Research Article

On Some New Generalized Difference Sequence Spaces of Nonabsolute Type

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We define a new triangle matrix $\widehat{W} = (w_{nk}^\lambda)$ by the composition of the matrices $\Lambda = (\lambda_{nk})$ and $B(r, s, t)$. Also, we introduce the sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ by using matrix domain of the matrix $\widehat{W}$ on the classical sequence spaces c_0 , c , ℓ_∞ , and ℓ_p , respectively, where $1 \leq p < \infty$. Moreover, we show that the space $\mu^\lambda(\widehat{B})$ is norm isomorphic to μ for $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$. Furthermore, we establish some inclusion relations concerning those spaces and determine α -, β -, and γ -duals of those spaces and construct the Schauder bases $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$. Finally, we characterize the classes $(\mu_1^\lambda(\widehat{B}) : \mu_2)$ of infinite matrices where $\mu_1 \in \{c, c_0, \ell_p\}$ and $\mu_2 \in \{\ell_\infty, c, c_0, \ell_p\}$.

1. Preliminaries, Background, and Notation

By a sequence space, we understand a linear subspace of the space ω of all complex sequences which contains ϕ , the set of all finitely nonzero sequences. We write ℓ_∞ , c , c_0 , and ℓ_p for the classical sequence spaces of all bounded, convergent, null, and absolutely p -summable sequences, respectively, where $1 \leq p < \infty$. Also by bs and cs , we denote the spaces of all bounded and convergent series, respectively. We assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to zero. We denote throughout that the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Let $A = (a_{nk})$ be an infinite matrix and X, Y two sequence spaces. Then, A defines a matrix mapping from X to Y and is denoted by $A : X \rightarrow Y$ if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n \in \mathbb{N}}$, the A -transform of x , is in Y , where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \quad (1)$$

By $(X : Y)$, denote the class of all matrices A such that $A : X \rightarrow Y$. Thus, $A \in (X : Y)$ if and only if the series on the right hand side of (1) converges for each $n \in \mathbb{N}$ and $x \in X$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence $x \in \omega$ is said to be A -summable to l if Ax converges to l , which is called the A -limit of x .

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x . Further, a triangle matrix U has a unique inverse $U^{-1} = V$ which is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Let $q = (q_k)$ be a sequence of positive reals and write

$$Q_n = \sum_{k=0}^n q_k, \quad (n \in \mathbb{N}). \quad (2)$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q = (q_k)$, and Euler mean of order r with $0 <$

$r < 1$ are, respectively, defined by the matrices $C = (c_{nk})$, $R^q = (r_{nk}^q)$, and $E^r = (e_{nk}^r)$, where

$$\begin{aligned} c_{nk} &= \begin{cases} \frac{1}{n+1}, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \\ r_{nk}^q &= \begin{cases} \frac{q_k}{Q_n}, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \\ e_{nk}^r &= \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k, & (0 \leq k \leq n), \\ 0, & (k > n) \end{cases} \end{aligned} \quad (3)$$

for all $k, n \in \mathbb{N}$. We write $\mathcal{U}$ for the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_k)$. Let $z, u, v \in \mathcal{U}$, and define the summation matrix $S = (s_{nk})$, the difference matrix $\Delta = (\Delta_{nk}^{(1)})$, and the generalized weighted mean or factorable matrix $G(u, v) = (g_{nk})$, $A_u^r = \{a_{nk}^r(u)\}$, $\Delta^{(m)} = (\Delta_{nk}^{(m)})$ by

$$\begin{aligned} s_{nk} &= \begin{cases} 1, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \\ \Delta_{nk}^{(1)} &= \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n), \\ 0, & (0 \leq k < n-1 \text{ or } k > n), \end{cases} \\ g_{nk} &= \begin{cases} u_n v_k, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \\ a_{nk}^r(u) &= \begin{cases} \frac{1+r^k}{n+1} u_k, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \\ \Delta_{nk}^{(m)} &= \begin{cases} (-1)^{n-k} \binom{m}{n-k}, & (\max\{0, n-m\} \leq k \leq n), \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases} \end{aligned} \quad (4)$$

for all $k, n \in \mathbb{N}$, where u_n and v_k depend only on n and k , respectively. Let r and s be nonzero real numbers, and define the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ by

$$b_{nk}(r, s) = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & (0 \leq k < n-1 \text{ or } k > n), \end{cases} \quad (5)$$

for all $k, n \in \mathbb{N}$. We note that if we choose $r = 1$ and $s = -1$ then the matrix $B(r, s)$ is reduced to the backward difference.

