Research Article

Coupled Fixed Point Theorems with Rational Type Contractive Condition in a Partially Ordered G -Metric Space

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Coupled fixed point theorems for a map satisfying mixed monotone property and a nonlinear, rational type contractive condition are established in a partially ordered G -metric space. The conditions for uniqueness of the coupled fixed point are discussed. We also present results for the existence of coupled coincidence points of two maps.

1. Introduction

The idea of weakening the contractive condition in a metric space by introducing partial order in the space and considering monotone functions satisfying contractive conditions was first developed by Ran and Reurings [1]. Later, this was extended by Bhaskar and Lakshmikantham [2] to prove a coupled fixed point theorem for functions satisfying mixed monotone property. Since then, there has been considerable interest in the development of coupled fixed point theorems in partially ordered metric spaces with a variety of contractive conditions [3–18].

Nonlinear contractive conditions were considered in [4, 6, 19]. In particular, a rational type contractive condition was considered by Jaggi [19] in a complete metric space and this was extended to a partially ordered complete metric space by Harjani et al. [6] to prove some fixed point theorems.

Some coupled fixed point theorems in partially ordered, complete G -metric spaces were developed by Choudhury and Maity [8] and Saadati et al. [9]. The contractive conditions used in [8] were extensions of that used by Bhaskar and Lakshmikantham [2] into a G -metric space. A new concept of an Ω distance was introduced in [9].

In this paper we develop a coupled fixed point theorem using a rational type, nonlinear contractive condition in a partially ordered complete G -metric space. The condition is similar to the rational type contractive condition of Ćirić et al. [3] and may be considered as a generalization of the condition

given in [3]. We also find conditions for the uniqueness of the coupled fixed point. Finally we consider the conditions for existence of coupled coincidence points.

We begin by introducing the basic definitions and notions used in the paper.

2. General Preliminaries

Throughout this work $\leq$ will denote a partial order relation on some given set. For any two elements x, y in some partially ordered set endowed with the partial order relation $\leq$, $x \leq y$ and $y \geq x$ are equivalent. Also by $x < y$ we mean $x \leq y$ and $x \neq y$.

Definition 1 (see [20]). Let X be a nonempty set and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (1) $G(x, y, z) = 0$ if $x = y = z$,
- (2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
- (3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
- (4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
- (5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized metric or more specifically a G -metric on X and the pair (X, G) is called a G -metric space.

Theorem 2 (see [20]). Let (X, G) be a G -metric space; then for any $x, y, z, a \in X$ it follows that

- (1) if $G(x, y, z) = 0$ then $x = y = z$;
- (2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
- (3) $G(x, y, y) \leq 2G(y, x, x)$;
- (4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

Definition 3 (see [20]). Let (X, G) be a G -metric space. The sequence $\{x_n\} \subseteq X$ is G -convergent to x if for any arbitrary $\epsilon > 0$ there is a positive integer N such that $G(x, x_n, x_n) < \epsilon$ for $n \geq N$, that is, if $\lim_{n \rightarrow \infty} G(x, x_n, x_n) = 0$.

Theorem 4 (see [20]). Let (X, G) be a G -metric space; then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$ the following are equivalent:

- (1) x_n is G -convergent to x ;
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (4) $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.

Theorem 5 (see [20]). Let (X, G) be a G -metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Remark 6. This means if $\{x_n\}$, $\{y_m\}$, and $\{z_l\}$ are sequences in X such that $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{m \rightarrow \infty} y_m = y$, and $\lim_{l \rightarrow \infty} z_l = z$, then $G(x_n, y_m, z_l) \rightarrow G(x, y, z)$ as $l, m, n \rightarrow \infty$.

Definition 7 (see [20]). Let (X, G) be a G -metric space. Then sequence $\{x_n\} \subseteq X$ is said to be G -Cauchy if, for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

Theorem 8 (see [20]). In a G -metric space (X, G) the following are equivalent:

- (1) the sequence x_n is G -Cauchy;
- (2) for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m > N$.

Definition 9 (see [20]). A G -metric space (X, G) is said to be G -complete if and only if every G -Cauchy sequence is G -convergent in (X, G) .

