Research Article

An Efficient Family of Optimal Eighth-Order Iterative Methods for Solving Nonlinear Equations and Its Dynamics

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The prime objective of this paper is to design a new family of optimal eighth-order iterative methods by accelerating the order of convergence of the existing seventh-order method without using more evaluations for finding simple root of nonlinear equations. Numerical comparisons have been carried out to demonstrate the efficiency and performance of the proposed method. Finally, we have compared new method with some existing eighth-order methods by basins of attraction and observed that the proposed scheme is more efficient.

1. Introduction

One of the prominent iterative methods for finding simple roots of a nonlinear equation, $f(x) = 0$, is Newton's method which is described as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1)$$

It is well known that the order of convergence of Newton's method is two. Solving a nonlinear equation is one of the most important and challenging tasks in numerical analysis. The vast literature is available for computing the solution of nonlinear equations or system of nonlinear equations; for instance, one can see [1–7]. During the last few years, multipoint methods have drawn the attention of many researchers. In [5] Petkovic et al. have presented a large collection of with and without memory multipoint methods for solving nonlinear equations. In the recent past, researchers have focused to optimize the existing methods without additional evaluation of function and derivative. In [8] Chun et al. have introduced the method of choosing weight functions in

iterative methods for simple root. Recently, Soleymani and Mousavi [9] introduced the following seventh-order method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f[x_n, y_n]} \cdot G(t_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]} \cdot H(t_n), \end{aligned} \quad (2)$$

where $t_n = f(y_n)/f(x_n)$ and $G(0) = G'(0) = 1$, $|G''(0)| < +\infty$; $H(0) = 1$, $H'(0) = 0$, $H''(0) = 2$, $|H^{(3)}(0)| < +\infty$. To compare the efficiency of different iterative methods the efficiency index is defined by $p^{1/n}$, where p is the order of convergence and n is the number of total function and derivative evaluations per iteration [7]. According to the conjecture of optimality, the optimal order of any multipoint iterative method without memory is given by 2^{n-1} where n is the total number of function evaluations [4]. Thus, the efficiency index of the method (2) is $7^{1/4} \approx 1.626$. Also, this method is not optimal, because this method requires four evaluations (three functions and one derivative) and for

TABLE 1: Functions and their roots.

Function ($f(x)$)	Root (α)
$f_1(x) = 10xe^{-x^2} - 1$	$\alpha_1 \approx 1.67963 \dots$
$f_2(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha_2 \approx 1.34742 \dots$
$f_3(x) = xe^{x^2} - (\sin x)^2 + 3 \cos x + 5$	$\alpha_3 \approx -1.20764 \dots$
$f_4(x) = x^4 + \sin\left(\frac{\pi}{x^2}\right) - 5$	$\alpha_4 = \sqrt{2}$
$f_5(x) = x^2 e^x - \sin x$	$\alpha_5 = 0$
$f_6(x) = \left(\sin x - \frac{\sqrt{2}}{2}\right)^2 (x + 1)$	$\alpha_6 = -1$
$f_7(x) = \sin 3x + x \cos x$	$\alpha_7 \approx 1.19776 \dots$

optimal its order should be $2^3 = 8$. The aim of this paper is to accelerate the order of convergence of the method (2) from seven to eight without adding more evaluations. Thus, it will agree with Kung-Traub conjecture as well as has higher efficiency index.

The rest of the paper is organized as follows: in Section 2, we propose a new family of optimal eighth-order iterative method for finding simple root of nonlinear equations. In Section 3, we employ some numerical examples to compare the performance of our new method with some existing eighth-order methods. Section 4 is devoted to the dynamical comparisons by basins of attraction in the complex plane of our proposed method with some existing methods. Finally, in the last section we give the conclusion.