For a sequence space X , the matrix domain X_A of an infinite matrix A in the space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}, \quad (6)$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic; that is, $X_A \cong X$.

Following Başar [1, page 51], we note that although in the most cases the new sequence space X_A generated by the limitation matrix A from a sequence space X is the expansion or the contraction of the original space X , it may be observed in some cases that those spaces overlap. Indeed, one can easily see that the inclusion, $X_S \subset X$, strictly holds for $X \in \{\ell_\infty, c, c_0\}$. As this, one can deduce that the inclusion $X \subset X_{\Delta^{(1)}}$ also strictly holds for $X \in \{\ell_\infty, c, c_0, \ell_p\}$. However, if we define $X = c_0 \oplus \text{span}\{z\}$ with $z = ((-1))^k$, that is, $x \in X$ if and only if $x := \beta + \alpha z$ for some $\beta \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n = (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in X$ but $Az = e \notin X$ which lead us to the consequences that $z \in X \setminus X_A$ and $e \in X_A \setminus X$, where $e = (1, 1, 1, \dots)$ and $e^{(n)}$ is a sequence whose only nonzero term is a 1 in n th place for each $n \in \mathbb{N}$. That is to say, the sequence spaces X_A and X overlap but neither contains the other.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Wang [2], Ng and Lee [3], Malkowsky [4], Altay and Başar [5], Malkowsky and Savaş [6], Başarır [7], Aydın and Başar [8], Başar et al. [9], Şengönül and Başar [10], Altay [11], Polat and Başar [12], and Malkowsky et al. [13]. Δ , Δ^2 and Δ^m are the transposes of the matrices $\Delta^{(1)}$, $\Delta^{(2)}$, and $\Delta^{(m)}$, respectively, and $c_0(u, p)$ are the spaces consisting of the sequences $x = (x_k)$ such that $ux = (u_k x_k)$ in the spaces $c_0(p)$ and $c(p)$ for $u \in \mathcal{U}$, respectively, studied by Başarır [7]. More recently, the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ has been used by Kirişçi and Başar [14] for generalizing the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$, and bv_p . Finally, the new technique for deducing certain topological properties, for example, AB -, KB -, and AD -properties, and determining the β - and γ -duals of the domain of a triangle matrix in a sequence space has been given by Altay and Başar [15].

Let r, s , and t be nonzero real numbers, and define the generalized difference matrix $\tilde{B} = B(r, s, t) = \{b_{nk}(r, s, t)\}$ by

$$b_{nk}(r, s, t) = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ t, & (k = n-2), \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases} \quad (7)$$

for all $n, k \in \mathbb{N}$. The inverse $B^{-1}(r, s, t) = \{d_{nk}(r, s, t)\}$ of $B(r, s, t) = \{b_{nk}(r, s, t)\}$ is given by

$$\begin{aligned} d_{nk}(r, s, t) &= \begin{cases} \frac{1}{r} \sum_{v=0}^{n-k} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{n-k-v} \\ \quad \times \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^v, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases} \end{aligned} \quad (8)$$

We should record here that $B(r, s, 0) = B(r, s)$, $B(1, -2, 1) = \Delta^{(2)}$, and $B(1, -1, 0) = \Delta^{(1)}$. So, the results related to the matrix domain of the triple band matrix $B(r, s, t)$ are more

general and more comprehensive than the consequences on the matrix domain of $B(r, s)$, $\Delta^{(2)}$ and $\Delta^{(1)}$ and include them. We assume throughout that $\lambda = (\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to ∞ ; that is

$$0 < \lambda_0 < \lambda_1 < \cdots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty. \quad (9)$$