Definition 10 (see [2]). Let $(X, \leq)$ be a partially ordered set and $T : X \times X \rightarrow X$. Then T is said to have mixed monotone property if $T(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y . That is, for all $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\implies T(x_1, y) \leq T(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 &\implies T(x, y_1) \geq T(x, y_2). \end{aligned} \quad (1)$$

Definition 11 (see [4]). Let $(X, \leq)$ be a partially ordered set and $T : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say T has the mixed g -monotone property if T is monotone g -nondecreasing in its

first argument and is monotone g -nonincreasing in its second argument. That is, for all $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \leq gx_2 &\implies T(x_1, y) \leq T(x_2, y), \\ y_1, y_2 \in X, \quad gy_1 \leq gy_2 &\implies T(x, y_1) \geq T(x, y_2). \end{aligned} \quad (2)$$

Definition 12 (see [2]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a map $T : X \times X \rightarrow X$ if $T(x, y) = x$ and $T(y, x) = y$.

Definition 13 (see [4]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the maps $T : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $T(x, y) = gx$ and $T(y, x) = gy$.

Definition 14 (see [4]). The maps $T : X \times X \rightarrow X$ and $g : X \rightarrow X$ are said to be commutative if $g(T(x, y)) = T(gx, gy)$.

Definition 15. For a map $T : X \times X \rightarrow X$, by $T^2(x, y)$, we mean $T(T(x, y), T(y, x))$. Similarly we define $T^n(x, y)$, $n \geq 3$.

3. Main Results

Our main results are presented in this section. We first develop a rational type contractive condition on a partially ordered, complete G -metric space and give a coupled fixed point theorem for a map satisfying this condition.

We start with a partially ordered set $(X, \leq)$ and suppose that there is a G -metric on X so that (X, G) is a complete G -metric space. We induce partial ordering on $X \times X$ by demanding that, for any $(x, y), (u, v) \in X \times X$, $(x, y) \leq (u, v) \iff x \leq u, y \geq v$.

Theorem 16. Let $(X, \leq)$ be a partially ordered set and let G be a generalized metric on X such that (X, G) is a complete G -metric space. Suppose $T : X \times X \rightarrow X$ is a continuous mapping on X having the mixed monotone property. Suppose also that for all $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \leq (u, v) \leq (w, z)$

$$\begin{aligned} &G(T(x, y), T(u, v), T(w, z)) \\ &\leq \alpha \left(G(x, T(x, y), T(x, y)) G(u, T(u, v), T(u, v)) \right. \\ &\quad \times G(w, T(w, z), T(w, z)) \times \left([G(x, u, w)]^2 \right)^{-1} \Big) \\ &\quad + \beta G(x, u, w), \end{aligned} \quad (3)$$

where $8\alpha + \beta < 1$. If there exists $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, then T has a coupled fixed point $(x^*, y^*) \in X \times X$. That is, (x^*, y^*) satisfies $x^* = T(x^*, y^*)$, $y^* = T(y^*, x^*)$.

Proof. Suppose that there exists $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$. We write $x_1 = T(x_0, y_0)$, $y_1 = T(y_0, x_0)$ and define $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$,

$n \geq 1$. From the conditions of the theorem and the mixed monotone property it easily follows that

$$\begin{aligned} x_1 &= T(x_0, y_0) \geq x_0, \\ x_2 &= T(x_1, y_1) \geq T(x_0, y_0) = x_1, \dots \end{aligned} \quad (4)$$

This gives

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \dots \quad (5)$$

Similarly proceeding with

$$\begin{aligned} y_1 &= T(y_0, x_0) \leq y_0, \\ y_2 &= T(y_1, x_1) \leq T(y_0, x_0) = y_1, \dots, \end{aligned} \quad (6)$$

we find

$$y_0 \geq y_1 \geq y_2 \geq \dots \geq y_n \geq y_{n+1} \geq \dots \quad (7)$$

Considering the sequence $\{x_n\}$ and using (3) we have

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &= G(T(x_n, y_n), T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1})) \\ &\leq \alpha \frac{G(x_n, x_{n+1}, x_{n+1}) G(x_{n-1}, x_n, x_n) G(x_{n-1}, x_n, x_n)}{[G(x_n, x_{n-1}, x_{n-1})]^2} \\ &\quad + \beta G(x_n, x_{n-1}, x_{n-1}). \end{aligned} \quad (8)$$