2. Improved Scheme and Convergence Analysis

In this section, the order of convergence of the method (2) will be improved. The order of convergence of the method (2) is seven by using four evaluations $[f(x_n), f'(x_n), f(y_n), f(z_n)]$, which is clearly not optimal. To build an optimal eighth-order method without using more evaluations, we consider

$$\begin{aligned}
 y_n &= x_n - A(t_1) \cdot \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - B(t_2) \cdot \frac{f(y_n)}{f[y_n, y_n]}, \\
 x_{n+1} &= z_n - \{P(t_2) + Q(t_3) + R(t_4)\} \cdot \frac{f(z_n)}{f[y_n, z_n]},
 \end{aligned} \tag{3}$$

where $t_1 = f(x_n)/f'(x_n)$, $t_2 = f(y_n)/f(x_n)$, $t_3 = f(z_n)/f(y_n)$, and $t_4 = f(z_n)/f(x_n)$. The weight functions should be chosen such that the order arrives at optimal level eight. Theorem 1 gives the conditions on weight functions to reach at the optimal order of convergence.

Theorem 1. Let the function $f : D \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ has sufficient number of continuous derivatives in a neighborhood D of simple root α of f . Then the method described by (3) has

eighth-order convergence, provided the weight functions $A(t_1)$, $B(t_2)$, $P(t_2)$, $Q(t_3)$, and $R(t_4)$ satisfy the following conditions:

$$\begin{aligned}
 A(0) &= 1, \quad A'(0) = 0, \quad A''(0) = 0, \quad |A^{(3)}(0)| < +\infty, \\
 B(0) &= 1, \quad B'(0) = 1, \quad |B^{(3)}(1)| < +\infty, \\
 R(0) &= 1 - P(0) - Q(0), \\
 P'(0) &= 0, \quad P''(0) = 2, \\
 P^{(3)}(0) &= 6B''(0) - 12, \quad |P^{(4)}(0)| < +\infty, \\
 Q'(0) &= 0, \quad |Q''(0)| < +\infty, \\
 R'(0) &= 2, \quad |R''(0)| < +\infty.
 \end{aligned} \tag{4}$$

Proof. Let $e_n = x_n - \alpha$ be the error in the n th iterate and $c_h = f^{(h)}(\alpha)/h!$, $h = 1, 2, 3, \dots$. We provide Taylor's series expansion of each term involved in (3). By Taylor expansion around the simple root in the n th iterate, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + \dots + O(e_n^{10}), \tag{5}$$

$$f'(x_n) = c_1 + 2c_2 e_n + \dots + O(e_n^9). \tag{6}$$

From (5) and (6), it can be easily found that

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{c_2 e_n^2}{c_1} + \dots + O(e_n^9). \tag{7}$$

By considering this relation and $A(0) = 1$, we obtain

$$\begin{aligned}
 y_n &= \alpha + \left\{ \frac{c_2}{c_1} - A'(0) \right\} e_n^2 \\
 &+ \left\{ -\frac{2c_2^2}{c_1^2} + \frac{2(c_3 + c_2 A'(0))}{c_1} - \frac{A''(0)}{2} \right\} e_n^3 \\
 &+ \dots + O(e_n^9).
 \end{aligned} \tag{8}$$

Taylor's expansion of $f(y_n)$ around the root α is given as

$$\begin{aligned}
 f(y_n) &= (c_2 - c_1 A'(0)) e_n^2 \\
 &+ \left\{ -\frac{2c_2^2}{c_1^2} + 2(c_3 + c_2 A'(0)) - \frac{1}{2} c_1 A''(0) \right\} e_n^3 \\
 &+ \dots + O(e_n^9).
 \end{aligned} \tag{9}$$

TABLE 2: Comparison of absolute value of the functions after third iteration.

