Research Article

Weighted L^p -Inequalities for the Dunkl Transform

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We give, for $1 < p \leq 2$, weighted L^p -inequalities for the Dunkl transform, using, respectively, the modulus of continuity of radial functions and the Dunkl convolution in the general case. As application, we obtain, in particular, the integrability of this transform on Besov-Lipschitz spaces.

1. Introduction

Dunkl theory is a far reaching generalization of Euclidean Fourier analysis. It started twenty years ago with Dunkl's seminal work [1] and was further developed by several mathematicians (see [2–6]) and later was applied and generalized in different ways by many authors (see [7–11]). The Dunkl operators are commuting differential-difference operators T_i , $1 \leq i \leq d$. These operators, attached to a finite root system R and a reflection group W acting on $\mathbb{R}^d$, can be considered as perturbations of the usual partial derivatives by reflection parts. These reflection parts are coupled by parameters, which are given in terms of a nonnegative multiplicity function k . The Dunkl kernel E_k has been introduced by Dunkl in [12]. For a family of weight functions w_k invariant under a reflection group W , we use the Dunkl kernel and the weighted Lebesgue measure $w_k(x)dx$ to define the Dunkl transform $\mathcal{F}_k$, which enjoys properties similar to those of the classical Fourier transform. If the parameter $k \equiv 0$ then $w_k(x) = 1$, so that $\mathcal{F}_k$ becomes the classical Fourier transform and the T_i , $1 \leq i \leq d$, reduce to the corresponding partial derivatives $\partial/\partial x_i$, $1 \leq i \leq d$ (see next section, Remark 1). The classical Fourier transform behaves well with the translation operator $f \mapsto f(\cdot - y)$, which leaves the Lebesgue measure on $\mathbb{R}^d$ invariant. However, the measure $w_k(x)dx$ is no longer invariant under the usual translation. Trimèche has introduced in [6] the Dunkl translation operators τ_x , $x \in \mathbb{R}^d$, on the space

of infinitely differentiable functions on $\mathbb{R}^d$. At the moment an explicit formula for the Dunkl translation $\tau_x(f)$ of a function f is unknown in general. However, such formula is known when the function f is radial (see next section). In particular, the boundedness of τ_x is established in this case. As a result one obtains a formula for the Dunkl convolution $*_k$. An important motivation to study Dunkl operators originates from their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics (see [13]).

Let f be a function in $L^p_k(\mathbb{R}^d)$, $1 \leq p < \infty$, where $L^p_k(\mathbb{R}^d)$ denote the space $L^p(\mathbb{R}^d, w_k(x)dx)$ with w_k the weight function associated with the Dunkl operators given by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d, \quad (1)$$

with R_+ a fixed positive root system (see next section).

The modulus of continuity $\omega_{p,k}(f)$ of first order of a radial function f in $L^p_k(\mathbb{R}^d)$ is defined by

$$\omega_{p,k}(f)(x) = \sup_{0 < t \leq x} \int_{S^{d-1}} \|\tau_{tu}(f) - f\|_{p,k} d\sigma(u), \quad x > 0, \quad (2)$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$. We use $\| \cdot \|_{p,k}$ as a shorthand for $\| \cdot \|_{L_k^p(\mathbb{R}^d)}$.

For $f \in L_k^p(\mathbb{R}^d)$, we set

$$\tilde{\omega}_{p,k}(f)(x) = \sup_{0 < t \leq x} \|f * \phi_t\|_{p,k}, \quad x > 0, \quad (3)$$

where ϕ is a radial function in $\mathcal{S}(\mathbb{R}^d)$ satisfying

$$\begin{aligned} \exists c > 0; \quad |\mathcal{F}_k(\phi)(y)| &> c, \\ \forall y \in \left\{ z \in \mathbb{R}^d : \frac{1}{2} \leq \|z\| \leq 1 \right\}, \end{aligned} \quad (4)$$

ϕ_t being the dilation of ϕ given by $\phi_t(y) = (1/t^{2(\gamma+d/2)})\phi(y/t)$ with $\gamma = \sum_{\xi \in R_+} k(\xi)$, $t \in (0, +\infty)$, and $y \in \mathbb{R}^d$.

Obviously, $\omega_{p,k}(f)(x)$ and $\tilde{\omega}_{p,k}(f)(x)$ are nondecreasing in x .

