Research Article

Inclusion Properties of Certain Subclasses of \( p \)-Valent Functions Associated with the Integral Operator

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The purpose of the present paper is to introduce two subclasses of \( p \)-valent functions by using the integral operator and to investigate various properties for these subclasses.

1. Introduction

Let \( \mathcal{A}(p) \) denote the class of functions of the following form:

\[
f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),
\]

which are analytic and \( p \)-valent in the open unit disc \( U = \{z \in \mathbb{C} : |z| < 1\} \). Let \( \mathcal{P}_k(p, \gamma) \) be the class of functions \( g \) analytic in \( U \) satisfying \( g(0) = p \) and

\[
\int_0^{2\pi} \left| \Re \left\{ g(z) \right\} - \gamma \right| \frac{d\theta}{p - \gamma} \leq k \pi, \quad (z = r e^{i\theta}; k \geq 2; 0 \leq \gamma < p).
\]

From (1), we have \( g \in \mathcal{P}_k(p, \gamma) \) if and only if there exists \( g_1, g_2 \in \mathcal{P}(p, \gamma) \) such that

\[
g(z) = \left( \frac{k}{4} + 1 \right) g_1(z) - \left( \frac{k}{4} - 1 \right) g_2(z), \quad (z \in U).
\]

It is known that \cite{4} the class \( \mathcal{P}_k(\gamma) \) is a convex set. Motivated essentially by Jung et al. \cite{5}, Liu and Owa \cite{6} introduced the integral operator \( Q_{\beta,p}^{\alpha} : \mathcal{A}(p) \to \mathcal{A}(p) \ (\alpha \geq 0; \beta > -p; p \in \mathbb{N}) \) as follows:

\[
Q_{\beta,p}^{\alpha} f(z) = \left( \frac{p + \alpha + \beta - 1}{p + \beta - 1} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) \, dt \quad (\alpha > 0),
\]

\[
f(z) \quad (\alpha = 0).
\]

For \( f \in \mathcal{A}(p) \) given by (1) and then from (4), we deduce that

\[
Q_{\beta,p}^{\alpha} f(z) = z^p + \sum_{j=1}^{\infty} \Gamma(\alpha + p + j) \frac{\Gamma(\beta + p + j)}{\Gamma(\beta + p)} a_{p+j} z^{p+j} \quad (\alpha \geq 0; \beta > -p).
\]

It is easily verified from (5) that (see \cite{6})

\[
Q_{\beta,p}^{\alpha+1} f(z) = (\alpha + \beta + p) Q_{\beta,p}^{\alpha} f(z) - (\alpha + \beta) Q_{\beta,p}^{\alpha} f(z).
\]
We note that (i) the one-parameter family of integral operator $Q_{\beta,1} = Q_{\beta}^+ \alpha$ was defined by Jung et al. [5] and studied by Aouf [7] and Gao et al. [8].

(ii) Consider

$$Q_{\alpha,\beta}^+ f(z) = F_{\alpha,\beta} (f) (z) = \frac{c + p}{z^{-c}} \int \frac{f(t)}{t} dt, \quad (c > -p),$$

(7)

where the operator $F_{\alpha,\beta}$ is the generalized Bernardi-Libera-Livingston integral operator (see [9]).

We have the following known subclasses $\mathcal{S}_k(p, \gamma)$ and $\mathcal{C}_k(p, \gamma)$ of the class $\mathcal{A}(p)$ for $0 \leq \gamma, \eta < p$, and $k \geq 2$ which are defined by

$$\mathcal{S}_k(p, \gamma) = \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(p, \gamma), z \in U \right\},$$

$$\mathcal{C}_k(p, \gamma) = \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(p, \gamma), z \in U \right\}.$$

(8)

Next, by using the integral operator $Q_{\beta,\alpha}^\nu p$, we introduce the following classes of analytic functions for $0 \leq \gamma < p$ and $k \geq 2$:

$$\mathcal{S}_k(p, \alpha ; \gamma) = \left\{ f \in \mathcal{A}(p) : Q_{\beta,\alpha}^\nu p f(z) \in \mathcal{S}_k(p, \gamma) \right\},$$

$$\mathcal{C}_k(p, \alpha ; \gamma) = \left\{ f \in \mathcal{A}(p) : Q_{\beta,\alpha}^\nu p f(z) \in \mathcal{C}_k(p, \gamma) \right\}.$$

(9)

We also note that

$$f \in \mathcal{C}_k(p, \alpha ; \gamma) \iff \frac{zf'(z)}{p} \in \mathcal{S}_k(p, \alpha ; \gamma).$$

(10)

In particular, we set $\mathcal{S}_k(1, \alpha ; \gamma) = \mathcal{S}_k(\alpha ; \gamma)$ and $\mathcal{C}_k(1, \alpha ; \gamma) = \mathcal{C}_k(\alpha ; \gamma)$.