The main purpose of the present paper is to introduce the sequence spaces $\mu^\lambda(\widehat{B})$ and to determine the α -, β -, and γ -duals of this space, where μ denotes throughout any of the classical spaces ℓ_∞, c, c_0 , or ℓ_p , and $\widehat{B}$ is the triple band matrix $B(r, s, t)$ and the sequence $\lambda = (\lambda_k)$ is defined in (9). Furthermore, the Schauder bases for the spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$ and $\ell_p^\lambda(\widehat{B})$ are given, and some topological properties of the spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ are examined. Finally, some classes of matrix mappings on the spaces $\mu^\lambda(\widehat{B})$ are characterized.

The paper is organized as follows. In Section 2, the BK -spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ of generalized difference sequence spaces are introduced and the Schauder bases of the spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ are given. In Section 3, some inclusion relations concerning these spaces are examined. In Section 4, the α -, β -, and γ -duals of the generalized difference sequence spaces $\mu^\lambda(\widehat{B})$ of nonabsolute type are determined. In Section 5, the classes $(\mu_1^\lambda(\widehat{B}) : \mu_2)$ of infinite matrices are characterized, where $\mu_1 \in \{c, c_0, \ell_p\}$ and $\mu_2 \in \{\ell_\infty, c, c_0, \ell_p\}$.

2. The Difference Sequence Spaces $\mu^\lambda(\widehat{B})$ of Nonabsolute Type

The difference sequence spaces have been studied by several authors in different ways [16–34]. In the present section, we introduce the spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ and show that these spaces are BK -spaces of nonabsolute type which are norm isomorphic to the spaces c_0, c, ℓ_∞ , and ℓ_p , respectively. Furthermore, we give the bases of the spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$.

We say that a sequence $x = (x_k) \in \omega$ is λ -convergent to the number $l \in \mathbb{C}$, called the λ -limit of x , if $(\Lambda x)_n \rightarrow l$ as $n \rightarrow \infty$ where

$$(\Lambda x)_n = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k; \quad (n \in \mathbb{N}). \quad (10)$$

In particular, we say that x is a λ -null sequence if $(\Lambda x)_n \rightarrow 0$ as $n \rightarrow \infty$. Further, we say that x is λ -bounded if $\sup_{n \in \mathbb{N}} |(\Lambda x)_n| < \infty$, [35]. Recently, Mursaleen and Noman

[35, 36] studied the sequence spaces c_0^λ , c^λ , ℓ_∞^λ , and ℓ_p^λ of nonabsolute type as follows:

$$\begin{aligned} c_0^\lambda &= \left\{ x = (x_k) \in \omega : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k = 0 \right\}, \\ c^\lambda &= \left\{ x = (x_k) \in \omega : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \text{ exists} \right\}, \\ \ell_\infty^\lambda &= \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}, \\ \ell_p^\lambda &= \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}. \end{aligned} \quad (11)$$

On the other hand, we define the matrix $\Lambda = (\lambda_{nk})$ for all $n, k \in \mathbb{N}$ by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & (0 \leq k \leq n) \\ 0, & (k > n). \end{cases} \quad (12)$$

Then, the space μ^λ can be restated with the notation of (6) that

$$c_0^\lambda = \{c_0\}_\Lambda, \quad c^\lambda = c_\Lambda, \quad \ell_\infty^\lambda = \{\ell_\infty\}_\Lambda, \quad \ell_p^\lambda = \{\ell_p\}_\Lambda. \quad (13)$$

