q)_k (x/y)^k}{(q, x/(yq^k); q)_k} \\ & \quad \times q^{-k(k-1)/2} f(yq^k). \end{aligned} \quad (29)$$

Proof. Applying Proposition 16, LHS of Lemma 18 comes to

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(a; q)_n (x/y)^n}{(q; q)_n} \\ & \times \sum_{k=0}^n \frac{(q; q)_n q^k}{(q; q)_k (q; q)_{n-k}} (-1)^k q^{k(k-1)/2-nk} f(yq^k) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (a; q)_k (x/y)^k}{(q; q)_k} q^{-k(k-1)/2} f(yq^k) \\ & \times \sum_{n=0}^{\infty} \frac{(aq^k; q)_n}{(q; q)_n} \left(\frac{x}{yq^k} \right)^n \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (a; q)_k (x/y)^k}{(q; q)_k} q^{-k(k-1)/2} f(yq^k) \\ & \times \frac{(ax/y; q)_{\infty}}{(x/(yq^k); q)_{\infty}} \\ &= \frac{(ax/y; q)_{\infty}}{(x/y; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k (a; q)_k (x/y)^k}{(q, x/(yq^k); q)_k} \\ & \quad \times q^{-k(k-1)/2} f(yq^k). \end{aligned} \quad (30)$$

We complete the proof. $\square$

Remark 19. To calculate the inner sum of the above, under the condition $a = q^{-G}$, q -binomial theorem is usable.

Lemma 20 (cf. [20, Lemma 1]). *If $|sc| < 1$, $|tc| < 1$, then*

$$\begin{aligned} & F(b; cD_q)_a \left\{ \frac{(aw; q)_{\infty}}{(as, at; q)_{\infty}} \right\} \\ &= \frac{(aw, bct; q)_{\infty}}{(as, at, ct; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} b, at, \frac{\omega}{s} \\ aw, bct \end{matrix}; q, sc \right). \end{aligned} \quad (31)$$

Proof. LHS of Lemma 20 equates to

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(b; q)_n c^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \left\{ \frac{(a\omega; q)_{\infty}}{(as; q)_{\infty}} \right\} \\
 & \quad \times D_q^{n-k} \left\{ \frac{1}{(atq^k; q)_{\infty}} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(b; q)_n c^n}{(q; q)_n} \\
 & \quad \times \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} \frac{s^k (\omega/s; q)_k (a\omega q^k; q)_{\infty}}{(as; q)_{\infty}} \frac{(tq^k)^{n-k}}{(atq^k; q)_{\infty}} \\
 &= \frac{(a\omega; q)_{\infty}}{(as, at; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(at, \omega/s; q)_k}{(q, a\omega; q)_k} s^k \\
 & \quad \times \sum_{n=k}^{\infty} \frac{(b; q)_n}{(q; q)_{n-k}} c^n t^{n-k} \\
 &= \frac{(a\omega; q)_{\infty}}{(as, at; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(at, \omega/s; q)_k}{(q, a\omega; q)_k} (sc)^k \\
 & \quad \times \sum_{m=0}^{\infty} \frac{(b; q)_{m+k}}{(q; q)_m} (tc)^m \\
 &= \frac{(a\omega; q)_{\infty}}{(as, at; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(at, \omega/s, b; q)_k}{(q, a\omega; q)_k} (sc)^k \\
 & \quad \times \sum_{m=0}^{\infty} \frac{(bq^k; q)_m}{(q; q)_m} (tc)^m \\
 &= \frac{(a\omega; q)_{\infty}}{(as, at; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(at, \omega/s, b; q)_k}{(q, a\omega; q)_k} (sc)^k \frac{(bctq^k; q)_{\infty}}{(ct; q)_{\infty}} \\
 &= \frac{(a\omega, bct; q)_{\infty}}{(as, at, ct; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} b, at, \frac{\omega}{s} \\ a\omega, bct \end{matrix}; q, sc \right). \tag{32}
 \end{aligned}$$

We complete the proof. $\square$

The special cases of Lemma 20 as taking $t = 0$, $\omega = t = 0$, and $\omega = 0$, respectively, will be frequently used.

Lemma 21. For $|sc| < 1$,

$$F(b; cD_q)_a \left\{ \frac{(a\omega; q)_{\infty}}{(as; q)_{\infty}} \right\} = \frac{(a\omega; q)_{\infty}}{(as; q)_{\infty}} {}_2\Phi_1 \left(\begin{matrix} b, \frac{\omega}{s} \\ a\omega \end{matrix}; q, sc \right). \tag{33}$$

Lemma 22. If $|sc| < 1$, then

$$F(b; cD_q)_a \left\{ \frac{1}{(as; q)_{\infty}} \right\} = \frac{(bcs; q)_{\infty}}{(as, cs; q)_{\infty}}. \tag{34}$$

Lemma 23. For $|sc| < 1$, $|tc| < 1$,

$$\begin{aligned}
 & F(b; cD_q)_a \left\{ \frac{1}{(as, at; q)_{\infty}} \right\} \\
 &= \frac{(bct; q)_{\infty}}{(as, at, ct; q)_{\infty}} {}_2\Phi_1 \left(\begin{matrix} b, at \\ bct \end{matrix}; q, sc \right). \tag{35}
 \end{aligned}$$

Throughout this paper, we also often use the following property.