The goal of this paper is to extend to the context of Dunkl theory the results obtained by Móricz in [14] for the Fourier transform on the real line. More precisely, we give, for $1 < p \leq 2$, sufficient conditions for weighted L^p -inequalities of the Dunkl transform $\mathcal{F}_k(f)$ of a radial function f using the modulus of continuity $\omega_{p,k}(f)$. In the general case, the difficulty arises in the application of the modulus of continuity for nonradial function f in L^p since whether the Dunkl translation τ_x can be defined on $L_k^1(\mathbb{R}^d)$ is still an open problem. To avoid this problem, we use the Dunkl convolution and $\tilde{\omega}_{p,k}(f)$. As application, we obtain, in particular, the integrability of this transform on Besov-Lipschitz spaces.

The contents of this paper are as follows.

In Section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.

In Section 3, we give sufficient conditions for weighted L^p -inequalities of the Dunkl transform of function which we apply in particular on Besov-Lipschitz spaces.

Along this paper we denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product on $\mathbb{R}^d$. We write, for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x \rangle}$ and we use c to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- (i) $\mathcal{C}^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$,
- (ii) $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{C}^\infty(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives,
- (iii) $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$ of compactly supported functions.

2. Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer to more details to the surveys [15].

Let W be a finite reflection group on $\mathbb{R}^d$, associated with a root system R . For $\alpha \in R$, we denote by $\mathbb{H}_\alpha$ the hyperplane

orthogonal to α . Given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} \mathbb{H}_\alpha$, we fix a positive subsystem $R_+ = \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$. We denote by k a nonnegative multiplicity function defined on R with the property that k is W -invariant. We associate with k the index

$$\gamma = \sum_{\xi \in R_+} k(\xi) \geq 0, \quad (5)$$

and the weight function w_k given by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d. \quad (6)$$

w_k is W -invariant and homogeneous of degree 2γ . Further, we introduce the Mehta-type constant c_k by

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2/2} w_k(x) dx \right)^{-1}. \quad (7)$$

For every $1 \leq p \leq +\infty$, we denote by $L_k^p(\mathbb{R}^d)$ the space $L^p(\mathbb{R}^d, w_k(x)dx)$ and we use $\| \cdot \|_{p,k}$ as a shorthand for $\| \cdot \|_{L_k^p(\mathbb{R}^d)}$.

By using the homogeneity of degree 2γ of w_k , it is shown in [4] that, for a radial function f in $L_k^1(\mathbb{R}^d)$, there exists a function F on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function F is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) w_k(x) dx &= \int_0^{+\infty} \left(\int_{S^{d-1}} f(r y) w_k(r y) d\sigma(y) \right) r^{d-1} dr \\ &= \int_0^{+\infty} \left(\int_{S^{d-1}} w_k(r y) d\sigma(y) \right) F(r) r^{d-1} dr \\ &= d_k \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr, \end{aligned} \quad (8)$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d/2-1} \Gamma(\gamma+d/2)}. \quad (9)$$

The Dunkl operators T_j , $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k , are the first order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x \rangle},$$

$$f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \quad (10)$$

where ρ_α is the reflection on the hyperplane $\mathbb{H}_\alpha$ and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, \dots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

Remark 1. In case $k \equiv 0$, the weighted function $w_k \equiv 1$ and the measure associated with the Dunkl operators coincide with the Lebesgue measure. The T_j reduce to the corresponding partial derivatives. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.

For $y \in \mathbb{C}^d$, the system

$$\begin{aligned} T_j u(x, y) &= y_j u(x, y), \quad 1 \leq j \leq d, \\ u(0, y) &= 1, \end{aligned} \quad (11)$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x, y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. We have, for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and, for $x, y \in \mathbb{R}^d$, $|E_k(x, iy)| \leq 1$.

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d. \quad (12)$$

We list some known properties of this transform.

- (i) The Dunkl transform of a function $f \in L_k^1(\mathbb{R}^d)$ has the following basic property:

$$\|\mathcal{F}_k(f)\|_{\infty, k} \leq \|f\|_{1, k}. \quad (13)$$

- (ii) The Dunkl transform is an automorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

- (iii) When both f and $\mathcal{F}_k(f)$ are in $L_k^1(\mathbb{R}^d)$, we have the inversion formula:

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d. \quad (14)$$

- (iv) (Plancherel's theorem) the Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric automorphism on $L_k^2(\mathbb{R}^d)$.