$ f $	Guess	$M_{8,1}$	$M_{8,2}$	$M_{8,3}$	$M_{8,4}$	$M_{8,5}$	$M_{8,6}$	$OM_{8,4}$
$ f_1 $	1.72	$0.2e - 617$	$0.2e - 617$	$0.2e - 636$	$0.2e - 602$	$0.1e - 660$	$0.2e - 654$	$0.4e - 688$
	1.5	$0.1e - 357$	$0.7e - 358$	$0.7e - 365$	$0.3e - 346$	$0.2e - 375$	$0.8e - 361$	$0.2e - 448$
	1.7	$0.5e - 796$	$0.5e - 796$	$0.1e - 794$	$0.2e - 762$	$0.4e - 828$	$0.5e - 809$	$0.4e - 866$
	1.1	$0.8e - 259$	$0.7e - 257$	$0.3e - 179$	$0.6e - 175$	$0.1e - 205$	$0.2e - 204$	$0.6e - 259$
$ f_2 $	1.1	NC	$0.6e - 52$	$0.1e - 177$	$0.9e - 116$	$0.7e - 165$	$0.5e - 299$	$0.3e - 127$
	1.8	$0.3e - 148$	$0.2e - 149$	$0.5e - 194$	$0.2e - 175$	$0.1e - 195$	$0.4e - 187$	$0.1e - 225$
	1.5	$0.6e - 347$	$0.6e - 347$	$0.2e - 377$	$0.1e - 348$	$0.9e - 437$	$0.1e - 390$	$0.4e - 436$
	2.0	$0.5e - 97$	$0.6e - 99$	$0.3e - 148$	$0.1e - 133$	$0.3e - 134$	$0.4e - 132$	$0.1e - 150$
$ f_3 $	-1.1	$0.5e - 234$	$0.1e - 235$	$0.4e - 337$	$0.8e - 285$	$0.1e - 325$	$0.6e - 433$	$0.1e - 301$
	-1.5	$0.5e - 124$	$0.7e - 125$	$0.2e - 182$	$0.2e - 161$	$0.2e - 253$	$0.2e - 205$	$0.2e - 254$
	-1.0	div.	$0.2e - 45$	$0.1e - 173$	$0.2e - 107$	$0.3e - 158$	$0.3e - 254$	$0.9e - 116$
	-1.3	$0.1e - 383$	$0.1e - 383$	$0.6e - 404$	$0.2e - 370$	$0.2e - 411$	$0.3e - 460$	$0.1e - 468$
$ f_4 $	1.0	$0.9e - 255$	$0.1e - 251$	$0.2e - 226$	$0.3e - 223$	$0.2e - 199$	$0.5e - 227$	$0.2e - 262$
	1.6	$0.8e - 338$	$0.7e - 338$	$0.9e - 356$	$0.2e - 329$	$0.3e - 370$	$0.5e - 430$	$0.1e - 441$
	1.5	$0.2e - 508$	$0.2e - 508$	$0.1e - 519$	$0.1e - 487$	$0.9e - 526$	$0.4e - 560$	$0.3e - 532$
	2.1	$0.1e - 105$	$0.2e - 107$	$0.1e - 146$	$0.9e - 134$	$0.1e - 144$	$0.1e - 143$	$0.3e - 159$
$ f_5 $	0.1	$0.1e - 272$	$0.3e - 273$	$0.1e - 340$	$0.4e - 303$	$0.6e - 349$	$0.1e - 338$	$0.1e - 364$
	0.5	$0.5e - 264$	$0.8e - 265$	$0.1e - 346$	$0.3e - 301$	$0.2e - 342$	$0.7e - 382$	$0.5e - 339$
	-0.1	$0.4e - 475$	$0.1e - 475$	$0.2e - 470$	$0.1e - 455$	$0.3e - 441$	$0.2e - 436$	$0.4e - 485$
	-0.5	$0.1e - 270$	$0.3e - 270$	$0.8e - 237$	$0.2e - 233$	$0.1e - 235$	$0.3e - 233$	$0.2e - 277$
$ f_6 $	-0.8	$0.3e - 158$	$0.1e - 162$	$0.4e - 258$	$0.9e - 214$	$0.4e - 248$	$0.6e - 288$	$0.3e - 254$
	-1.2	$0.3e - 422$	$0.4e - 423$	$0.7e - 398$	$0.9e - 387$	$0.3e - 385$	$0.1e - 380$	$0.2e - 435$
	-0.9	$0.1e - 425$	$0.1e - 425$	$0.1e - 456$	$0.1e - 424$	$0.4e - 500$	$0.4e - 457$	$0.6e - 526$
	-1.5	$0.1e - 324$	$0.8e - 324$	$0.7e - 262$	$0.1e - 258$	$0.7e - 273$	$0.1e - 272$	$0.3e - 336$
$ f_7 $	1.0	$0.1e - 446$	$0.2e - 448$	$0.1e - 374$	$0.1e - 371$	$0.2e - 378$	$0.7e - 374$	$0.2e - 505$
	0.8	$0.6e - 81$	NC	$0.3e - 130$	$0.4e - 114$	$0.7e - 159$	$0.1e - 116$	$0.6e - 150$
	1.8	NC	NC	$0.2e - 38$	$0.3e - 20$	NC	NC	$0.6e - 29$
	0.3	$0.1e - 226$	$0.3e - 235$	$0.8e - 244$	$0.7e - 215$	$0.7e - 277$	$0.1e - 278$	$0.2e - 416$