The following lemma will be required in our investigation.

**Lemma 1** (see [10]). Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u, v)$ be a complex-valued function satisfying the following conditions:

(i) $\Psi(u, v)$ is continuous in a domain $D \in \mathbb{C}^2$;

(ii) $(0, 1) \in D$ and $\Psi(0, 1) > 0$;

(iii) $\Re\{\Psi(u, v_i)\} > 0$ whenever $(u, v_i) \in D$ and $v_i \leq -(1/2)(1 + u_i^2)$.

If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $U$ such that $(h(z), \frac{z h'(z)}{h(z)}) \in D$ and $\Re\{\Psi(h(z), \frac{z h'(z)}{h(z)})\} > 0$ for $z \in U$, then $\Re\{\Psi(h(z), \frac{z h'(z)}{h(z)})\} > 0$ in $U$.

**Lemma 2** (see [11]). Let $p(z)$ be analytic in $U$ with $p(0) = a$ and $\Re\{p(z)\} > 0$, $z \in U$. Then, for $s > 0$ and $\mu \in \mathbb{C} \setminus \{-1\}$,

$$\Re \left\{ p(z) + \frac{szp'(z)}{p(z) + \mu} \right\} > 0, \quad (|z| < r_0),$$

(11)

where $r_0$ is given by

$$r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{1/2}}}, \quad A = 2(s + 1)^2 + |\mu|^2 - 1,$$

(12)

and this radius is the best possible.

**Lemma 3** (see [12]). Let $\psi$ be convex and let $g$ be starlike in $U$. Then, for $F$ analytic in $U$ with $F(0) = 1$, $(\psi \ast F g)/(\psi \ast g)$ is contained in the convex hull of $F(U)$.

In this paper, we obtain several inclusion properties of the classes $\mathcal{S}_k(p, \alpha ; \gamma)$ and $\mathcal{C}_k(p, \alpha ; \gamma)$ associated with the operator $Q_{\beta,\alpha}^\nu p$.

2. **Main Results**

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2, \alpha \geq 0, \beta > 0, 0 \leq \gamma < p$, and $p \in \mathbb{N}$.

**Theorem 4.** One has

$$\mathcal{S}_k(p, \alpha + 1; \gamma) \subset \mathcal{S}_k(p, \alpha; \gamma).$$

(13)

**Proof.** We begin by setting

$$\frac{z(Q_{\beta,\alpha}^\nu p f(z))'}{Q_{\beta,\alpha}^\nu p f(z)} = (p - \gamma) h(z) + \gamma$$

(14)

$$= \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ (p - \gamma) h_1(z) + \gamma \right\} - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (p - \gamma) h_2(z) + \gamma \right\},$$

where $h_i$ is analytic in $U$ with $h_i(0) = 1, i = 1, 2$. Using the identity (6) in (14) and differentiating the resulting equation with respect to $z$, we obtain

$$\frac{z(Q_{\beta,\alpha}^\nu p f(z))'}{Q_{\beta,\alpha}^\nu p f(z)} = \gamma + (p - \gamma) h(z)$$

$$+ \frac{(p - \gamma) z h'(z)}{(p - \gamma) h(z) + \gamma + \alpha + \beta} \in \mathcal{P}_k(p, \gamma).$$

(15)

This implies that

$$h_i(z) + \frac{z h_i'(z)}{(p - \gamma) h_i(z) + \gamma + \alpha + \beta} \in \mathcal{P}_k(p, \gamma) \quad (z \in U; i = 1, 2).$$

(16)

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z)$ and $v = z h_i'(z)$:

$$\Psi(u, v) = u + \frac{v}{(p - \gamma) u + \gamma + \alpha + \beta}.$$
Clearly, the first two conditions of Lemma 1 are satisfied. Now, we verify condition (iii) as follows:

\[ \Re \{ \Psi (it_2, v_i) \} = \Re \left\{ \frac{v_1}{(p - y) i t_2 + y + \alpha + \beta} \right\} \]

\[ \leq - \frac{(y + \alpha + \beta) (1 + u_2^2)}{2 \left[ (p - y)^2 u_2^2 + (y + \alpha + \beta)^2 \right]} < 0. \]

Therefore applying Lemma 1, \( h_i \in \mathcal{P} \) \((i = 1, 2)\) and consequently \( h \in \mathcal{P}_k \) for \( z \in \mathcal{U} \). This completes the proof of Theorem 4.