Since the Dunkl transform $\mathcal{F}_k(f)$ is of strong-type $(1, \infty)$ and $(2, 2)$, then, by interpolation, we get, for $f \in L_k^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$ and p' such that $1/p + 1/p' = 1$, the Hausdorff-Young inequality

$$\|\mathcal{F}_k(f)\|_{p', k} \leq c \|f\|_{p, k}. \quad (15)$$

The Dunkl transform of a radial function in $L_k^1(\mathbb{R}^d)$ is also radial and could be expressed via the Hankel transform. More

precisely, according to ([4], proposition 2.4), we have the following results:

$$\begin{aligned} \int_{S^{d-1}} E_k(ix, y) w_k(y) d\sigma(y) &= d_k j_{\gamma+d/2-1}(\|x\|), \quad (16) \\ \mathcal{F}_k(f)(x) &= \int_0^{+\infty} \left(\int_{S^{d-1}} E_k(-ix, y) w_k(y) d\sigma(y) \right) F(r) r^{2\gamma+d-1} dr \\ &= c_k^{-1} \mathcal{H}_{\gamma+d/2-1}(F)(\|x\|), \quad x \in \mathbb{R}^d, \end{aligned} \quad (17)$$

where F is the function defined on $[0, +\infty)$ by $F(\|x\|) = f(x)$, $x \in \mathbb{R}^d$, $\mathcal{H}_{\gamma+d/2-1}$ is the Hankel transform of order $\gamma + d/2 - 1$, and $j_{\gamma+d/2-1}$ is the normalized Bessel function of the first kind and order $\gamma + d/2 - 1$.

The integral representation of j_α , $\alpha > -1/2$, is given by

$$j_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} \cos(\lambda xt) dt \quad \text{for } x \in \mathbb{R}, \quad \lambda \in \mathbb{C}. \quad (18)$$

We note that

$$\frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} dt = 1. \quad (19)$$

Trimèche has introduced in [6] the Dunkl translation operators τ_x , $x \in \mathbb{R}^d$, on $\mathcal{S}^\infty(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y) \mathcal{F}_k(f)(y). \quad (20)$$

Notice that, for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$

$$\begin{aligned} \tau_x &\text{ is a continuous linear mapping} \\ &\text{from } \mathcal{S}^\infty(\mathbb{R}^d) \text{ into } \mathcal{S}^\infty(\mathbb{R}^d). \end{aligned} \quad (21)$$

As an operator on $L_k^2(\mathbb{R}^d)$, τ_x is bounded. A priori it is not at all clear whether the translation operator can be defined for L^p -functions with p different from 2. However, according to ([11], Theorem 3.7), the operator τ_x can be extended to the space of radial functions in $L_k^p(\mathbb{R}^d)$, $1 \leq p \leq 2$, and we have, for a radial function f in $L_k^p(\mathbb{R}^d)$,

$$\|\tau_x(f)\|_{p, k} \leq \|f\|_{p, k}. \quad (22)$$

The Dunkl convolution product $*_k$ of two functions f and g in $L_k^2(\mathbb{R}^d)$ is given by

$$(f *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y) g(y) w_k(y) dy, \quad x \in \mathbb{R}^d. \quad (23)$$

The Dunkl convolution product is commutative and, for $f, g \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\mathcal{F}_k(f * g) = \mathcal{F}_k(f) \mathcal{F}_k(g). \quad (24)$$

It was shown in ([11], Theorem 4.1) that when g is a bounded radial function in $L_k^1(\mathbb{R}^d)$, then

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) \tau_x(g)(-y) w_k(y) dy, \quad x \in \mathbb{R}^d, \quad (25)$$

initially defined on the intersection of $L_k^1(\mathbb{R}^d)$ and $L_k^2(\mathbb{R}^d)$ extends to $L_k^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, as a bounded operator. In particular,

$$\|f * g\|_{p,k} \leq \|f\|_{p,k} \|g\|_{1,k}. \quad (26)$$

For $\beta > 0$, $1 \leq p < +\infty$ and using the modulus of continuity of first order of function introduced in Section 1, we define the weighted Besov-Lipschitz spaces (see [16] for the classical case) denoted by $\mathcal{B}_p^{\beta,k}$ as the subspace of radial functions f in $L_k^p(\mathbb{R}^d)$ satisfying

$$\sup_{x>0} \frac{\omega_{p,k}(f)(x)}{x^\beta} < +\infty. \quad (27)$$

3. Weighted L^p -Estimates for the Dunkl Transform with Sufficient Conditions

In this section, we give sufficient conditions for weighted L^p -estimates of the Dunkl transform of function which we apply on Besov-Lipschitz spaces.