Now using inequality (3) from Theorem 2, $G(x, y, y) \leq 2G(y, x, x)$. Setting $x = x_{n-1}$ and $y = x_n$ in this we find

$$G(x_{n-1}, x_n, x_n) \leq 2G(x_n, x_{n-1}, x_{n-1}). \quad (9)$$

Using this (8) becomes

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &\leq 4\alpha \frac{G(x_n, x_{n+1}, x_{n+1}) G(x_{n-1}, x_n, x_n) G(x_{n-1}, x_n, x_n)}{[G(x_{n-1}, x_n, x_n)]^2} \\ &\quad + \beta G(x_n, x_{n-1}, x_{n-1}) \\ &= 4\alpha G(x_n, x_{n+1}, x_{n+1}) + \beta G(x_n, x_{n-1}, x_{n-1}) \\ &\leq 8\alpha G(x_{n+1}, x_n, x_n) + \beta G(x_n, x_{n-1}, x_{n-1}), \end{aligned} \quad (10)$$

where in the last step we have used inequality (3) from Theorem 2. Rearranging and simplifying this we get

$$G(x_{n+1}, x_n, x_n) \leq \left(\frac{\beta}{1-8\alpha} \right) G(x_n, x_{n-1}, x_{n-1}). \quad (11)$$

Evidently the condition that (11) is contractive is $\beta/(1-8\alpha) < 1$, that is,

$$8\alpha + \beta < 1. \quad (12)$$

Similarly, considering $G(y_{n+1}, y_n, y_n) = G(T(y_n, x_n), T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1}))$ and arguing as above we find

$$G(y_{n+1}, y_n, y_n) \leq \left(\frac{\beta}{1-8\alpha} \right) G(y_n, y_{n-1}, y_{n-1}). \quad (13)$$

With condition (12), we find (13) is contractive.

Let $d_n = G(x_{n+1}, x_n, x_n)$ and $k = (\beta/(1-8\alpha))$. Using (11) we get

$$d_n \leq k d_{n-1} \leq k^2 d_{n-2} \leq \dots \leq k^n d_0. \quad (14)$$

If $d_0 = 0$, then $G(T(x_0, y_0), x_0, x_0) = G(x_1, x_0, x_0) = d_0 = 0$. But this means $T(x_0, y_0) = x_0$. Similarly writing $\delta_n = G(y_{n+1}, y_n, y_n)$ we find

$$\delta_n \leq k \delta_{n-1} \leq k^2 \delta_{n-2} \leq \dots \leq k^n \delta_0. \quad (15)$$

If $\delta_0 = 0$ in addition to $d_0 = 0$, we deduce similarly that $T(y_0, x_0) = y_0$. So if $d_0 = \delta_0 = 0$, (x_0, y_0) is a coupled fixed point.

However if $d_0 > 0$, for $m > n$, using inequality (5) of Definition 1 we get

$$\begin{aligned} G(x_m, x_n, x_n) &\leq G(x_{n+1}, x_n, x_n) + G(x_m, x_{n+1}, x_{n+1}) \\ &\leq G(x_{n+1}, x_n, x_n) + G(x_{n+2}, x_{n+1}, x_{n+1}) \\ &\quad + G(x_m, x_{n+2}, x_{n+2}) \\ &\vdots \\ &\leq G(x_{n+1}, x_n, x_n) + G(x_{n+2}, x_{n+1}, x_{n+1}) \\ &\quad + G(x_{n+3}, x_{n+2}, x_{n+2}) + G(x_m, x_{m-1}, x_{m-1}) \\ &= d_n + d_{n+1} + d_{n+2} + \dots + d_{m-1} \\ &\leq [k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1}] d_0 \\ &= k^n [1 + k + k^2 + \dots + k^{m-n-1}] d_0 \\ &< \frac{k^n}{1-k} d_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ since } k < 1. \end{aligned} \quad (16)$$

So $\{x_n\}$ is a G -Cauchy sequence in X . Next, considering $G(y_m, y_n, y_n)$ and arguing as above we can show that $\{y_n\}$ is also a G -Cauchy sequence in X . G -Completeness of (X, G) now implies that there are points $x^*, y^* \in X$ such that $x_n \rightarrow x^*$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$.