More recently, Sönmez [37] has defined the sequence spaces $\ell_\infty(\widehat{B})$, $c(\widehat{B})$, $c_0(\widehat{B})$, and $\ell_p(\widehat{B})$ as follows:

$$\begin{aligned} \ell_\infty(\widehat{B}) &= \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |rx_k + sx_{k-1} + tx_{k-2}| < \infty \right\}, \\ c(\widehat{B}) &= \left\{ x = (x_k) \in \omega : \right. \\ &\quad \left. \exists l \in \mathbb{C} \ni \lim_{k \rightarrow \infty} |rx_k + sx_{k-1} + tx_{k-2} - l| = 0 \right\}, \\ c_0(\widehat{B}) &= \left\{ x = (x_k) \in \omega : \lim_{k \rightarrow \infty} |rx_k + sx_{k-1} + tx_{k-2}| = 0 \right\}, \\ \ell_p(\widehat{B}) &= \left\{ x = (x_k) \in \omega : \sum_k |rx_k + sx_{k-1} + tx_{k-2}|^p < \infty \right\}. \end{aligned} \quad (14)$$

In fact, the sequence spaces $\ell_\infty(\widehat{B})$, $c(\widehat{B})$, $c_0(\widehat{B})$, and $\ell_p(\widehat{B})$ can be considered as the set of all sequences whose $\widehat{B}(r, s, t)$ -transforms are in the spaces ℓ_∞, c, c_0 , and ℓ_p , respectively. That is,

$$\begin{aligned} \ell_\infty(\widehat{B}) &= \{\ell_\infty\}_{B(r,s,t)}, & c(\widehat{B}) &= c_{B(r,s,t)}, \\ c_0(\widehat{B}) &= \{c_0\}_{B(r,s,t)}, & \ell_p(\widehat{B}) &= \{\ell_p\}_{B(r,s,t)}. \end{aligned} \quad (15)$$

Now, we introduce the difference sequence spaces $\ell_\infty^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $c_0^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ as follows:

$$\begin{aligned} \ell_\infty^\lambda(\widehat{B}) &= \left\{ x = (x_n) \in \omega : \right. \\ &\quad \left. \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \right. \right. \\ &\quad \left. \left. \times (rx_k + sx_{k-1} + tx_{k-2}) \right| < \infty \right\}, \\ c^\lambda(\widehat{B}) &= \left\{ x = (x_n) \in \omega : \right. \\ &\quad \left. \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \right. \\ &\quad \left. \times (rx_k + sx_{k-1} + tx_{k-2}) \text{ exists} \right\}, \\ c_0^\lambda(\widehat{B}) &= \left\{ x = (x_n) \in \omega : \right. \\ &\quad \left. \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \right. \\ &\quad \left. \times (rx_k + sx_{k-1} + tx_{k-2}) = 0 \right\}, \\ \ell_p^\lambda(\widehat{B}) &= \left\{ x = (x_n) \in \omega : \right. \\ &\quad \left. \sum_n \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \right. \right. \\ &\quad \left. \left. \times (rx_k + sx_{k-1} + tx_{k-2}) \right|^p < \infty \right\}. \end{aligned} \quad (16)$$

Now, we define the triangle matrix $\widehat{W} = (w_{nk}^\lambda) = \Lambda \widehat{B}$; that is,

$$w_{nk}^\lambda = \begin{cases} (r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})) \times (\lambda_n)^{-1}, & (k < n-1), \\ \frac{r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1})}{\lambda_n}, & (k = n-1), \\ \frac{r(\lambda_n - \lambda_{n-1})}{\lambda_n}, & (k = n), \\ 0, & (\text{otherwise}) \end{cases} \quad (17)$$

for all $k, n \in \mathbb{N}$. Further, for any sequence $x = (x_k)$ we define the sequence $y = \{y_k(\lambda)\}$ which will be used, as the $\widehat{W}$ -transform of x ; that is,

$$\begin{aligned} y_k(\lambda) &= \sum_{j=0}^{k-2} \frac{r(\lambda_j - \lambda_{j-1}) + s(\lambda_{j+1} - \lambda_j) + t(\lambda_{j+2} - \lambda_{j+1})}{\lambda_k} x_j \\ &\quad + \frac{r(\lambda_{k-1} - \lambda_{k-2}) + s(\lambda_k - \lambda_{k-1})}{\lambda_k} x_{k-1} \\ &\quad + \frac{r(\lambda_k - \lambda_{k-1})}{\lambda_k} x_k; \quad (k \in \mathbb{N}). \end{aligned} \quad (18)$$