Throughout this section, we denote by p' the conjugate $p/(p-1)$ of p for $1 < p \leq 2$. According to (22) and (26), we recall the following.

- (i) The modulus of continuity $\omega_{p,k}(f)$ of first order of a radial function f in $L_k^p(\mathbb{R}^d)$ is defined by

$$\omega_{p,k}(f)(x) = \sup_{0 < t \leq x} \int_{S^{d-1}} \|\tau_{tu}(f) - f\|_{p,k} d\sigma(u), \quad x > 0. \quad (28)$$

- (2) We set, for $f \in L_k^p(\mathbb{R}^d)$,

$$\bar{\omega}_{p,k}(f)(x) = \sup_{0 < t \leq x} \|f * \phi_t\|_{p,k}, \quad x > 0, \quad (29)$$

where ϕ is a radial function in $\mathcal{S}(\mathbb{R}^d)$ satisfying

$$\begin{aligned} \exists c > 0; \quad |\mathcal{F}_k(\phi)(y)| &> c, \\ \forall y \in \left\{ z \in \mathbb{R}^d : \frac{1}{2} \leq \|z\| \leq 1 \right\}, \end{aligned} \quad (30)$$

ϕ_t being the dilation of ϕ given by $\phi_t(y) = (1/t^{2(\gamma+d/2)})\phi(y/t)$, for all $t \in (0, +\infty)$ and $y \in \mathbb{R}^d$.

For $\theta \geq 1$, we introduce a class of nonnegative w_k -locally integrable radial functions on $\{x \in \mathbb{R}^d : \|x\| \geq 1\}$, which we denote by G_θ . We said that a function g belongs to the class G_θ if there exists a constant $\kappa_\theta \geq 1$ such that for $\eta = 1, 2, \dots$

$$\begin{aligned} &\left(\int_{1 \leq \|z\| \leq 2} g(2^\eta z)^\theta w_k(z) dz \right)^{1/\theta} \\ &\leq \kappa_\theta \int_{1/2 \leq \|z\| \leq 1} g(2^\eta z) w_k(z) dz. \end{aligned} \quad (31)$$

If we set

$$C_\eta = \{y \in \mathbb{R}^d : 2^\eta \leq \|y\| < 2^{\eta+1}\}, \quad \text{for } \eta = 0, 1, 2, \dots, \quad (32)$$

then using (8) and the change of variables $y = 2^\eta z$, we can write (31) in the form

$$\begin{aligned} &\left(\int_{C_\eta} g(y)^\theta w_k(y) dy \right)^{1/\theta} \\ &\leq \kappa_\theta 2^{\eta((1-\theta)/\theta)(2\gamma+d)} \int_{C_{\eta-1}} g(y) w_k(y) dy. \end{aligned} \quad (33)$$

Example 2. If $g \equiv \text{const}$, then $g \in G_\theta$, where we choose

$$\kappa_\theta \geq 2^{2\gamma+d} \left(\frac{2^{2\gamma+d} - 1}{2\gamma + d} \right)^{1/\theta-1}. \quad (34)$$

Theorem 3. Let $1 < p \leq 2$ and let f be a radial function in $L_k^p(\mathbb{R}^d)$. Then for $1 \leq q \leq p'$ and $g \in G_{p/(p-qp+q)}$, one has

$$\begin{aligned} &\int_{\|y\| \geq 2} g(y) |\mathcal{F}_k(f)(y)|^q w_k(y) dy \\ &\leq C \int_{\|y\| \geq 1} g(y) \|y\|^{-(q/p')(2\gamma+d)} \\ &\quad \times \omega_{p,k}^q(f) \left(\frac{\pi}{\|y\|} \right) w_k(y) dy, \end{aligned} \quad (35)$$

where C is a constant depending only on p and q .

Since the proof of Theorem 3 is too long, we introduce the following technical lemma.