We next show that (x^*, y^*) is a coupled fixed point of T . Using the fact that T is continuous on X and G as a metric is continuous in each of its variables, we have

$$\begin{aligned} G(T(x^*, y^*), x^*, x^*) &= G\left(\lim_{n \rightarrow \infty} T(x_n, y_n), x^*, x^*\right) \\ &= G\left(\lim_{n \rightarrow \infty} x_{n+1}, x^*, x^*\right) \\ &= \lim_{n \rightarrow \infty} G(x_{n+1}, x^*, x^*) \\ &= 0 \quad \text{since } x_n \rightarrow x^* \text{ as } n \rightarrow \infty. \end{aligned} \quad (17)$$

But this means $T(x^*, y^*) = x^*$. Similarly by considering $G(T(y^*, x^*), y^*, y^*)$ and repeating the arguments used to derive (17) we can show that $T(y^*, x^*) = y^*$. This proves (x^*, y^*) is a coupled fixed point of T . $\square$

The $\mathcal{W}$ map was introduced by Chen [21] and a subclass of $\mathcal{W}$ functions was defined by Chakrabarti in [22].

Definition 17 (see [22]). We call $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a function of class $\mathcal{W}_\beta$ if there is a β such that $0 < \beta < 1$, and the following conditions are satisfied:

- (1) $\phi(t) \leq \beta t$ for all $t > 0$ and $\phi(0) = 0$,
- (2) $\lim_{t_n \rightarrow t} \inf \phi(t_n) \leq \beta t$ for all $t > 0$.

Using Definition 17 we obtain the following as a generalization of Theorem 16.

Theorem 18. Let $(X, \leq)$ be a partially ordered set and let G be a metric on X such that (X, G) is a complete G -metric space. Suppose $T : X \times X \rightarrow X$ is a continuous mapping on X having the mixed monotone property. For some given $0 < \alpha, \beta < 1$, let $\phi \in \mathcal{W}_\alpha$ and $\psi \in \mathcal{W}_\beta$ where $8\alpha + \beta < 1$. Suppose also that for all $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \leq (u, v) \leq (w, z)$

$$\begin{aligned} & G(T(x, y), T(u, v), T(w, z)) \\ & \leq \phi \left(G(x, T(x, y), T(x, y)), G(u, T(u, v), T(u, v)) \right. \\ & \quad \times G(w, T(w, z), T(w, z)) \left([G(x, u, w)]^2 \right)^{-1} \Big) \\ & \quad + \psi(G(x, u, w)). \end{aligned} \quad (18)$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, then T has a coupled fixed point $(x^*, y^*) \in X \times X$. That is, (x^*, y^*) satisfies $x^* = T(x^*, y^*)$, $y^* = T(y^*, x^*)$.

Proof. Since $\phi \in \mathcal{W}_\alpha$ and $\psi \in \mathcal{W}_\beta$, it follows from Definition 17 that $\phi(t) \leq \alpha t$ and $\psi(t) \leq \beta t$ for all $t > 0$. Inequality (18) now becomes equivalent to inequality (3) of Theorem 16 and the proof is immediate. $\square$

Example 19. Let $X = [0, \infty)$ and consider the function $G : X \times X \times X \rightarrow \mathbb{R}^+$ defined by

$$G(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases} \quad (19)$$

Then (X, G) is a complete G -metric space [23]. We define a partial order $\leq$ on X by the following: for any $x, y \in X$, $x \leq y$ if $x \geq y$. Also let $F : X \times X \rightarrow X$ be defined by

$$F(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Suppose $x, y, u, v, w, z \in X$ satisfy $z \leq v \leq y \leq x \leq u \leq w$ with nonzero x, u, w , but are otherwise arbitrary. Then we

have $w \leq u \leq x \leq y \leq v \leq z$. So the left side of (3) is $G(0, 0, 0) = 0$. The right side of (3)

$$\begin{aligned} & = \alpha \frac{G(x, 0, 0) G(u, 0, 0) G(w, 0, 0)}{[G(x, u, w)]^2} + \beta G(x, u, w) \\ & = \alpha \frac{xuw}{x^2} + \beta w \\ & = \frac{\alpha uw + \beta xw}{x} > 0 \quad \text{with } \alpha = \frac{1}{16}, \beta = \frac{1}{3}. \end{aligned} \quad (21)$$

If $x_0 = 0$ and $y_0 = 1$ then $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. So all conditions of Theorem 16 are satisfied. Easily we find that $(0, 1)$ is a coupled fixed point of F . Similarly $(1, 0)$ is a coupled fixed point.

In the next theorem, we provide conditions under which the coupled fixed point of the map T established in Theorem 16 is unique.