In fact, the sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ can be considered as the set of all sequences whose $\widehat{W}$ -transforms are in the spaces c_0 , c , ℓ_∞ , and ℓ_p , respectively. That is,

$$\begin{aligned} c_0^\lambda(\widehat{B}) &= \{c_0\}_{\widehat{W}}, & c^\lambda(\widehat{B}) &= c_{\widehat{W}}, \\ \ell_\infty^\lambda(\widehat{B}) &= \{\ell_\infty\}_{\widehat{W}}, & \ell_p^\lambda(\widehat{B}) &= \{\ell_p\}_{\widehat{W}}. \end{aligned} \quad (19)$$

Since the proof may also be obtained in a similar way as the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces. Now, we may begin with the following theorem which is essential in the study.

Theorem 1. (i) The difference sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, and $\ell_\infty^\lambda(\widehat{B})$ are BK-spaces with the norm $\|x\|_{c_0^\lambda(\widehat{B})} = \|x\|_{c^\lambda(\widehat{B})} = \|x\|_{\ell_\infty^\lambda(\widehat{B})} = \|\widehat{W}x\|_\infty$; that is,

$$\|x\|_{c_0^\lambda(\widehat{B})} = \|x\|_{c^\lambda(\widehat{B})} = \|x\|_{\ell_\infty^\lambda(\widehat{B})} = \sup_{n \in \mathbb{N}} |(\widehat{W}x)_n|. \quad (20)$$

(ii) Let $1 \leq p < \infty$. Then $\ell_p^\lambda(\widehat{B})$ is a BK-space with the norm $\|x\|_{\ell_p^\lambda(\widehat{B})} = \|\widehat{W}x\|_{\ell_p}$; that is,

$$\|x\|_{\ell_p^\lambda(\widehat{B})} = \left(\sum_n |(\widehat{W}x)_n|^p \right)^{1/p}. \quad (21)$$

Proof. Since (19) holds and c_0 , c , ℓ_∞ , and ℓ_p are BK-spaces with respect to their natural norms (see [1, pages 16-17]) and the matrix $\widehat{W}$ is a triangle, Theorem 4.3.12 of Wilansky [38, page 63] gives the fact that $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ are BK-spaces with the given norms. This completes the proof. $\square$

Remark 2. Let $|x| = (|x_k|)$. Then the absolute property does not hold on the space $\mu^\lambda(\widehat{B})$; that is, $\|x\|_{\mu^\lambda(\widehat{B})} \neq \| |x| \|_{\mu^\lambda(\widehat{B})}$. This can be shown for at least one sequence in those spaces. Hence $\mu^\lambda(\widehat{B})$ is the sequence space of nonabsolute type.

Theorem 3. The sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ of nonabsolute type are norm isomorphic to the spaces c_0 , c , ℓ_∞ , and ℓ_p , respectively; that is, $c_0^\lambda(\widehat{B}) \cong c_0$, $c^\lambda(\widehat{B}) \cong c$, $\ell_\infty^\lambda(\widehat{B}) \cong \ell_\infty$, and $\ell_p^\lambda(\widehat{B}) \cong \ell_p$.