Lemma 24. If $|sc| < 1$, $|tc| < 1$, then

$$\begin{aligned}
 & F(b; cD_q)_a \left\{ \frac{a^M (a\omega; q)_{\infty}}{(as, at; q)_{\infty}} \right\} \\
 &= \frac{a^M (q; q)_M (a\omega, bct; q)_{\infty}}{(as, at, ct; q)_{\infty}} \\
 & \quad \times \sum_{k=0}^{\infty} \frac{(c/a)^k (b, as, at; q)_k}{(q, a\omega, bct; q)_k (q; q)_{M-k}} \\
 & \quad \times {}_3\Phi_2 \left(\begin{matrix} bq^k, atq^k, \frac{\omega}{s} \\ a\omega q^k, bctq^k \end{matrix}; q, sc \right). \tag{36}
 \end{aligned}$$

Proof. We find that LHS of Lemma 24 equates to

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(b; q)_n c^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{a^M\} \\
 & \quad \times D_q^{n-k} \left\{ \frac{(a\omega q^k; q)_{\infty}}{(asq^k, atq^k; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(q; q)_M a^{M-k}}{(q; q)_k (q; q)_{M-k}} \\
 & \quad \times \sum_{n=k}^{\infty} \frac{(b; q)_n c^n q^{k(k-n)}}{(q; q)_{n-k}} D_q^{n-k} \left\{ \frac{(a\omega q^k; q)_{\infty}}{(asq^k, atq^k; q)_{\infty}} \right\} \\
 &= a^M (q; q)_M \sum_{k=0}^{\infty} \frac{(c/a)^k (b; q)_k}{(q; q)_k (q; q)_{M-k}} F \left(bq^k; \left(\frac{c}{q^k} \right) D_q \right) \\
 & \quad \times \left\{ \frac{(a\omega q^k; q)_{\infty}}{(asq^k, atq^k; q)_{\infty}} \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{a^M(q; q)_M (aw, bct; q)_\infty}{(as, at, ct; q)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{(c/a)^k (b, as, at; q)_k}{(q, aw, bct; q)_k (q; q)_{M-k}} \\
&\times {}_3\Phi_2 \left(\begin{matrix} bq^k, & atq^k, & \frac{\omega}{s} \\ & awq^k, & bctq^k \end{matrix}; q, sc \right).
\end{aligned} \tag{37}$$

This completes the proof. $\square$

3. Proof of Theorems

Proof of Theorem 1. LHS of (3) is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} F(a; xD_q)_y \{y^{(m+n)}\} F(b; XD_q)_Y \{Y^{(n)}\} \frac{z^n}{(q; q)_n} \\
&= F(a; xD_q)_y F(b; XD_q)_Y \left\{ \frac{y^m}{(Yyz; q)_\infty} \right\} \\
&= F(a; xD_q)_y \left\{ \frac{y^m (ybXz; q)_\infty}{(yYz, yXz; q)_\infty} \right\} \\
&= \frac{y^m (q; q)_m (byXz, axXz; q)_\infty}{(yYz, yXz, xXz; q)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{(x/y)^k (a, yYz, yXz; q)_k}{(q, byXz, axXz; q)_k (q; q)_{m-k}} \\
&\times {}_3\Phi_2 \left(\begin{matrix} aq^k, & yXzq^k, & \frac{bX}{Y} \\ & byXzq^k, & axXzq^k \end{matrix}; q, xYz \right).
\end{aligned} \tag{38}$$

We complete the proof. $\square$

Proof of Theorem 2. LHS of (4) is equal to

$$\begin{aligned}
&F(a; xD_q)_y \left\{ \frac{1}{(yt : q)_\infty} F(b; XD_q)_Y \left\{ \frac{1}{(Y\gamma\gamma, Yz; q)_\infty} \right\} \right\} \\
&= \frac{(bXz; q)_\infty}{(Yz, Xz; q)_\infty} \sum_{m=0}^{\infty} \frac{(b, Yz; q)_m (X\gamma)^m}{(q; bXz)_m} \\
&\times F(a; xD_q)_y \left\{ \frac{y^m}{(yt, yY\gamma : q)_\infty} \right\};
\end{aligned} \tag{39}$$

