Research Article
Harmonic Subtangent Structures

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The concept of harmonic subtangent structures on almost subtangent metric manifolds is introduced and a Bochner-type formula is proved for this case. Conditions for a subtangent harmonic structure to be preserved by harmonic maps are also given.

1. Introduction

Inspired by the paper of Jianming [1], we introduce the notion of harmonic almost subtangent structure and underline the connection between harmonic subtangent structures and harmonic maps. It is well known that harmonic maps play an important role in many areas of mathematics. They often appear in nonlinear theories because of the nonlinear nature of the corresponding partial differential equations. In theoretical physics, harmonic maps are also known as sigma models. Remark also that harmonic maps between manifolds endowed with different geometrical structures have been studied in many contexts: Ianus and Pastore treated the case of contact metric manifolds [2], Bejan and Benyounes the almost para-Hermitian manifolds [3], Sahin the locally conformal Kähler manifolds [4], Ianus et al. the quaternionic Kähler manifolds [5], Jaiswal the Sasakian manifolds [6], Fetcu the complex Sasakian manifolds [7], Li the Finsler manifolds [8], and so forth. Fotiadis studied the noncompact case, describing the problem of finding a harmonic map between noncompact manifolds [9].

Let $M$ be a smooth, $m$-dimensional real manifold for which we denote by $C^\infty(M)$ the real algebra of smooth real functions on $M$, by $\Gamma(TM)$ the Lie algebra of vector fields on $M$, and by $T^1_0(M)$ the $C^\infty(M)$-module of tensor fields of $(r,s)$-type on $M$. An element of $T^1_0(M)$ is usually called vector 1-form or affinor.

Recall the concept of almost tangent geometry.

**Definition 1** (see [10]). $J \in T^1_0(M)$ is called almost tangent structure on $M$ if it has a constant rank and

$$\text{Im } J = \ker J. \quad (1)$$

The pair $(M,J)$ is called almost tangent manifold.

The name is motivated by the fact that (1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. Denoting rank $J = n$ it results in $m = 2n$. If in addition, we assume that $J$ is integrable, that is,

$$N_J (X,Y) := [JX,JY] - J [JX,Y] - J [X,JY] + J^2 [X,Y] = 0, \quad (2)$$

then $J$ is called tangent structure and $(M,J)$ is called tangent manifold.

From [11] we deduce some aspects of tangent manifolds:

(i) the distribution $\text{Im } J(= \ker J)$ defines a foliation;

(ii) there exist local coordinates $(x, y) = (x^i, y^j)_{1 \leq i \leq n}$ on $M$ such that $J = \partial / \partial y^j \otimes dx^i$; that is,

$$J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^j}, \quad J \left( \frac{\partial}{\partial y^j} \right) = 0. \quad (3)$$
We call \((x, y)\) canonical coordinates and the change of canonical coordinates \((x, y) \rightarrow (\tilde{x}, \tilde{y})\) is given by
\[
\begin{align*}
\tilde{x}^i &= \tilde{x}^i (x), \\
\tilde{y}^j &= \frac{\partial \tilde{x}^i}{\partial x^a} y^a + B^i (x).
\end{align*}
\]
(4)
So another description can be obtained in terms of \(G\)-structures. Namely, a tangent structure is a \(G\)-structure with a pseudo-Riemannian metric \(g\) compatible with \(J\). So another description can be obtained in terms of \(G\)-structures. Namely, a tangent structure is a \(G\)-structure with a pseudo-Riemannian metric \(g\) compatible with \(J\).

### Definition 2
An almost subtangent structure \(J\) is called harmonic if \(\Delta J = 0\).

If \(M\) is compact, from the definition it follows that \(J\) is harmonic if and only if \(dJ = 0\) and \(\delta J = 0\) which is equivalent to \((\nabla_X J)Y = (\nabla_Y J)X\), for any \(X, Y \in \Gamma(TM)\) and trace\((\nabla J) = 0\), \(\nabla\) being the Levi-Civita connection associated with the pseudo-Riemannian structure \(g\).