**Theorem 5.** One has

\[ \mathcal{C}_k (p, \alpha + 1; \gamma) \subset \mathcal{C}_k (p, \alpha; \gamma). \] (19)

**Proof.** Applying (10) and Theorem 4, we observe that

\[ f \in \mathcal{C}_k (p, \alpha + 1; \gamma) \Rightarrow \frac{zf'}{p} \in \mathcal{S}_k (p, \alpha + 1; \gamma) \Rightarrow \frac{zf'}{p} \in \mathcal{S}_k (p, \alpha; \gamma) \]

\[ \Rightarrow f \in \mathcal{C}_k (p, \alpha; \gamma), \]

which evidently proves Theorem 5.

**Theorem 6.** If \( f \in \mathcal{S}_k (p, \alpha; \gamma) \), then \( F_{pcp} (f) \in \mathcal{S}_k (p, \alpha; \gamma) \) \((\gamma \geq 0)\), where the generalized Libera integral operator \( F_{pcp} \) is defined by (7).

**Proof.** Let \( f \in \mathcal{S}_k (p, \alpha; \gamma) \) and set

\[ z \left( Q_{\beta, p}^a F_{pcp} (f) (z) \right)' = \frac{(p - y) h (z) + y}{(p, \alpha, \gamma) \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ (p - y) h_1 (z) + \gamma \right\} - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (p - y) h_2 (z) + y + \gamma \right\}. \]

(21)

where \( h \) is analytic in \( \mathcal{U} \) with \( h(0) = 1 \). From (21), we have

\[ z \left( Q_{\beta, p}^a F_{pcp} (f) (z) \right)' = (c + p) Q_{\beta, p}^a f (z) - c Q_{\beta, p}^a F_{pcp} (f) (z). \] (22)

Then, by using (21) and (22), we obtain

\[ (c + p) \frac{Q_{\beta, p}^a f (z)}{Q_{\beta, p}^a F_{pcp} (f) (z)} = \frac{(p - y) h (z) + y + c}{(p, \gamma) \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ (p - y) h_1 (z) + \gamma \right\} - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (p - y) h_2 (z) + y + \gamma \right\}. \] (23)

Taking the logarithmic differentiation on both sides of (23) with respect to \( z \) and multiplying by \( z \), we have

\[ \frac{1}{p - y} \left\{ \frac{z \left( Q_{\beta, p}^a f (z) \right)'}{Q_{\beta, p}^a f (z)} - \gamma \right\} = h (z) + \frac{zh' (z)}{(p - y) h (z) + y + c} \in \mathcal{P}_k. \] (24)

This implies that

\[ \left\{ h_1 (z) + \frac{zh'_1 (z)}{(p - y) h (z) + y + c} \right\} \in \mathcal{P}_k, \] \( (z \in \mathcal{U} \cup i = 1, 2) \).

(25)

We form the functional \( \Psi (u, \nu) \) by choosing \( u = h_1 (z) \) and \( \nu = zh'_1 (z) \):

\[ \Psi (u, \nu) = u + \frac{\nu}{(p - y) u + y + c}. \] (26)

Then clearly \( \Psi (u, \nu) \) satisfies all the properties of Lemma 1. Hence, \( h_1 \in \mathcal{P} \) \((i = 1, 2)\) and consequently \( h \in \mathcal{P}_k \) for \( z \in \mathcal{U} \), which implies that \( F_{pcp} (f) \in \mathcal{S}_k (p, \alpha; \gamma) \).

Next, we derive an inclusion property for the subclass \( \mathcal{C}_k (\alpha; \gamma) \) involving \( F_{pcp} (f) \), which is given by the following theorem.

**Theorem 7.** If \( f \in \mathcal{C}_k (p, \alpha; \gamma) \), then \( F_{pcp} (f) \in \mathcal{C}_k (p, \alpha; \gamma) \) \((\gamma \geq 0)\), where \( F_{pcp} \) is defined by (7).

**Proof.** By applying Theorem 6, it follows that

\[ f \in \mathcal{C}_k (p, \alpha; \gamma) \Rightarrow \frac{zf'}{p} \in \mathcal{S}_k (p, \alpha; \gamma) \]

\[ \Rightarrow F_{pcp} \left( \frac{zf'}{p} \right) \in \mathcal{S}_k (p, \alpha; \gamma) \]

(by Theorem 5)

\[ \Rightarrow \frac{z}{(c + p) Q_{\beta, p}^a f (z) - c Q_{\beta, p}^a F_{pcp} (f) (z)} \in \mathcal{S}_k (p, \alpha; \gamma) \]

\[ \Rightarrow F_{pcp} (f) \in \mathcal{C}_k (p, \alpha; \gamma), \]

which proves Theorem 7.