then using Lemma 24 and simplifying, we can get the desired result. $\square$

Proof of Theorem 3. LHS of (5) is equal to

$$\begin{aligned}
&F(a; xD_q)_y \left\{ y^n F(b; XD_q)_Y \left\{ \frac{Y^m}{(Yyz; q)_\infty} \right\} \right\} \\
&= Y^m (q; q)_m \sum_{k=0}^{\infty} \frac{(X/Y)^k (b; q)_k}{(q; q)_k (q; q)_{m-k}} F(a; xD_q)_y \\
&\times \left\{ \frac{y^n (ybXzq^k; q)_\infty}{(yYzq^k, yXz; q)_\infty} \right\} \\
&= \frac{y^n Y^m (q; q)_n (q; q)_m (ybXz, xaXz; q)_\infty}{(yYz, yXz, xXz; q)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{(X/Y)^k (b, yYz; q)_k}{(q, ybXz; q)_k (q; q)_{m-k}} \\
&\times \sum_{j=0}^{\infty} \frac{(x/y)^j (a, yYzq^k, yXz; q)_j}{(q, ybXzq^k, xaXz; q)_j (q; q)_{n-j}} \\
&\times {}_3\Phi_2 \left(\begin{matrix} aq^j, & yXzq^j, & \frac{bX}{Y} \\ & byXzq^{k+j}, & axXzq^j \end{matrix}; q, xYzq^k \right).
\end{aligned} \tag{40}$$

This completes the proof. $\square$

Proof of Theorem 4. LHS of (6) is equal to

$$\begin{aligned}
&\sum_{m_1, m_2, m_3=0}^{\infty} F(a_1; x_1 D_q)_{y_1} \{y_1^{m_1+m_2+m_3}\} F(a_2; x_2 D_q)_{y_2} \\
&\times \{y_2^{m_1+m_3}\} F(a_3; x_3 D_q)_{y_3} \{y_3^{m_1+m_2}\} \\
&\times \frac{t_1^{m_1} t_2^{m_2} t_3^{m_3}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} \\
&= F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} F(a_3; x_3 D_q)_{y_3} \\
&\times \left\{ \frac{1}{(t_1 y_1 y_2 y_3, t_2 y_1 y_3, t_3 y_1 y_2; q)_\infty} \right\} \\
&= F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} \\
&\times \left\{ \frac{1}{(t_3 y_1 y_2; q)_\infty} F(a_3; x_3 D_q)_{y_3} \right. \\
&\quad \times \left. \left\{ \frac{1}{(t_1 y_1 y_2 y_3, t_2 y_1 y_3; q)_\infty} \right\} \right\} \\
&= F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} \\
&\times \left\{ \frac{(a_3 x_3 t_2 y_1; q)_\infty}{(t_2 y_1 y_3, x_3 t_2 y_1; q)_\infty} \sum_{j_1=0}^{\infty} \frac{(a_3, t_2 y_1 y_3; q)_{j_1}}{(q, a_3 x_3 t_2 y_1; q)_{j_1}} \right. \\
&\quad \times \left. \frac{(x_3 t_1 y_1 y_2)^{j_1}}{(t_3 y_1 y_2, t_1 y_1 y_2 y_3; q)_\infty} \right\}
\end{aligned}$$

$$= F(a_1; x_1 D_q)_{y_1} \times \left\{ \frac{(a_3 x_3 t_2 y_1; q)_\infty}{(t_2 y_1 y_3, x_3 t_2 y_1; q)_\infty} \sum_{j_1=0}^{\infty} \frac{(a_3, t_2 y_1 y_3; q)_{j_1}}{(q, a_3 x_3 t_2 y_1; q)_{j_1}} \right. \\ \left. \times F(a_2; x_2 D_q)_{y_2} \left\{ \frac{(x_3 t_1 y_1 y_2)^{j_1}}{(t_3 y_1 y_2, t_1 y_1 y_2 y_3; q)_\infty} \right\} \right\}$$

$$= \sum_{j_1=0}^{\infty} \frac{(a_3; q)_{j_1} (x_3 t_1 y_2)^{j_1} (q; q)_{j_1}}{(q; q)_{j_1}}$$

$$\times \sum_{j_2=0}^{\infty} \frac{(x_2/y_2)^{j_2} (a_2; q)_{j_2}}{(q; q)_{j_2} (q; q)_{j_1-j_2}}$$

$$\times \sum_{j_3=0}^{\infty} \frac{(x_2 t_3)^{j_3} (a_2 q^{j_2}; q)_{j_3}}{(q; q)_{j_3}} F(a_1; x_1 D_q)_{y_1} \\ \times \left\{ \left(y_1^{j_1+j_3} (a_3 x_3 t_2 y_1 q^{j_1}, a_2 x_2 t_1 y_1 y_3 q^{j_2+j_3}; q)_\infty \right) \right. \\ \left. \times \left((t_2 y_1 y_3 q^{j_1}, t_3 y_1 y_2 q^{j_2}, t_1 y_1 y_2 y_3 q^{j_2+j_3}, \right. \right. \\ \left. \left. x_3 t_2 y_1, x_2 t_1 y_1 y_3; q)_\infty \right)^{-1} \right\}. \quad (41)$$