Theorem 20. Suppose that the conditions of Theorem 16 are valid. In addition suppose that for each $(x, y), (u, v) \in X \times X$ there is a $(w, z) \in X \times X$ which is comparable to (x, y) and (u, v) . Then T has a unique coupled fixed point.

Proof. Suppose that $(x^*, y^*), (x', y') \in X \times X$ are coupled fixed points.

Case 1. If (x^*, y^*) and (x', y') are comparable,

$$\begin{aligned} & G(T(x^*, y^*), T(x', y'), T(x', y')) \\ & \leq \alpha \left(G(x^*, T(x^*, y^*), T(x^*, y^*)) \right. \\ & \quad \times [G(x', T(x', y'), T(x', y'))]^2 \\ & \quad \times \left([G(x^*, x', x')]^2 \right)^{-1} \Big) \\ & \quad + \beta G(x^*, x', x') \\ & = \alpha \frac{G(x^*, x^*, x^*) [G(x', x', x')]^2}{[G(x^*, x', x')]^2} + \beta G(x^*, x', x') \\ & = \beta G(x^*, x', x'). \end{aligned} \quad (22)$$

This is equivalent to $G(x^*, x', x') \leq \beta G(x^*, x', x')$. However this is a contradiction since $\beta < 1$. So we must have $x^* = x'$. Similarly, considering $G(T(y^*, x^*), T(y', x'), T(y', x'))$ we easily show that $y^* = y'$. This shows that $(x^*, y^*) = (x', y')$, so the coupled fixed point is unique.

Case 2. If (x^*, y^*) and (x', y') are not comparable, by the condition of the theorem there is a $(u, v) \in X \times X$ comparable

to (x^*, y^*) and (x', y') . If there is a positive integer n_0 such that $T^{n_0}(u, v) = (x^*, y^*)$, then

$$\begin{aligned} T^{n_0}(u, v) &= (x^*, y^*) \\ T^{n_0+1}(u, v) &= T(x^*, y^*) = x^* \\ T^{n_0+2}(u, v) &= T^2(x^*, y^*) = T(T(x^*, y^*), T(y^*, x^*)) \\ &= T(x^*, y^*) = x^* \\ &\vdots \end{aligned} \quad (23)$$

So $T^n(u, v) = x^*$ for $n \geq n_0$ and hence $T^n(u, v) \rightarrow x^*$ as $n \rightarrow \infty$.

On the other hand, if no such n_0 exists, we have that, for any $n \geq 1$,

$$\begin{aligned} &G(T^n(u, v), x^*, x^*) \\ &= G(T^n(u, v), T^n(x^*, y^*), T^n(x^*, y^*)) \\ &\leq \alpha \left(G(T^{n-1}(u, v), T^n(u, v), T^n(u, v)) \right. \\ &\quad \times [G(T^{n-1}(x^*, y^*), T^n(x^*, y^*), T^n(x^*, y^*))]^2 \\ &\quad \times \left([G(T^{n-1}(u, v), T^{n-1}(x^*, y^*), \right. \\ &\quad \left. T^{n-1}(x^*, y^*))]^2 \right)^{-1} \\ &\quad \left. + \beta G(T^{n-1}(u, v), T^{n-1}(x^*, y^*), T^{n-1}(x^*, y^*)) \right), \end{aligned} \quad (24)$$

where we have used the fact that $T^n(x, y) = T(T^{n-1}(x, y), T^{n-1}(y, x))$ for any $(x, y) \in X \times X$. Since (x^*, y^*) is a coupled fixed point of T , $T^n(x^*, y^*) = x^*$ for all $n \geq 1$ and from (24), we now deduce that

$$\begin{aligned} &G(T^n(u, v), x^*, x^*) \\ &\leq \alpha \frac{G(T^{n-1}(u, v), T^n(u, v), T^n(u, v)) [G(x^*, x^*, x^*)]^2}{[G(T^{n-1}(u, v), x^*, x^*)]^2} \\ &\quad + \beta G(T^{n-1}(u, v), x^*, x^*) \\ &= \beta G(T^{n-1}(u, v), x^*, x^*) \\ &\vdots \\ &= \beta^{n-1} G(T(u, v), x^*, x^*) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (25)$$

since $\beta < 1$. This proves $T^n(u, v) \rightarrow x^*$ as $n \rightarrow \infty$. Similarly we can show that $T^n(v, u) \rightarrow y^*$ as $n \rightarrow \infty$. Replacing x^* with x' and y^* with y' and repeating the above arguments we can deduce $T^n(u, v) \rightarrow x'$ and $T^n(v, u) \rightarrow y'$ as $n \rightarrow \infty$. But this means $x^* = x'$ and $y^* = y'$. So the $(x^*, y^*) = (x', y')$ and the coupled fixed point is unique. $\square$

We next establish the conditions under which two maps $T : X \times X \rightarrow X$ and $g : X \rightarrow X$ have a coupled coincidence point.