### Proposition 3
On a compact almost subtangent manifold, any harmonic almost subtangent structure \(J\) is integrable (i.e., it is a subtangent structure).

**Proof.** Let \(X, Y \in \Gamma(TM)\). Then
\[
(dJ) (X, Y) := (\nabla J) (X, Y) - (\nabla J) (Y, X) = [X, JY] + \nabla_{XY} X - [Y, JX] - \nabla_{JX} Y - J [X, Y],
\]
(8)
As \(\Delta J = 0\) implies \(dJ = 0\), we get
\[
0 = (dJ) (JX, Y) + (dJ) (X, JY)
\]
(9)
which shows the integrability of \(J\).

### Remark 4
As expected, the harmonicity of an almost sub-

tangent structure is not always preserved under conformal transformations. Indeed, let \(J\) be a harmonic subtangent structure (with respect to \(\Delta\)) and for a smooth positive function \(f\) on the 2\(n\)-dimensional manifold \(M\), let \(\tilde{g} = f^2 g\). Then the Levi-Civita connection associated with \(\tilde{g}\) is \(\tilde{\nabla}_X Y = \nabla_X Y + (1/f) X(f) \cdot Y + (1/f) Y(f) \cdot X - (1/f) g(X, Y) \cdot \text{grad}_g (f)\), for any \(X, Y \in \Gamma(TM)\). The necessary and sufficient condition for \(J\) to be harmonic (with respect to \(\Delta\)) is
\[
\left( \tilde{\nabla}_X J \right) Y = \left( \tilde{\nabla}_Y J \right) X, \quad \text{for any } X, Y \in \Gamma(TM),
\]
(10)
but
\[
\left( \tilde{\nabla}_X J \right) Y = \left( \nabla_X J \right) Y + \frac{1}{f} \left[ (JY) (f) \cdot X - Y (f) \cdot JX - g(X, Y) \cdot \text{grad}_g (f) \right],
\]
(11)
so the first relation is equivalent to
\[
X (f) \cdot JY - (JX) (f) \cdot Y = Y (f) \cdot JX - (JY) (f) \cdot X, \quad \text{for any } X, Y \in \Gamma(TM).
\]
(12)
Taking \( \{E_i\}_{1 \leq i \leq 2n} \) an orthonormal frame field on \((M, g)\) with \( \nabla E_i E_j = 0, 1 \leq i, j \leq 2n \), and computing

\[
\text{trace}(\nabla f) = \text{trace}(\nabla f) + \frac{1}{f} \sum_{i=1}^{2n} \left[ \left( E_i \right)(f) \cdot E_i - E_i(f) \cdot E_i \right] + J \left( \text{grad}_g(f) \right) - g(\langle E_i, E_i \rangle \text{grad}_g(f)),
\]

the second relation is equivalent to

\[
\sum_{i=1}^{2n} [E_i(f) \cdot E_i - (E_i)(f) \cdot E_i] = \sum_{i=1}^{2n} [J(\text{grad}_g(f)) - g(\langle E_i, E_i \rangle \text{grad}_g(f))].
\]

In conclusion, \( J \) is also harmonic with respect to \( \Delta \) if and only if

\[
df \otimes J - J \otimes df = (df \circ J) \otimes I - I \otimes (df \circ J)
\]

\[
\text{trace}[df \otimes J - (df \circ J) \otimes I] = \frac{2n}{f} \left[ J(\text{grad}_g(f)) - \text{trace}[g(\langle I \times J \rangle) \cdot \text{grad}_g(f)] \right].
\]

Now we want to see how a Bochner-type formula can be written on an almost subtangent metric manifold.

We know that for any tangent bundle-valued differential form, \( T \in \Gamma(T^* M \otimes TM) \), the following Weitzenböck formula holds [14]:

\[
\Delta T = - \nabla^2 T - S,
\]

where \( \nabla^2 T := \sum_{i=1}^{2n} V_i^E V_i T - \sum_{i=1}^{2n} V_i^E T \) and \( S(X) := \sum_{i=1}^{2n} R_{E_i X}^E T, X \in \Gamma(TM) \), for \( \{E_i\}_{1 \leq i \leq 2n} \) an orthonormal frame field and \( R_{X Y} := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} X, Y \in \Gamma(TM) \), the Riemann curvature tensor field.