Setting

$$f(y_1) = \left(y_1^{j_1+j_3} (a_3 x_3 t_2 y_1 q^{j_1}, a_2 x_2 t_1 y_1 y_3 q^{j_2+j_3}; q)_\infty \right) \\ \times \left((t_2 y_1 y_3 q^{j_1}, t_3 y_1 y_2 q^{j_2}, t_1 y_1 y_2 y_3 q^{j_2+j_3}, \right. \\ \left. x_3 t_2 y_1, x_2 t_1 y_1 y_3; q)_\infty \right)^{-1}, \quad (42)$$

then by applying Lemma 18 and simplifying, we get

$$F(a_1; x_1 D_q)_{y_1} \{f(y_1)\} \\ = \left(y_1^{j_1+j_3} \left(\frac{a_1 x_1}{y_1}, a_3 x_3 t_2 y_1, a_2 x_2 t_1 y_1 y_3; q \right)_\infty \right) \\ \times \left(\left(\frac{x_1}{y_1}, t_2 y_1 y_3, t_3 y_1 y_2, t_1 y_1 y_2 y_3, \right. \right. \\ \left. \left. x_3 t_2 y_1, x_2 t_1 y_1 y_3; q \right)_\infty \right)^{-1}$$

$$\times \sum_{j_4=0}^{\infty} \frac{(a_1, x_3 t_2 y_1, x_2 t_1 y_1 y_3; q)_{j_4} (-1)^{j_4} q^{-(j_4-1)j_4/2}}{(q, x_1/(y_1 q^{j_4}); q)_{j_4}} \\ \times \left(\frac{x_1 q^{j_1+j_3}}{y_1} \right)^{j_4} \frac{(t_3 y_1 y_2; q)_{j_2+j_4} (t_2 y_1 y_3; q)_{j_1+j_4}}{(a_3 x_3 t_2 y_1; q)_{j_1+j_4}} \\ \times \frac{(t_1 y_1 y_2 y_3; q)_{j_2+j_3+j_4}}{(a_2 x_2 t_1 y_1 y_3; q)_{j_2+j_3+j_4}}; \quad (43)$$

substituting it into (41), we complete the theorem. $\square$

Proof of Theorem 5. LHS of (7) is equal to

$$F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} F(a_3; x_3 D_q)_{y_3} \\ \times \left\{ \frac{1}{(y_1 y_2 t_1, y_1 y_2 t_2, y_1 y_3 t_3; q)_\infty} \right\} \\ = F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} \\ \times \left\{ \frac{(a_3 x_3 y_1 t_3; q)_\infty}{(y_1 y_3 t_3, y_1 x_3 t_3; q)_\infty} \frac{1}{(y_1 y_2 t_1, y_1 y_2 t_2; q)_\infty} \right\} \\ = \sum_{j_1=0}^{\infty} \frac{(a_2; q)_{j_1} (t_1 x_2)^{j_1}}{(q; q)_{j_1}} F(a_1; x_1 D_q)_{y_1} \\ \times \left\{ \left(y_1^{j_1} (a_3 y_1 x_3 t_3, a_2 y_1 x_2 t_2 q^{j_1}; q)_\infty \right) \right. \\ \times \left((y_1 y_3 t_3, y_1 x_3 t_3, y_1 y_2 t_1, \right. \\ \left. y_1 y_2 t_2 q^{j_1}, y_1 x_2 t_2; q)_\infty \right)^{-1} \left. \right\}. \quad (44)$$

Applying Lemma 18 and simplifying, we get the theorem. This completes the proof. $\square$

Proof of Theorems 6 and 7. Using Lemma 17, then Theorem 6 can be obtained by Lemma 20 directly.

Using Lemma 17 and then applying Lemma 18, we can get Theorem 7. $\square$

Proof of Theorem 8. LHS of (10) is equal to

$$F(a; x D_q)_y \left\{ \frac{(y t_1, y t_4; q)_\infty}{(y t_2, y t_3; q)_\infty} \right\} \\ = \sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} D_q^n \left\{ \frac{(y t_1, y t_4; q)_\infty}{(y t_2, y t_3; q)_\infty} \right\}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} \\
&\quad \times \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \left\{ \frac{(yt_1; q)_{\infty}}{(yt_2; q)_{\infty}} \right\} \\
&\quad \times D_q^{n-k} \left\{ \frac{(yt_4 q^k; q)_{\infty}}{(yt_3 q^k; q)_{\infty}} \right\} \\
&= \frac{(yt_1, yt_4; q)_{\infty}}{(yt_2, yt_3; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; a)_n (xt_3)^n}{(q; q)_n} \\
&\quad \times \sum_{k=0}^n \frac{(q; q)_n (t_2/t_3)^k}{(q; q)_k (q; q)_{n-k}} \frac{(t_1/t_2, yt_3; q)_k (t_4/t_3; q)_{n-k}}{(yt_4; q)_k} \\
&= \frac{(yt_1, yt_4; q)_{\infty}}{(yt_2, yt_3; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(a, t_1/t_2, yt_3; q)_j (xt_2)^j}{(q, yt_1, yt_4; q)_j} \\
&\quad \times {}_2\Phi_1 \left(\begin{matrix} aq^j, & \frac{t_4}{t_3} \\ & yt_4 q^j \end{matrix}; q, xt_3 \right);
\end{aligned} \tag{45}$$