Lemma 4. Let f be a radial function of $L_k^p(\mathbb{R}^d)$, where $1 < p \leq 2$, and let F be a function defined on $[0, +\infty[$ by $F(\|x\|) = f(x)$ for all $x \in \mathbb{R}^d$. Then

$$\left(\int_{2^\eta}^{2^{\eta+1}} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} r^{2\gamma+d-1} dr \right)^{1/p'} \quad (36)$$

$$\leq c \omega_{p,k}(f) \left(\frac{\pi}{2^{\eta+1}} \right),$$

where $\eta = 1, 2, \dots$ and c is the constant defined by

$$c = \frac{4\Gamma(\gamma+d/2) \sin^2(\pi/8)}{\sqrt{\pi}\Gamma(\gamma+d/2-1/2)} \int_{1/2}^1 (1-t^2)^{\gamma+d/2-3/2} dt. \quad (37)$$

Proof. Let f be a radial function in $L_k^p(\mathbb{R}^d)$ for $1 < p \leq 2$; then, by (20), we have

$$\mathcal{F}_k(\tau_{xu}(f) - f)(y) = (E_k(ixu, y) - 1) \mathcal{F}_k f(y), \quad (38)$$

$u \in S^{d-1}$, $x \in [0, +\infty)$, and a.e. $y \in \mathbb{R}^d$.

We can assert by Hausdorff-Young's inequality that

$$\begin{aligned} & \|\mathcal{F}_k(\tau_{xu}(f) - f)\|_{p',k} \\ &= \left(\int_{\mathbb{R}^d} |\mathcal{F}_k f(y)|^{p'} |E_k(ixu, y) - 1|^{p'} w_k(y) dy \right)^{1/p'} \\ &\leq c \|\tau_{xu}(f) - f\|_{p,k}. \end{aligned} \quad (39)$$

On the other hand, from (8) and (17), we get

$$\begin{aligned} & c_k^{p'} \int_{\mathbb{R}^d} |\mathcal{F}_k f(y)|^{p'} |E_k(ixu, y) - 1|^{p'} w_k(y) dy \\ &= \int_0^{+\infty} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} \\ &\quad \times \left(\int_{S^{d-1}} |E_k(irxu, z) - 1|^{p'} w_k(z) d\sigma(z) \right) r^{2\gamma+d-1} dr, \end{aligned} \quad (40)$$

where $r = \|y\|$ and F is the function defined on $(0, +\infty)$ by $F(\|y\|) = f(y)$ for all $y \in \mathbb{R}^d$.

By (9), (16), and Hölder's inequality, we have

$$\begin{aligned} & d_k |j_{\gamma+d/2-1}(rx) - 1| \\ &= \left| \int_{S^{d-1}} [E_k(ixru, z) - 1] w_k(z) d\sigma(z) \right| \\ &\leq \left(\int_{S^{d-1}} w_k(z) d\sigma(z) \right)^{1/p} \\ &\quad \times \left(\int_{S^{d-1}} |E_k(ixru, z) - 1|^{p'} w_k(z) d\sigma(z) \right)^{1/p'} \\ &\leq d_k^{1/p} \left(\int_{S^{d-1}} |E_k(ixru, z) - 1|^{p'} w_k(z) d\sigma(z) \right)^{1/p'}. \end{aligned} \quad (41)$$

According to (39), it follows that

$$\begin{aligned} & \int_0^{+\infty} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} |j_{\gamma+d/2-1}(rx) - 1|^{p'} r^{2\gamma+d-1} dr \\ &\leq c \|\tau_{xu}(f) - f\|_{p,k}^{p'}. \end{aligned} \quad (42)$$

Integrating the two members over S^{d-1} , this yields

$$\begin{aligned} & \int_0^{+\infty} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} |j_{\gamma+d/2-1}(rx) - 1|^{p'} r^{2\gamma+d-1} dr \\ &\leq c \omega_{p,k}^{p'}(f)(x). \end{aligned} \quad (43)$$

From (18) and (19), we get

$$\begin{aligned} & |j_{\gamma+d/2-1}(rx) - 1| \\ &= \frac{4\Gamma(\gamma+d/2)}{\sqrt{\pi}\Gamma(\gamma+d/2-1/2)} \int_0^1 (1-t^2)^{\gamma+d/2-3/2} \sin^2\left(\frac{rxt}{2}\right) dt; \end{aligned} \quad (44)$$

then, we obtain

$$\begin{aligned} & \frac{4\Gamma(\gamma+d/2)}{\sqrt{\pi}\Gamma(\gamma+d/2-1/2)} \int_{1/2}^1 (1-t^2)^{\gamma+d/2-3/2} \sin^2\left(\frac{rxt}{2}\right) dt \\ &\leq |j_{\gamma+d/2-1}(rx) - 1|. \end{aligned} \quad (45)$$