Here div. = divergent; NC = not convergent.

In view of (5) and (9), we get

$$\begin{aligned}
 \frac{f(y_n)}{f(x_n)} &= \left\{ \frac{c_2}{c_1} - A'(0) \right\} e_n \\
 &+ \left\{ -\frac{3c_2^2}{c_1^2} + \frac{2c_3 + 3c_2 A'(0)}{c_1} - \frac{A''(0)}{2} \right\} e_n^2 \quad (10) \\
 &+ \cdots + O(e_n^9), \\
 f[x_n, y_n] &= \frac{f(x_n) - f(y_n)}{x_n - y_n} = c_1 + c_2 e_n \\
 &+ \left\{ \frac{c_2^2}{c_1} + c_3 - c_2 A'(0) \right\} e_n^2 \quad (11) \\
 &+ \cdots + O(e_n^9).
 \end{aligned}$$

Using (9), (10), (11) and $A'(0) = 0, B(0) = 1, B'(0) = 1$, in the second step of (3), we find

$$\begin{aligned}
 z_n &= \alpha \\
 &+ \frac{c_2 (c_1 (-2c_3 + c_1 A''(0)) - c_2^2 (-6 + B''(0)))}{2c_1^3} e_n^4 \\
 &+ \frac{1}{12c_1^4} (-3c_1^2 (-4c_3 + c_1 A''(0)) (-2c_3 + c_1 A''(0)) \\
 &+ 3c_1 c_2^2 (-4c_3 + c_1 A''(0)) \\
 &\times (-20 + 3B''(0)) + 2c_1^2 c_2 (-12c_4 + c_1 A^{(3)}(0)) \\
 &+ c_2^4 (54 (-4 + B''(0) - 2B^{(3)}(0)))) e_n^5 \\
 &+ \cdots + O(e_n^9). \quad (12)
 \end{aligned}$$

TABLE 3: Comparison of number of iterations and total number of function evaluations.

f	Guess	$M_{8,1}$	$M_{8,2}$	$M_{8,3}$	$M_{8,4}$	$M_{8,5}$	$M_{8,6}$	$OM_{8,4}$
f_1	1.72	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)
	1.5	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)
	1.7	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)
	1.1	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)
f_2	1.1	NC	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	1.8	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	1.5	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)	3 (12)	2 (8)
	2.0	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
f_3	-1.1	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)	2 (8)
	-1.5	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	-1.0	div.	4 (16)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	-1.3	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)	2 (8)	2 (8)
f_4	1.0	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	1.6	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)	2 (8)
	1.5	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)
	2.1	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
f_5	0.1	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	0.5	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	-0.1	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)	2 (8)
	-0.5	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
f_6	-0.8	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	-1.2	2 (8)	2 (8)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)
	-0.9	2 (8)	2 (8)	2 (8)	3 (12)	2 (8)	2 (8)	2 (8)
	-1.5	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
f_7	1.0	2 (8)	2 (8)	3 (12)	3 (12)	3 (12)	3 (12)	2 (8)
	0.8	3 (12)	NC	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)
	1.8	NC	NC	4 (16)	4 (16)	NC	NC	4 (16)
	0.3	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)	3 (12)