we complete the proof. $\square$

Proof of Theorem 9. LHS of (11) equates to

$$\begin{aligned}
&F(a; xD_q)_y F(a_0; x_0 D_q)_{y_0} F(a_1; x_1 D_q)_{y_1} \\
&\quad \times F(a_2; x_2 D_q)_{y_2} F(a_3; x_3 D_q)_{y_3} \\
&\quad \times \left\{ \frac{y^s y_0^r}{(yy_0 t, yy_1 t_1, yy_2 t_2, yy_3 t_3; q)_{\infty}} \right\} \\
&= F(a; xD_q)_y F(a_0; x_0 D_q)_{y_0} F(a_1; x_1 D_q)_{y_1} \\
&\quad \times F(a_2; x_2 D_q)_{y_2} \left\{ \frac{y^s y_0^r}{(yy_0 t, yy_1 t_1, yy_2 t_2; q)_{\infty}} \right. \\
&\quad \quad \left. \times \frac{(a_3 y x_3 t_3; q)_{\infty}}{((yy_3 t_3, yx_3 t_3; q)_{\infty})} \right\} \\
&= F(a; xD_q)_y F(a_0; x_0 D_q)_{y_0} F(a_1; x_1 D_q)_{y_1} \\
&\quad \times \left\{ \frac{y^s y_0^r}{(yy_0 t, yy_1 t_1; q)_{\infty}} \frac{(a_2 y x_2 t_2; q)_{\infty}}{((yy_2 t_2, yx_2 t_2; q)_{\infty})} \right. \\
&\quad \quad \left. \times \frac{(a_3 y x_3 t_3; q)_{\infty}}{((yy_3 t_3, yx_3 t_3; q)_{\infty})} \right\} \\
&= F(a; xD_q)_y F(a_0; x_0 D_q)_{y_0} \left\{ \frac{y^s y_0^r}{(yy_0 t; q)_{\infty}} \right\} \\
&\quad \times \left\{ \frac{(a_1 y x_1 t_1; q)_{\infty}}{((yy_1 t_1, yx_1 t_1; q)_{\infty})} \frac{(a_2 y x_2 t_2; q)_{\infty}}{((yy_2 t_2, yx_2 t_2; q)_{\infty})} \right. \\
&\quad \quad \left. \times \frac{(a_3 y x_3 t_3; q)_{\infty}}{((yy_3 t_3, yx_3 t_3; q)_{\infty})} \right\}.
\end{aligned} \tag{46}$$

If we set

$$\begin{aligned}
f(y) &= F(a_0; x_0 D_q)_{y_0} \left\{ \frac{y^s y_0^r}{(yy_0 t; q)_{\infty}} \right\} \\
&\quad \times \left\{ \frac{(a_1 y x_1 t_1; q)_{\infty}}{((yy_1 t_1, yx_1 t_1; q)_{\infty})} \frac{(a_2 y x_2 t_2; q)_{\infty}}{((yy_2 t_2, yx_2 t_2; q)_{\infty})} \right. \\
&\quad \quad \left. \times \frac{(a_3 y x_3 t_3; q)_{\infty}}{((yy_3 t_3, yx_3 t_3; q)_{\infty})} \right\},
\end{aligned} \tag{47}$$

using Lemma 24, we get $f(y)$; then employing Lemma 18, we can get the desired result. $\square$

Proof of Theorem 10. LHS of (12) equates to

$$\begin{aligned}
&F(a_1; x_1 D_q)_{y_1} F(a_2; x_2 D_q)_{y_2} F(a_3; x_3 D_q)_{y_3} \\
&\quad \times F(a_4; x_4 D_q)_{y_4} \left\{ \frac{(t_1 y_2 y_3 y_4, t_4 y_1 y_2 y_3; q)_{\infty}}{(t_2 y_1 y_3 y_4, t_3 y_1 y_2 y_4; q)_{\infty}} \right\};
\end{aligned} \tag{48}$$

then using Lemma 18 four times we can get the result. $\square$

4. Some Other Cases and a Homogeneous q -Mehler's Formula

In this section we obtain Mehler's formula involving a ${}_3\Phi_2$ series and then give an interesting relationship between the polynomial multiplications.