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\]

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\text{trace}[df \otimes J - (df \circ J) \otimes I] = \frac{2n}{f} \left[ J(\text{grad}_g(f)) - \text{trace}[g(\langle I \times J \rangle) \cdot \text{grad}_g(f)] \right].
\]
Consider $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ a smooth map and let

$$\tau (\Phi) := \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* E_i - \Phi_* \left( \nabla_{E_i} E_i \right) \right]$$

be the tension field of $\Phi$, where $\{E_i\}_{1 \leq i \leq 2n}$ is an orthonormal frame field on $(M, g)$.

**Proposition 8.** Let $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \tilde{J} \circ \Phi_*$. Then

$$\tilde{J} (\tau (\Phi)) - \Phi_* \left( \text{trace} (\nabla J) \right) + \text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right)$$

$$= \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right],$$

for $\{E_i\}_{1 \leq i \leq 2n}$ an orthonormal frame field on the 2n-dimensional manifold $(M, g)$.

**Proof.** Express trace $(\nabla J) = \sum_{i=1}^{2n} (\nabla J)(E_i)$ and replace it in the left side of the relation. $\square$

**Proposition 9.** Let $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \tilde{J} \circ \Phi_*$. If for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\tilde{\nabla}_{\phi_i} X \tilde{J}) \circ \Phi_*$, then

$$\text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right)$$

$$= \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right].$$

**Proof.** For any $X, Y \in \Gamma(TM)$, $\tilde{\nabla}_{\phi_i} X \tilde{J} (\Phi_* Y) - \Phi_* (\nabla_X Y) = \tilde{J} (\tilde{\nabla}_{\phi_i} X \tilde{J} \Phi_* Y) - \Phi_* (J (\nabla_X Y))$ and for $X = Y = E_i$,

$$\tilde{J} (\tau (\Phi)) - \Phi_* \left( \text{trace} (\nabla J) \right) + \text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right)$$

$$= \tilde{J} \left( \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right] \right).$$

$\square$

**Definition 10.** A smooth map $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ is said to be harmonic if its tension field $\tau (\Phi)$ vanishes.

**Proposition 11.** Let $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \tilde{J} \circ \Phi_*$. If $\Phi$ is harmonic map, then

$$\Phi_* \left( \text{trace} (\nabla J) \right) = \text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right)$$

$$- \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right].$$

for $\{E_i\}_{1 \leq i \leq 2n}$ an orthonormal frame field on the 2n-dimensional manifold $(M, g)$.

Moreover, if for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\tilde{\nabla}_{\phi_i} X \tilde{J}) \circ \Phi_*$, then

$$\sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right] = 0,$$

for $\{E_i\}_{1 \leq i \leq 2n}$ an orthonormal frame field on the 2n-dimensional manifold $(M, g)$.

**Corollary 12.** Let $\Phi : (M, g, J) \to (\tilde{M}, \tilde{g}, \tilde{J})$ be a smooth map between almost subtangent metric manifolds such that $\Phi_* \circ J = \tilde{J} \circ \Phi_*$ and $J$ is harmonic subtangent structure.

1. If for any $X \in \Gamma(TM)$, $\Phi_* \circ \nabla_X J = (\tilde{\nabla}_{\phi_i} X \tilde{J}) \circ \Phi_*$, then

$$\text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right) = 0.$$

Moreover, if $\Phi$ is surjective submersion, then $\tilde{J}$ is harmonic subtangent structure, too.

2. If $\Phi$ is harmonic map, then

$$\text{trace} \left( (\tilde{\nabla}) \circ \Phi_* \left( \nabla J \right) \right)$$

$$= \sum_{i=1}^{2n} \left[ \tilde{\nabla}_{\phi_i} E_i \Phi_* \left( J E_i \right) - \Phi_* \left( \nabla_{E_i} E_i \right) \right].$$

for $\{E_i\}_{1 \leq i \leq 2n}$ an orthonormal frame field on the 2n-dimensional manifold $(M, g)$.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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