Let $x = \pi/2^{\eta+1}$, $\eta = 1, 2, \dots$. For $t \in [1/2, 1[$ and $y \in C_\eta$, we have

$$\frac{\pi}{8} \leq \frac{rxt}{2} \leq \frac{\pi}{2}, \quad (46)$$

which gives that

$$\begin{aligned} c &= \frac{4\Gamma(\gamma+d/2) \sin^2(\pi/8)}{\sqrt{\pi}\Gamma(\gamma+d/2-1/2)} \\ &\quad \times \int_{1/2}^1 (1-t^2)^{\gamma+d/2-3/2} dt \\ &\leq |j_{\gamma+d/2-1}\left(r \frac{\pi}{2^{\eta+1}}\right) - 1|. \end{aligned} \quad (47)$$

Hence, from (43), we find that

$$\begin{aligned} & \left(\int_{2^\eta}^{2^{\eta+1}} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} r^{2\gamma+d-1} dr \right)^{1/p'} \\ &\leq c \omega_{p,k}(f) \left(\frac{\pi}{2^{\eta+1}} \right), \end{aligned} \quad (48)$$

which proves the result. $\square$

Proof of Theorem 3. Take $g \in G_{p/(p-qp+q)}$ and put $\tilde{g}(\|y\|) = g(y)$, for $y \in \mathbb{R}^d$. Applying Hölder's inequality and using Lemma 4, it follows from (8) and (33) that

$$\begin{aligned} & \int_{2^\eta}^{2^{\eta+1}} \tilde{g}(r) |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^q r^{2\gamma+d-1} dr \\ &\leq \left(\int_{2^\eta}^{2^{\eta+1}} |\mathcal{H}_{\gamma+d/2-1}(F)(r)|^{p'} r^{2\gamma+d-1} dr \right)^{q/p'} \\ &\quad \times \left(\int_{2^\eta}^{2^{\eta+1}} \tilde{g}^{p'/(p'-q)}(r) r^{2\gamma+d-1} dr \right)^{(p'-q)/p'} \\ &\leq c \kappa_{p'/(p'-q)} 2^{-\eta(q/p')(2\gamma+d)} \omega_{p,k}^q(f) \left(\frac{\pi}{2^{\eta+1}} \right) \\ &\quad \times \int_{2^{\eta-1}}^{2^\eta} \tilde{g}(r) r^{2\gamma+d-1} dr. \end{aligned} \quad (49)$$

Then, we deduce

$$\begin{aligned} & \int_2^{+\infty} \tilde{g}(r) \left| \mathcal{H}_{\gamma+d/2-1}(F)(r) \right|^q r^{2\gamma+d-1} dr \\ & \leq c \kappa_{p'/(p'-q)} \sum_{\eta=1}^{+\infty} 2^{-\eta(q/p')(2\gamma+d)} \omega_{p,k}^q(f) \left(\frac{\pi}{2^{\eta+1}} \right) \\ & \quad \times \int_{2^{\eta-1}}^{2^\eta} \tilde{g}(r) r^{2\gamma+d-1} dr. \end{aligned} \quad (50)$$

Using the monotonicity of $\omega_{p,k}(f)(x)$ in $x > 0$ and the fact that $\pi/2^{\eta+1} \leq \pi/r$, $2^{-\eta(q/p')(2\gamma+d)} \leq r^{-(q/p')(2\gamma+d)}$, we obtain

$$\begin{aligned} & \int_2^{+\infty} \tilde{g}(r) \left| \mathcal{H}_{\gamma+d/2-1}(F)(r) \right|^q r^{2\gamma+d-1} dr \\ & \leq c \kappa_{p'/(p'-q)} \sum_{\eta=1}^{+\infty} \int_{2^{\eta-1}}^{2^\eta} \tilde{g}(r) r^{-(q/p')(2\gamma+d)} \\ & \quad \times \omega_{p,k}^q(f) \left(\frac{\pi}{r} \right) r^{2\gamma+d-1} dr \\ & \leq c \kappa_{p'/(p'-q)} \int_1^{+\infty} \tilde{g}(r) r^{-(q/p')(2\gamma+d)} \\ & \quad \times \omega_{p,k}^q(f) \left(\frac{\pi}{r} \right) r^{2\gamma+d-1} dr. \end{aligned} \quad (51)$$