Theorem 25 (cf. [1, Equation (1.17)] or [7, Lemma 12]). *One has*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \phi_n^{(a)}(x, y) \phi_n^{(b)}(X, Y) \frac{t^n}{(q; q)_n} \\
&= \frac{(axYt, byXt; q)_{\infty}}{(xYt, yXt, yYt; q)_{\infty}} {}_3\Phi_2 \left(\begin{matrix} a, & b, & yYt \\ axYt, & byXt & ; q, xXt \end{matrix} \right).
\end{aligned} \tag{49}$$

Proof. LHS of Theorem 25 is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \phi_n^{(b)}(X, Y) \frac{t^n}{(q; q)_n} F(a; xD_q)_y \{y^n\} \\
&= F(a; xD_q)_y \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(b; q)_k X^k Y^{n-k} (ty)^n}{(q; q)_k (q; q)_{n-k}} \right\} \\
&= F(a; xD_q)_y \left\{ \frac{(btXy; q)_{\infty}}{(tXy; q)_{\infty}} \frac{1}{(tYy; q)_{\infty}} \right\} \\
&= \frac{(axYt, byXt; q)_{\infty}}{(xYt, yXt, yYt; q)_{\infty}} \\
&\quad \times {}_3\Phi_2 \left(\begin{matrix} a, & b, & yYt \\ axYt, & byXt & ; q, xXt \end{matrix} \right).
\end{aligned} \tag{50}$$

This completes the proof. $\square$

Remark 26. Theorem 25 can be reduced to the common q -Mehler's formula (cf. [1, Equation (1.6)]) by setting $a = b = 0$ and $y = Y = 1$. Employing this formula, we can show that the polynomials $\Psi_n(a, b; x_1, y_1; z)$ defined by Andrews [4] can be expressed by $\phi_n^{(a)}(x, y)$. Using Sear's nonterminating ${}_3\Phi_2$ transformation formula [15, page 241, Equation III.10]

$$\begin{aligned} & {}_3\Phi_2\left(\begin{matrix} A, & B, & C \\ D, & E \end{matrix}; q, \frac{DE}{ABC}\right) \\ &= \frac{(B, DE/AB, DE/BC; q)_\infty}{(D, E, DE/ABC; q)_\infty} \\ &\quad \times {}_3\Phi_2\left(\begin{matrix} \frac{D}{B}, & \frac{E}{B}, & \frac{DE}{ABC} \\ \frac{DE}{AB}, & \frac{DE}{BC} \end{matrix}; q, B\right), \end{aligned} \quad (51)$$

then the right-hand side of (49) reduces to

$$\begin{aligned} & \frac{(a, xyXYt^2, bxXt; q)_\infty}{(xYt, yXt, yYt, xXt; q)_\infty} \\ & \times {}_3\Phi_2\left(\begin{matrix} \frac{byXt}{a}, & xYt, & xXt \\ xyXYt^2, & bxXt \end{matrix}; q, a\right). \end{aligned} \quad (52)$$

Then setting $xX = z$, $yX = y_1$, and $xY = x_1$, we have (cf. [4, Theorem 4.1])

$$\begin{aligned} & \sum_{n=0}^{\infty} \phi_n^{(a)}(x, y) \phi_n^{(b)}(X, Y) \frac{t^n}{(q; q)_n} \\ &= \frac{(a, bzt, x_1y_1t^2; q)_\infty}{(t, y_1t, zt, x_1t; q)_\infty} \\ & \times {}_3\Phi_2\left(\begin{matrix} \frac{bty_1}{a}, & zt, & x_1t \\ bzt, & x_1y_1t^2 \end{matrix}; q, xXt\right). \end{aligned} \quad (53)$$

Comparison of the above identity with Theorem 4.1 that appeared in [4] gives the following.

Corollary 27 (cf. [4, Corollary 3.1]). *If $xX = z$, $yX = y_1$, and $xY = x_1$, then*

$$\Psi_n(a, b; x_1, y_1; z) = \phi_n^{(a)}(x, y) \phi_n^{(b)}(X, Y). \quad (54)$$

Combined with Theorem 6, if we replace (x, y, X, Y, a, b, t) by $(x_1, y_1, 1, t_2/t_1, a_1, t_3/t_1, t_1)$, respectively, and then equate the coefficient of t_1^N , we have the following.

Corollary 28. *One has*

$$\begin{aligned} & \frac{\phi_N^{(a)}(x_1, y_1) \phi_N^{(b)}(1, Y)}{(q; q)_N} \\ &= \sum_{m_1+m_2+m_3=N} \phi_{m_1+m_2+m_3}^{(a)}(x_1, y_1) \\ & \quad \times \frac{Y^{m_2} b^{m_3} (-1)^{m_3} q^{(m_3-1)m_3/2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}}. \end{aligned} \quad (55)$$