**Theorem 8.** If \( f \in \mathcal{C}_k (p, \alpha + 1; \gamma) \) for \( z \in \mathcal{U} \), then \( f \in \mathcal{C}_k (p, \alpha + 1; \gamma) \) for

\[ |z| < r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{1/2}}}, \] (28)

where \( A = 2(s + 1)^2 + \mu^2 = 2, \mu = ((\gamma + \alpha + \beta)/(p - y)) \neq -1 \) and \( s = (1/(p - y)) \). This radius is the best possible.

**Proof.** Let \( f \in \mathcal{C}_k (p, \alpha + 1; \gamma) \) for \( z \in \mathcal{U} \) and let

\[ \frac{z}{(c + p) Q_{\beta, p}^a f (z) - c Q_{\beta, p}^a F_{pcp} (f) (z)} \]

\[ = \frac{1}{p - y} \left\{ \frac{z \left( Q_{\beta, p}^a f (z) \right)'}{Q_{\beta, p}^a f (z)} - \gamma \right\} = h (z) + \frac{zh' (z)}{(p - y) h (z) + y + c} \in \mathcal{S}_k. \]

\[ = \frac{k}{4} + \frac{1}{2} \left\{ (p - y) h_1 (z) + \gamma \right\} \]

\[ - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (p - y) h_2 (z) + y + \gamma \right\}, \] (29)
where \( h_i \) is analytic in \( U \) with \( h_i(0) = 1 \) and \( \Re\{h_i(z)\} > 0 \) for \( i = 1, 2 \). Using the identity (6) in (29) and differentiating the resulting equation with respect to \( z \), we obtain

\[
\frac{1}{p - y} \left\{ z\left(\frac{Q_{\beta,p} f(z)}{Q_{\beta,p} G(z)}\right)' - y \right\} = h(z) + \frac{(1/(p - y)) \, z \, h'(z)}{h(z) + ((y + \alpha + \beta)/(p - y))}
\]

Applying Lemma 2 with \( s = ((y + \alpha + \beta)/(p - y)) \) and \( \mu = ((y + \alpha + \beta)/(p - y)) \neq 1 \), we get

\[
\Re \left\{ h_1(z) + \frac{(1/(p - y)) \, z \, h'_1(z)}{h_1(z) + ((y + \alpha + \beta)/(p - y))} \right\} > 0 \quad \text{for } |z| < r_0,
\]

where \( r_0 \) is given by (28). This completes the proof of Theorem 8.

**Theorem 9.** Let \( \phi \) be a convex function and \( f \in \delta_2(\alpha; \gamma) \). Then \( G \in \delta_2(\alpha; \gamma) \), where \( G = \phi \ast f \).

**Proof.** Let \( \phi = \phi \ast f \). Then

\[
Q_{\beta,p}^\alpha G(z) = Q_{\beta,p}^\alpha (\phi \ast f)(z) = \phi(z) \ast Q_{\beta,p}^\alpha f(z).
\]

Also, \( f \in \delta_2(\alpha; \gamma) \). Therefore, \( Q_{\beta,p}^\alpha f \in \delta_2(\gamma) \). By logarithmic differentiation of (32) and after some simplification, we obtain

\[
\frac{z \left( Q_{\beta,p}^\alpha G(z) \right)'}{pQ_{\beta,p} G(z)} = \frac{\phi(z) \ast F(z) Q_{\beta,p}^\alpha f(z)}{\phi(z) \ast Q_{\beta,p}^\alpha f(z)},
\]

where \( F = z(Q_{\beta,p}^\alpha f(z))'/pQ_{\beta,p}^\alpha f(z) \) is analytic in \( U \) and \( F(0) = 1 \). From Lemma 3, we can see that \( z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p} G(z) \) is contained in the convex hull of \( F(U) \). Since \( z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p} G(z) \) is analytic in \( U \) and

\[
F(U) = \Omega = \left\{ w : z\left( \frac{Q_{\beta,p}^\alpha w(z)}{pQ_{\beta,p}^\alpha w(z)} \right) \in \mathcal{D}(\gamma) \right\},
\]

then \( z(Q_{\beta,p}^\alpha G(z))'/pQ_{\beta,p} G(z) \) lies in \( \Omega \); this implies that \( G = \phi \ast f \in \delta_2(\alpha; \gamma) \).

**Remark 10.** Putting \( p = 1 \) in the above results, we obtain corresponding results for the operator \( Q_{\beta}^\alpha \).

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


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