Note: the choice of the parameters in the proposed method (19) is discussed in Section 4 based on the dynamical behavior for some test cases.

TABLE 4: Comparison of dynamical behavior of our proposed method (19) for different values of the parameters α , β , γ , and δ .

Values of parameters	Method	$P_1(z)$	$P_2(z)$	$P_3(z)$	$P_4(z)$	Average
$\alpha = 1, \beta = -1, \gamma = 1, \delta = 1$	$OM_{8,1}$	4	3	3	3	13/4
$\alpha = -1, \beta = -1, \gamma = -1, \delta = -1$	$OM_{8,2}$	3	3	4	3	13/4
$\alpha = 0, \beta = 0, \gamma = 3, \delta = 0$	$OM_{8,3}$	2	2	2	2	8/4
$\alpha = 0, \beta = 3, \gamma = 0, \delta = 1$	$OM_{8,4}$	1	1	1	1	4/4

By virtue of the above equation, we have

$$\begin{aligned}
 f(z_n) = & \frac{c_2 \left(c_1 \left(-2c_3 + c_1 A''(0) \right) - c_2^2 \left(-6 + B''(0) \right) \right)}{2c_1^2} e_n^4 \\
 & + \frac{1}{12c_1^3} \left(-3c_1^2 \left(-4c_3 + c_1 A''(0) \right) \right. \\
 & \quad \times \left(-2c_3 + c_1 A''(0) \right) \\
 & \quad + 3c_1 c_2^2 \left(-4c_3 + c_1 A''(0) \right) \\
 & \quad \times \left(-20 + 3B''(0) \right) \\
 & \quad + 2c_1^2 c_2 \left(-12c_4 + c_1 A^{(3)}(0) \right) \\
 & \quad \left. + c_2^4 \left(54 \left(-4 + B''(0) - 2B^{(3)}(0) \right) \right) \right) e_n^5 \\
 & + \cdots + O(e_n^9).
 \end{aligned} \tag{13}$$

FIGURE 1: Basins of attraction for polynomial $P_1(z) = (z^4 - 10z^2 + 9)$.

With the help of the above expression, one can derive

$$f[y_n, z_n] = \frac{f(y_n) - f(z_n)}{y_n - z_n} = c_1 + \frac{c_2^2}{c_1} e_n^2 + \cdots + O(e_n^9), \quad (14)$$

$$\begin{aligned} \frac{f(z_n)}{f(y_n)} &= \frac{c_1(-2c_3 + c_1 A''(0)) - c_2^2(-6 + B''(0))}{2c_1^2} e_n^2 \\ &+ \frac{1}{6c_1^3} \{3c_1 c_2(-4c_3(-6 + B''(0)) \\ &+ c_1 A''(0)(-5 + B''(0))) \\ &+ c_1^2(-12c_4 + c_1 A^{(3)}(0)) \\ &+ c_2^3(-72 + 21B''(0) - B^{(3)}(0))\} e_n^3 \\ &+ \cdots + O(e_n^9), \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{f(z_n)}{f(x_n)} &= \frac{c_2(c_1(-2c_3 + c_1 A''(0)) - c_2^2(-6 + B''(0)))}{2c_1^3} e_n^3 \\ &+ \frac{1}{12c_1^4} \{-3c_1^2(-4c_3 + c_1 A''(0))(-2c_3 + c_1 A''(0)) \\ &+ 3c_1 c_2^2(-12c_3(-7 + B''(0)) \\ &+ c_1 A''(0)(-22 + 3B''(0))) \\ &+ 2c_1^2 c_2(-12c_4 + c_1 A^{(3)}(0)) \\ &+ c_2^4(60B''(0) - 2(126 + B^{(3)}(0)))\} e_n^4 \\ &+ \cdots + O(e_n^9). \end{aligned} \quad (16)$$