Theorem 29 (cf. [1, Equation (1.17)]). *If $m, n \in N$, then*

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \phi_{m+n}^{(a_1)}(x_1, y_1) \phi_m^{(a_2)}(x_2, y_2) \phi_n^{(a_3)} \\ & \times (x_3, y_3) \frac{t^m z^n}{(q; q)_m (q; q)_n} \\ &= \frac{(a_1 x_1 x_3 z, a_2 x_2 y_1 t, a_3 x_3 y_1 z; q)_\infty}{(x_1 x_3 z, x_2 y_1 t, x_3 y_1 z, y_1 y_2 t, y_1 y_3 z; q)_\infty} \\ & \times \sum_{j_1, j_2, j_3=0}^{\infty} \frac{(x_1 y_3 z)^{j_1} (x_1 x_2 t)^{j_2} (x_1 y_2 t)^{j_3}}{(q; q)_{j_1} (q; q)_{j_2} (q; q)_{j_3}} \\ & \times \frac{(a_2 x_2 / y_2, x_2 y_1 t; q)_{j_3} (a_3 x_3 z / (x_2 t); q)_{j_2}}{(a_2 x_2 y_1 t; q)_{j_3}} \\ & \times \frac{(y_1 y_3 z; q)_{j_2+j_3} (a_1, x_3 y_1 z; q)_{j_1+j_2+j_3}}{(a_3 x_3 y_1 z; q)_{j_2+j_3} (a_1 x_1 x_3 z; q)_{j_1+j_2+j_3}}, \end{aligned} \quad (56)$$

provided that $\max\{|x_1 y_3 z|, |x_3 y_1 z|, |x_1 y_2 t|, |x_2 y_1 t|, |y_1 y_2 t|, |y_1 y_3 z|, |a_1 x_1 x_3 z|\} < 1$.

Proof. Applying Lemma 17 and then using q -binomial theorem, we get that the LHS of (56) comes to

$$\begin{aligned} & F(a_1, x_1 D_q)_{y_1} F(a_2, x_2 D_q)_{y_2} F(a_3, x_3 D_q)_{y_3} \\ & \times \left\{ \frac{1}{(y_1 y_2 t, y_1 y_3 z; q)_\infty} \right\}. \end{aligned} \quad (57)$$

Twice applying Lemma 22, the above identity is equal to

$$\begin{aligned} & F(a_1, x_1 D_q)_{y_1} \left\{ \frac{(a_2 x_2 y_1 t, a_3 x_3 y_1 z; q)_\infty}{(y_1 y_2 t, x_2 y_1 t, y_1 y_3 z, x_3 y_1 z; q)_\infty} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n x_1^n}{(q; q)_n} D_q^n \\ & \times \left\{ \frac{(a_2 x_2 y_1 t, a_3 x_3 y_1 z; q)_\infty}{(y_1 y_2 t, x_2 y_1 t, y_1 y_3 z, x_3 y_1 z; q)_\infty} \right\}. \end{aligned} \quad (58)$$

For (58), by using q -Leibnitz rule three times and using q -binomial theorem, we can achieve the proof after simplification. $\square$

Taking $a_1 = a_2 = a_3 = 0$, $y_1 = y_2 = y_3 = 1$, we have the following.

Corollary 30 (cf. [9, Equation (4.4)]). *One has*

$$\begin{aligned} & \sum_{m,n=0}^{\infty} H_{m+n}(x_1) H_m(x_2) H_n(x_3) \frac{t^m z^n}{(q; q)_m (q; q)_n} \\ &= \frac{(x_1 x_3 z^2; q)_{\infty}}{(x_1 z, x_1 x_3 z, x_2 t, x_3 z, t, z; q)_{\infty}} \\ & \times \sum_{j_1, j_2=0}^{\infty} \frac{(x_2 t; q)_{j_2} (z, x_3 z; q)_{j_1+j_2} (x_1 t)^{j_1+j_2} x_2^{j_1}}{(q; q)_{j_1} (q; q)_{j_2} (x_1 x_3 z^2; q)_{j_1+j_2}}. \end{aligned} \quad (59)$$

Remark 31. Corollary 30 is equivalent to [9, Equation (4.4)] after applying q -Heine transformation formula to simplify. Note that the process of proof is only using q -Leibnitz rule, so we can remove the restrictive condition $a = q^{-F}, b = q^{-G}$ in [7, Theorem 3]. Employing the same method, we recover Theorem 7 as follows.