Theorem 21. Let $(X, \leq)$ be a partially ordered set and let G be a metric on X such that (X, G) is a complete G -metric space. Let $T : X \times X \rightarrow X$ and $g : X \rightarrow X$ be continuous mapping on X such that T has the mixed g -monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commutes with T , and, for $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \leq (u, v) \leq (w, z)$ and $gx \leq gu \leq gw, gy \geq gv \geq gz$,

$$\begin{aligned} &G(T(x, y), T(u, v), T(w, z)) \\ &\leq \alpha \left(G(gx, T(x, y), T(x, y)) G(gu, T(u, v), T(u, v)) \right. \\ &\quad \times G(gw, T(w, z), T(w, z)) \left([G(gx, gu, gw)]^2 \right)^{-1} \\ &\quad \left. + \beta G(gx, gu, gw) \right), \end{aligned} \quad (26)$$

where $8\alpha + \beta < 1$. If there exists $x_0, y_0 \in X$ such that $gx_0 \leq T(x_0, y_0)$ and $gy_0 \geq T(y_0, x_0)$, then T has a coupled coincidence point $(x^*, y^*) \in X \times X$. That is (x^*, y^*) satisfies $gx^* = T(x^*, y^*)$, $gy^* = T(y^*, x^*)$.

Proof. Since $T(X \times X) \subseteq g(X)$ we can choose $x_1, y_1 \in X$ such that $gx_1 = T(x_0, y_0)$, $gy_1 = T(y_0, x_0)$. For similar reasons, $x_2, y_2 \in X$ can be found such that $gx_2 = T(x_1, y_1)$, $gy_2 = T(y_1, x_1)$. Due to the mixed g -monotone property of T , we have $gx_0 \leq gx_1 \leq gx_2$ and $gy_2 \leq gy_1 \leq gy_0$. In general, it can be shown that, [4] for $n \geq 0$,

$$\begin{aligned} gx_n &= T(x_{n-1}, y_{n-1}) \leq gx_{n+1} = T(x_n, y_n), \\ gy_{n+1} &= T(y_n, x_n) \leq gy_n = T(x_{n-1}, y_{n-1}). \end{aligned} \quad (27)$$

Now by the same arguments used to deduce (10) we have

$$\begin{aligned} &G(gx_{n+1}, gx_n, gx_n) \\ &\leq 4\alpha \frac{G(gx_n, gx_{n+1}, gx_{n+1}) [G(gx_{n-1}, gx_n, gx_n)]^2}{[G(gx_{n-1}, gx_n, gx_n)]^2} \\ &\quad + \beta G(gx_n, gx_{n-1}, gx_{n-1}) \\ &= 4\alpha G(gx_n, gx_{n+1}, gx_{n+1}) + \beta G(gx_n, gx_{n-1}, gx_{n-1}) \\ &\leq 8\alpha G(gx_{n+1}, gx_n, gx_n) \\ &\quad + \beta G(gx_n, gx_{n-1}, gx_{n-1}). \end{aligned} \quad (28)$$

This gives

$$G(gx_{n+1}, gx_n, gx_n) \leq \left(\frac{\beta}{1 - 8\alpha} \right) G(gx_n, gx_{n-1}, gx_{n-1}). \quad (29)$$