Finally, using (10), (15), (16), (12), (13), (14) and $R(0) = 1 - P(0) - Q(0)$, $P'(0) = 0$, $Q'(0) = 0$, $P''(0) = 2$, $R'(0) = 2$,

FIGURE 2: Basins of attraction for polynomial $P_2(z) = (z^3 + 4z^2 - 10)$.

$A''(0) = 0$, $P^{(3)}(0) = 6B''(0) - 12$, in the last step of (3), we get the final error expression which is given by

$$\begin{aligned}
 e_{n+1} = & \frac{c_2}{48c_1^7} \{2c_1c_3 + c_2^2(-6 + B''(0))\} \\
 & \times \{12c_1^2c_3^2Q''(0) + 12c_1c_2^2c_3 \\
 & \times (8 + (-6 + B''(0))Q''(0)) \\
 & + 4c_1^2c_2(-6c_4 + c_1A^{(3)}(0)) \\
 & + c_2^4(108Q''(0) + 3B''(0) \\
 & \times (8 + (-12 + B''(0))Q''(0)) \\
 & - 8(9 + B^{(3)}(0) + P^{(4)}(0)))\} e_n^8 + O(e_n^9).
 \end{aligned} \tag{17}$$

Thus, theorem is proved. $\square$

Particular Case. Let

$$\begin{aligned}
 A(t_1) &= 1 + \alpha t_1^3, \\
 B(t_2) &= 1 + t_2 + \beta t_2^2, \\
 P(t_2) &= t_2^2 + 2(\beta - 1)t_2^3, \\
 Q(t_3) &= \gamma t_3^2, \\
 R(t_4) &= 1 + 2t_4 + \delta t_4^2,
 \end{aligned} \tag{18}$$

where α, β, γ , and $\delta \in \mathbb{R}$; then the method becomes

$$\begin{aligned}
 y_n &= x_n - \{1 + \alpha t_1^3\} \cdot \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - \{1 + t_2 + \beta t_2^2\} \cdot \frac{f(y_n)}{f[x_n, y_n]},
 \end{aligned}$$

FIGURE 3: Basins of attraction for non-polynomial $P_3(z) = (z^3 - 1/2z + 1)$.

$$\begin{aligned}
 x_{n+1} = z_n - & \left\{ t_2^2 + 2(\beta - 1)t_2^3 \right\} + \left\{ \gamma t_3^2 \right\} \\
 & + \left\{ 1 + 2t_4 + \delta t_4^2 \right\} \cdot \frac{f(z_n)}{f[y_n, z_n]},
 \end{aligned} \tag{19}$$

where $t_1 = f(x_n)/f'(x_n)$, $t_2 = f(y_n)/f(x_n)$, $t_3 = f(z_n)/f(y_n)$, and $t_4 = f(z_n)/f(x_n)$. Then its error expression becomes

$$\begin{aligned}
 e_{n+1} = & \frac{c_2}{c_1^2} \left\{ c_1 c_3 + c_2^2 (-3 + \beta) \right\} \\
 & \cdot \left\{ \alpha c_1^3 c_2 + (-3 + 2\beta + (-3 + \beta^2)\gamma) c_2^4 \right. \\
 & \quad \left. + 2(2 + (-3 + \beta)\gamma) c_1 c_2^2 c_3 + c_1^2 (\gamma c_3^2 - c_2 c_4) \right\} \\
 & \times e_n^8 + O(e_n^9).
 \end{aligned} \tag{20}$$

Remark 2. By taking different values of α , β , γ , and δ one can get a number of eighth-order iterative methods. Clearly its

efficiency index is $8^{1/4} \approx 1.682$ which is more than $7^{1/4} \approx 1.626$ of the method (2).