By (8), (17), and (51), we conclude that

$$\begin{aligned} & \int_{\|y\| \geq 2} g(y) |\mathcal{F}_k(f)(y)|^q w_k(y) dy \\ & \leq C \int_{\|y\| \geq 1} g(y) \|y\|^{-(q/p')(2\gamma+d)} \omega_{p,k}^q(f) \left(\frac{\pi}{\|y\|} \right) w_k(y) dy, \end{aligned} \quad (52)$$

where $C = c \kappa_{p'/(p'-q)}$ is a constant depending only on p and q . This completes the proof of the theorem. $\square$

In particular, for the case $g \equiv \text{const}$ and $f \in \mathcal{BD}_p^{\beta,k}$, we obtain, as application of Theorem 3, the following result which is of special interest.

Corollary 5. Let $\beta > 0$, $1 < p \leq 2$, and $f \in \mathcal{BD}_p^{\beta,k}$; then consider the following.

(1) For $0 < \beta \leq 2(\gamma + d/2)/p$, one has

$$\begin{aligned} & \mathcal{F}_k(f) \in L_k^q(\mathbb{R}^d) \\ & \text{provided that } \frac{2(\gamma + d/2)p}{\beta p + 2(\gamma + d/2)(p-1)} < q \leq p'. \end{aligned} \quad (53)$$

(2) For $\beta > 2(\gamma + d/2)/p$, one has $\mathcal{F}_k(f) \in L_k^1(\mathbb{R}^d)$.

Proof. Let $\beta > 0$, $1 < p \leq 2$, and $f \in \mathcal{BD}_p^{\beta,k}$.

(i) Suppose that $0 < \beta \leq 2(\gamma + d/2)/p$. Using Theorem 3 with $g \equiv \text{const}$, it yields

$$\begin{aligned} & \int_{\|x\| \geq 2} |\mathcal{F}_k(f)(x)|^q w_k(x) dx \\ & \leq c \int_1^{+\infty} r^{-(q/p')(2\gamma+d)-\beta q+2\gamma+d-1} dr < +\infty. \end{aligned} \quad (54)$$

Since $L_k^{p'}(B(0, 2), w_k(x)dx) \subset L_k^q(B(0, 2), w_k(x)dx)$, where $B(0, 2)$ is the ball of radius 2 centered at 0, we deduce that $\mathcal{F}_k(f)$ is in $L_k^q(\mathbb{R}^d)$.

(ii) Assume now $\beta > 2(\gamma + d/2)/p$. Using Theorem 3 with $g \equiv \text{const}$ and $q = 1$, we get

$$\begin{aligned} & \int_{\|x\| \geq 2} |\mathcal{F}_k(f)(x)| w_k(x) dx \\ & \leq c \int_1^{+\infty} r^{-(2\gamma+d)/p'-\beta+2\gamma+d-1} dr < +\infty. \end{aligned} \quad (55)$$

Since $L_k^{p'}(B(0, 2), w_k(x)dx) \subset L_k^1(B(0, 2), w_k(x)dx)$, we deduce that $\mathcal{F}_k(f)$ is in $L_k^1(\mathbb{R}^d)$. This completes the proof. $\square$

For $f \in L_k^p(\mathbb{R}^d)$, $1 < p \leq 2$, if we replace $\omega_{p,k}(f)(x)$ by $\tilde{\omega}_{p,k}(f)(x)$ in Theorem 3, we obtain analogous results in the following.

Theorem 6. Let $1 < p \leq 2$ and let f be a function in $L_k^p(\mathbb{R}^d)$. Then for $1 \leq q \leq p'$ and $g \in G_{p/(p-qp+q)}$, one has

$$\begin{aligned} & \int_{\|y\| \geq 2} g(y) |\mathcal{F}_k(f)(y)|^q w_k(y) dy \\ & \leq C \int_{\|y\| \geq 1} g(y) \|y\|^{-(q/p')(2\gamma+d)} \tilde{\omega}_{p,k}^q(f) \left(\frac{1}{\|y\|} \right) w_k(y) dy, \end{aligned} \quad (56)$$

where C is a constant depending only on p and q .