Since $8\alpha + \beta < 1$, $k = \beta/(1 - 8\alpha) < 1$. Writing $d_n = G(gx_{n+1}, gx_n, gx_n)$ we find from (28) that

$$d_n \leq kd_{n-1} \leq k^2 d_{n-2} \leq \dots \leq k^n d_0. \quad (30)$$

If $d_0 = 0$, then $G(gx_1, gx_0, gx_0) = G(T(x_0, y_0), gx_0, gx_0) = 0$ giving $gx_0 = T(x_0, y_0)$, so x_0 is a coincidence point. However if $d_0 > 0$, we have for $m > n$,

$$\begin{aligned}
 & G(gx_m, gx_n, gx_n) \\
 & \leq G(gx_{n+1}, gx_n, gx_n) + G(gx_m, gx_{n+1}, gx_{n+1}) \\
 & \leq G(gx_{n+1}, gx_n, gx_n) + G(gx_{n+2}, gx_{n+1}, gx_{n+1}) \\
 & \quad + G(gx_m, gx_{n+2}, gx_{n+2}) \\
 & \vdots \\
 & \leq G(gx_{n+1}, gx_n, gx_n) + G(gx_{n+2}, gx_{n+1}, gx_{n+1}) \\
 & \quad + \cdots + G(gx_m, gx_{m-1}, gx_{m-1}) \\
 & = d_n + d_{n+1} + d_{n+2} + \cdots + d_{m-1} \\
 & \leq [k^n + k^{n+1} + \cdots + k^{m-1}] d_0 \\
 & < \frac{k^n}{1-k} d_0 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty
 \end{aligned} \tag{31}$$

since $k < 1$. This shows that $\{gx_n\}$ is a Cauchy sequence in (X, G) and G -completeness of (X, G) ensures a point $x^* \in X$ such that $gx_n \rightarrow x^*$ as $n \rightarrow \infty$.

Replacing x_n by y_n for all $n \geq 0$ we get the analogue of (29):

$$G(gy_{n+1}, gy_n, gy_n) \leq \left(\frac{\beta}{1-8\alpha} \right) G(gy_n, gy_{n-1}, gy_{n-1}). \tag{32}$$

Next writing $\delta_n = G(gy_{n+1}, gy_n, gy_n)$ we find as in (30) that

$$\delta_n \leq k\delta_{n-1} \leq k^2\delta_{n-2} \leq \cdots \leq k^n\delta_0. \tag{33}$$

As before, if $\delta_0 = 0$, $G(gy_1, gy_0, gy_0) = G(T(y_0, x_0), gy_0, gy_0) = 0$, giving $T(y_0, x_0) = gy_0$ so that y_0 is a coincidence point. If in addition $d_0 = 0$, we have $gx_0 = T(x_0, y_0)$ and $gy_0 = T(y_0, x_0)$. So (x_0, y_0) is a coupled coincidence point.

Proceeding as in (31) we can show further that $\{gy_n\}$ is a Cauchy sequence in (X, G) and due to G -completeness of (X, G) there is a point $y^* \in X$ such that $gy_n \rightarrow y^*$ as $n \rightarrow \infty$.

Finally we prove that (x^*, y^*) is a coupled coincident point. Since T and g commute, we have

$$\begin{aligned}
 g(gx_{n+1}) &= g(T(x_n, y_n)) = T(gx_n, gy_n), \\
 g(gy_{n+1}) &= g(T(y_n, x_n)) = T(gy_n, gx_n).
 \end{aligned} \tag{34}$$

Taking limits as $n \rightarrow \infty$ in (34) and noting that T and g are, respectively, continuous on $X \times X$ and X , we get

$$\begin{aligned}
 gx^* &= \lim g(gx_{n+1}) = \lim g(T(x_n, y_n)) = \lim T(gx_n, gy_n), \\
 gy^* &= \lim g(gy_{n+1}) = \lim g(T(y_n, x_n)) = \lim T(gy_n, gx_n).
 \end{aligned} \tag{35}$$

Next we observe that G as metric is continuous in all its variables. This finally leads to

$$\begin{aligned}
 G(T(x^*, y^*), gx^*, gx^*) &= G(\lim T(gx_n, gy_n), gx^*, gx^*) \\
 &= G(gx^*, gx^*, gx^*) \\
 &= 0.
 \end{aligned} \tag{36}$$

So $gx^* = T(x^*, y^*)$. Similarly, we show that $gy^* = T(y^*, x^*)$. This proves (x^*, y^*) is a coupled coincidence point. $\square$

4. Conclusion

To summarize, we have introduced a rational type contractive condition in a G -metric space and proved some coupled fixed point theorems for maps satisfying mixed monotone property. We established the conditions for uniqueness of the coupled fixed point. Conditions for the existence of coupled coincidence points of two maps are also deduced.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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