3. Results and Discussion

This section deals with the numerical comparisons of the proposed method (19). We have taken four particular cases of the proposed method (19) by considering different values of α , β , γ , and δ . The result of the comparisons of these methods based on the dynamical behavior is given in Table 4. From Table 4, we found that for $\alpha = \gamma = 0$, $\beta = 3$, and $\delta = 1$ method $OM_{8,4}$ is more efficient than the other ones. In order to check the effectiveness of the proposed iterative method $OM_{8,4}$ we have considered seven test nonlinear functions which are taken from [10]. The test nonlinear functions and their roots are listed in Table 1. In recent days, higher-order methods are very important because numerical applications use high precision computations. Due to this reason all the computations reported have been performed on the programming package *MATHEMATICA8* using 1000 digits floating point arithmetic using “*SetAccuracy*” command. The results of

FIGURE 4: Basins of attraction for nonpolynomial $P_4(z) = (z^4 - 1/z)$.

the comparisons are given in Tables 2 and 3. The Computer characteristics during numerical calculations are Microsoft Windows 8 Intel Core i5-3210M CPU@ 2.50 GHz with 4.00 GB of RAM, 64-bit Operating System throughout this paper. Here, we compare the performance of our new eighth-order method ($OM_{8,4}$) with the methods (34) ($M_{8,1}$), (35) ($M_{8,2}$) of [11]; NM2 ($M_{8,3}$), NM3 ($M_{8,4}$) of [12]; (11) ($M_{8,5}$), (15) ($M_{8,6}$) of [10]. Table 2 represents the value of $|f(x_n)|$ calculated for the total number of function evaluations twelve (TNFE-12) for each scheme. Table 3 exhibits the number of iterations and total number of function evaluations using the stopping criteria $|f(x_{n+1})| < \epsilon$ where $\epsilon = 10^{-50}$. It can be observed from Tables 2 and 3 that our method $OM_{8,4}$ is competitive.

4. Dynamical Analysis

In this section, we study the dynamic comparison of some higher-order simple root methods in the complex plane using basins of attraction. The basin of attraction is a way to view how an algorithm behaves as a function of the various starting

point. It is another way to compare the iterative methods. Recently, different iterative methods have been compared by their basins of attraction in [13–17]. In order to compare these methods we consider a rectangle $D = [-3, 3] \times [-3, 3] \in \mathbb{C}$ and assign a color to each point $z_0 \in D$, according to the root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. We have used stopping criteria 10^{-4} and maximum number of iterations 200 for each method. In this section, the following test polynomials and nonpolynomials have been considered for comparison:

- (i) $P_1(z) = (z^4 - 10z^2 + 9)$
- (ii) $P_2(z) = (z^3 + 4z^2 - 10)$
- (iii) $P_3(z) = (z^3 - 1/2z + 1)$
- (iv) $P_4(z) = (z^4 - 1/z)$.

Here, we try to find the optimal value of the parameters used in the proposed method (19). In order to obtain the optimal value, we have given rank 1 for best and rank 4 for the worst performance in each polynomial. The average value of

FIGURE 5: Basins of attraction for polynomial $P_1(z) = (z^4 - 10z^2 + 9)$.

FIGURE 6: Basins of attraction for polynomial $P_2(z) = (z^3 + 4z^2 - 10)$.

FIGURE 7: Basins of attraction for nonpolynomial $P_3(z) = (z^3 - 1/2z + 1)$.