Proof. Let f be a function in $L_k^p(\mathbb{R}^d)$ for $1 < p \leq 2$. Then by (24), we have, for $x \in (0, +\infty)$, $\mathcal{F}_k(f * \phi_x)(y) = \mathcal{F}_k(f)(y) \mathcal{F}_k(\phi_x)(y)$, a.e. $y \in \mathbb{R}^d$. Since $\mathcal{F}_k(\phi_x)(y) = \mathcal{F}_k(\phi)(xy)$, we get $\mathcal{F}_k(f * \phi_x)(y) = \mathcal{F}_k(f)(y) \mathcal{F}_k(\phi)(xy)$. Using Hausdorff-Young's inequality, we obtain

$$\begin{aligned} & \|\mathcal{F}_k(f * \phi_x)\|_{p',k} \\ & = \left(\int_{\mathbb{R}^d} |\mathcal{F}_k f(y)|^{p'} |\mathcal{F}_k(\phi)(xy)|^{p'} w_k(y) dy \right)^{1/p'} \\ & \leq c \|f * \phi_x\|_{p,k} \\ & \leq c \tilde{\omega}_{p,k}^q(f)(x). \end{aligned} \quad (57)$$

Let $x = 1/2^{\eta+1}$ and $\eta = 1, 2, \dots$. For $y \in C_\eta$, we have

$$\frac{1}{2} \leq x \|y\| \leq 1; \quad (58)$$

this gives, using (57) and the property of the function ϕ ,

$$\left(\int_{C_\eta} |\mathcal{F}_k f(y)|^{p'} w_k(y) dy \right)^{1/p'} \leq c \tilde{\omega}_{p,k}^q(f) \left(\frac{1}{2^{\eta+1}} \right). \quad (59)$$

Now, take $g \in G_{p/(p-qp+q)}$. Applying Hölder's inequality, it follows from (33) and (59) that

$$\begin{aligned} & \int_{C_\eta} g(y) |\mathcal{F}_k f(y)|^q w_k(y) dy \\ & \leq \left(\int_{C_\eta} |\mathcal{F}_k f(y)|^{p'} w_k(y) dy \right)^{q/p'} \\ & \quad \times \left(\int_{C_\eta} g^{p'/(p'-q)}(y) w_k(y) dy \right)^{(p'-q)/p'} \\ & \leq c \kappa_{p'/(p'-q)} 2^{-\eta(q/p')(2\gamma+d)} \tilde{\omega}_{p,k}^q(f) \left(\frac{1}{2^{\eta+1}} \right) \\ & \quad \times \int_{C_{\eta-1}} g(y) w_k(y) dy. \end{aligned} \quad (60)$$

Then, we deduce

$$\begin{aligned} & \int_{\|y\| \geq 2} g(y) |\mathcal{F}_k f(y)|^q w_k(y) dy \\ & \leq c \kappa_{p'/(p'-q)} \sum_{\eta=1}^{+\infty} 2^{-\eta(q/p')(2\gamma+d)} \tilde{\omega}_{p,k}^q(f) \left(\frac{1}{2^{\eta+1}} \right) \\ & \quad \times \int_{C_{\eta-1}} g(y) w_k(y) dy. \end{aligned} \quad (61)$$

As in the proof of Theorem 3, using the monotonicity of $\tilde{\omega}_{p,k}(f)(x)$ in $x > 0$, we conclude that

$$\begin{aligned} & \int_{\|y\| \geq 2} g(y) |\mathcal{F}_k f(y)|^q w_k(y) dy \\ & \leq C \int_{\|y\| \geq 1} g(y) \|y\|^{-(q/p')(2\gamma+d)} \tilde{\omega}_{p,k}^q(f) \left(\frac{1}{\|y\|} \right) w_k(y) dy, \end{aligned} \quad (62)$$

where $C = c \kappa_{p'/(p'-q)}$ is a constant depending only on p and q . This completes the proof. $\square$

Remark 7. As consequence of Theorem 6, we deduce, in the particular case when $g \equiv \text{const}$ and $f \in L_k^p(\mathbb{R}^d)$ satisfying

$$\sup_{x>0} \frac{\tilde{\omega}_{p,k}(f)(x)}{x^\beta} < +\infty, \quad (63)$$

similar results to those obtained in Corollary 5.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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