Theorem 32. *If $m_i \in N$, $\max\{|yt_i|, |xt_i|\} < 1$, $i = 1, 2, \dots, n$, then*

$$\begin{aligned} & \sum_{m_1, \dots, m_n=0}^{\infty} \phi_{m_1+\dots+m_n}^{(a)}(x, y) \frac{t_1^{m_1} \dots t_n^{m_n}}{(q; q)_{m_1} \dots (q; q)_{m_n}} \\ &= \frac{(axt_1; q)_{\infty}}{(xt_1, yt_1, yt_2, \dots, yt_n; q)_{\infty}} \\ & \times \sum_{j_1, j_2, \dots, j_{n-1}=0}^{\infty} \frac{(a, yt_1; q)_{j_1+j_2+\dots+j_{n-1}} x^{j_1+j_2+\dots+j_{n-1}}}{(axt_1; q)_{j_1+j_2+\dots+j_{n-1}} (q; q)_{j_1} (q; q)_{j_2}} \\ & \times (yt_2; q)_{j_2+\dots+j_{n-1}}, \dots, (yt_{n-2}; q)_{j_{n-2}+j_{n-1}} \\ & \times (yt_{n-1}; q)_{j_{n-1}} t_2^{j_1} t_3^{j_2}, \dots, t_n^{j_{n-1}} \\ & \times ((q; q)_{j_3}, \dots, (q; q)_{j_{n-1}})^{-1}. \end{aligned} \quad (60)$$

Proof. Using Lemma 17, LHS of (60) is equal to

$$\begin{aligned} & F(a; xD_q)_y \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_1; q)_{\infty}} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k} D_q^k \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_1; q)_{\infty}} \right\} \\ &= \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k} \\ & \times \sum_{j_1=0}^k \begin{bmatrix} k \\ j_1 \end{bmatrix} q^{j_1(j_1-k)} D_q^{j_1} \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_2; q)_{\infty}} \right\} \\ & \times D_q^{k-j_1} \left\{ \frac{1}{(yt_1^{j_1}; q)_{\infty}} \right\} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k} \\ & \times \sum_{j_1=0}^k \begin{bmatrix} k \\ j_1 \end{bmatrix} D_q^{j_1} \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_2; q)_{\infty}} \right\} \\ & \times \frac{t_1^{k-j_1} (yt_1; q)_{j_1}}{(yt_1; q)_{\infty}}. \end{aligned} \quad (61)$$

Exchanging the order of summation and using q -binomial, theorem (61) is equal to

$$\begin{aligned} & \frac{(axt_1; q)_{\infty}}{(xt_1, yt_1; q)_{\infty}} \sum_{j_1=0}^{\infty} \frac{(a, yt_1; q)_{j_1} x^{j_1}}{(q, axt_1; q)_{j_1}} \\ & \times D_q^{j_1} \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_2; q)_{\infty}} \right\} \\ &= \frac{(axt_1; q)_{\infty}}{(xt_1, yt_1; q)_{\infty}} \sum_{j_1=0}^{\infty} \frac{(a, yt_1; q)_{j_1} x^{j_1}}{(q, axt_1; q)_{j_1}} \\ & \times D_q^{j_1} \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_3; q)_{\infty}} \right. \\ & \left. \times \frac{1}{(yt_2; q)_{\infty}} \right\}. \end{aligned} \quad (62)$$

Applying q -Leibnitz and exchanging the order, (62) is equal to

$$\begin{aligned} & \frac{(axt_1; q)_{\infty}}{(xt_1, yt_1, yt_2; q)_{\infty}} \\ & \times \sum_{j_2=0}^{\infty} \frac{(yt_2; q)_{j_2}}{(q; q)_{j_2}} D_q^{j_2} \left\{ \frac{1}{(yt_n, yt_{n-1}, \dots, yt_3; q)_{\infty}} \right\} \\ & \times \sum_{j_1=0}^{\infty} \frac{(a, yt_1; q)_{j_1+j_2} x^{j_1+j_2} t_2^{j_2}}{(q; q)_{j_2} (axt_1; q)_{j_1+j_2}}. \end{aligned} \quad (63)$$

By induction, similar proof can be performed to get the desired result. $\square$

Taking $a = 0$, $y = 1$ in (60), we have the following.

Corollary 33. *If $m_i \in N$, $\max\{|t_i|, |xt_i|\} < 1$, $i = 1, 2, \dots, n$, then*

$$\begin{aligned} & \sum_{m_1, \dots, m_n=0}^{\infty} H_{m_1+\dots+m_n}(x) \frac{t_1^{m_1} \dots t_n^{m_n}}{(q; q)_{m_1} \dots (q; q)_{m_n}} \\ &= \frac{1}{(xt_1, t_1, t_2, \dots, t_n; q)_{\infty}} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{j_1, j_2, \dots, j_{n-1}=0}^{\infty} \frac{(t_1; q)_{j_1+j_2+\dots+j_{n-1}} x^{j_1+j_2+\dots+j_{n-1}}}{(q; q)_{j_1} (q; q)_{j_2}} \\
& \quad \times \left((t_2; q)_{j_2+\dots+j_{n-1}} \cdots (t_{n-2}; q)_{j_{n-2}+\dots+j_{n-1}} \right. \\
& \quad \times (t_{n-1}; q)_{j_{n-1}} t_2^{j_1} t_3^{j_2} \cdots t_n^{j_{n-1}} \left. \right) \\
& \quad \times \left((q; q)_{j_3} \cdots (q; q)_{j_{n-1}} \right)^{-1}.
\end{aligned} \quad (64)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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