FIGURE 8: Basins of attraction for nonpolynomial $P_4(z) = (z^4 - 1/z)$.

the ranks for each method has been mentioned in Table 4. The method with least average value will be the best performer. In Figure 1, it can be easily observed that for polynomial (i) the dynamical behavior of the method $OM_{8,4}$ is superior than the methods $OM_{8,1}$, $OM_{8,2}$, and $OM_{8,3}$. So, we have given rank 1 to method $OM_{8,4}$ and rank 2 to method $OM_{8,3}$. The method $OM_{8,2}$ is slightly better than $OM_{8,1}$ so we have assigned rank 3 to method $OM_{8,2}$ and rank 4 to method $OM_{8,1}$.

Based on Figure 2, for polynomial (ii) again $OM_{8,4}$ is best, followed by method $OM_{8,3}$. The methods $OM_{8,4}$ and $OM_{8,3}$ have given the rank 1 and 2, respectively. The behavior of methods $OM_{8,1}$ and $OM_{8,2}$ is almost similar with rank 3.

According to Figure 3, $OM_{8,4}$ and $OM_{8,3}$ have been given the ranks 1 and 2, respectively, for nonpolynomial (iii). Methods $OM_{8,1}$ and $OM_{8,2}$ are given the ranks 3 and 4, respectively.

From Figure 4, it can be seen that again $OM_{8,4}$ is best with rank 1. Method $OM_{8,3}$ has rank 2. The dynamical behaviors of methods $OM_{8,1}$ and $OM_{8,2}$ have rank 3 due to their almost similar basins of attraction.

Finally, from Table 4 we observe that the average rank for the method $OM_{8,4}$ is minimum. The method $OM_{8,4}$ is better than others in terms of less chaotic behavior and large basins of attraction for finding the solutions. Hence, we will consider it for comparison with some other well established existing eighth-order methods.

Polynomial (i), has roots 3, -3 , -1 , and 1. According to Figure 5, we found that $OM_{8,4}$ is best and other methods $M_{8,3}$, $M_{8,4}$, $M_{8,1}$, and $M_{8,2}$ do not perform well. Methods $M_{8,5}$ and $M_{8,6}$ show chaotic behavior. Polynomial (ii) has roots $-2.68262 + 0.358259I$, $-2.68262 - 0.358259I$, and 1.36523. According to Figure 6, we found that $OM_{8,4}$ is best and other methods $M_{8,3}$, $M_{8,4}$, $M_{8,1}$, and $M_{8,2}$ do not perform well. Again methods $M_{8,5}$ and $M_{8,6}$ show chaotic behavior.

Nonpolynomial (iii) has roots -1.1299 , $0.336674 + 0.925153I$, $0.336674 - 0.925153I$, and 0.456553. Based on Figure 7, we observe that method $OM_{8,4}$ dominates other methods.

Nonpolynomial (iv) has roots 1, $-0.809017 + 0.587785I$, $-0.809017 - 0.587785I$, $0.309017 + 0.951057I$, and $0.309017 - 0.951057I$. In Figure 8, again $OM_{8,4}$ performs better than other methods. Methods $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,4}$, $M_{8,5}$, and $M_{8,6}$ have large number of diverging points (black points).

In this section, we observe that some schemes have simple boundary of basins and other have complex boundary of basins. We also found that some existing iterative methods have chaotic behavior. Based of Figures 5, 6, 7, and 8 we also observe that methods $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,4}$, $M_{8,5}$, and $M_{8,6}$ have more diverging points (black area) in comparison with $OM_{8,4}$. Finally, our method $OM_{8,4}$ has less number of diverging points and large basins of attraction.

5. Conclusion

In this paper, we have developed a new family of optimal eighth-order iterative methods to find simple root of nonlinear equations. New proposed family is obtained by improving the order of convergence of the existing seventh-order method without adding more evaluations and using the

weight function technique. Numerical and dynamical comparisons have also presented to show the performance of the new method. From numerical and graphical comparisons, we can conclude that the new family is efficient and give tough competition to some existing eighth-order methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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