Proof. We prove the theorem for the space $c_0^\lambda(\widehat{B})$. To prove our assertion we should show the existence of a linear bijection between the spaces $c_0^\lambda(\widehat{B})$ and c_0 . Let $T : c_0^\lambda(\widehat{B}) \rightarrow c_0$ be defined by (18). Then, $Tx = y = \widehat{W}x \in c_0$ for every $x \in c_0^\lambda(\widehat{B})$ and the linearity of T is clear. Further, it is trivial that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Moreover, let $y = (y_k) \in c_0$ and we define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=0}^k \frac{1}{r} \sum_{v=0}^{k-j} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{k-j-v} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^v \times \sum_{i=j-1}^j (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i. \quad (22)$$

Then we obtain

$$\begin{aligned} rx_k(\lambda) + sx_{k-1}(\lambda) + tx_{k-2}(\lambda) \\ = \sum_{j=k-1}^k (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} y_j, \quad \forall k \in \mathbb{N}. \end{aligned} \quad (23)$$

Hence, we get for every $n \in \mathbb{N}$ that

$$\begin{aligned} (\widehat{W}x)_n &= \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) (rx_k + sx_{k-1} + tx_{k-2}) \\ &= \frac{1}{\lambda_n} \sum_{k=0}^n \sum_{j=k-1}^k (-1)^{k-j} \lambda_j y_j = y_n. \end{aligned} \quad (24)$$

This shows that $\widehat{W}x = y$ and since $y \in c_0$, we conclude that $\widehat{W}x \in c_0$. Thus, we deduce that $x \in c_0^\lambda(\widehat{B})$ and $Tx = y$. Hence T is surjective.

Moreover one can easily see for every $x \in c_0^\lambda(\widehat{B})$ that

$$\|Tx\|_\infty = \|y\|_\infty = \|\widehat{W}x\|_\infty = \|x\|_{c_0^\lambda(\widehat{B})} \quad (25)$$

which means that T is norm preserving. Consequently T is a linear bijection which shows that the spaces $c_0^\lambda(\widehat{B})$ and c_0 are linearly isomorphic, as desired. $\square$

Let $(X, \|\cdot\|)$ be a normed space. A sequence (b_k) of points of X is called a Schauder basis for X if and only if, for each $x \in X$, there exists a unique sequence (α_k) of scalars such that $x = \sum_k \alpha_k b_k$; that is,

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{k=0}^n \alpha_k b_k \right\| = 0. \quad (26)$$

Because of the isomorphism T , which is defined in the proof of Theorem 3, the inverse image of the basis of the spaces c_0, c , and ℓ_p are the basis of new spaces $c_0^\lambda(\widehat{B}), c^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$, respectively. Therefore, we have the following.

Theorem 4. Let $\alpha_k(\lambda) = (\widehat{W}x)_k$ for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} (\widehat{W}x)_k = l$. Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n \in \mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(\lambda) = \begin{cases} \frac{d_{nk}(r, s, t) \lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{d_{n,k+1}(r, s, t) \lambda_k}{\lambda_{k+1} - \lambda_k}, & (n > k) \\ \frac{1}{r} \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})}, & (n = k) \\ 0, & (n < k). \end{cases} \quad (27)$$

Then, the following statements hold.

- (i) The sequence $\{b^{(k)}(\lambda)\}_{k \in \mathbb{N}}$ is a basis for the spaces $c_0^\lambda(\widehat{B})$ and $\ell_p^\lambda(\widehat{B})$; any $x \in c_0^\lambda(\widehat{B})$ or $x \in \ell_p^\lambda(\widehat{B})$ has a unique representation of the form $x = \sum_k \alpha_k(\lambda) b^{(k)}(\lambda)$.
- (ii) The sequence $\{b, b^{(0)}(\lambda), b^{(1)}(\lambda), b^{(2)}(\lambda), \dots\}$ is a basis for the space $c^\lambda(\widehat{B})$ and any $x \in c^\lambda(\widehat{B})$ has a unique representation of the form $x = lb + \sum_k [\alpha_k(\lambda) - l] b^{(k)}(\lambda)$, where $b = (b_k) = (\sum_{j=0}^k d_{kj})$.

3. The Inclusion Relations

In the present section, we prove some inclusion relations concerning the spaces $c_0^\lambda(\widehat{B}), c^\lambda(\widehat{B}), \ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$.

Theorem 5. The inclusion $c_0^\lambda(\widehat{B}) \subset c^\lambda(\widehat{B})$ strictly holds.

Proof. It is obvious that the inclusion $c_0^\lambda(\widehat{B}) \subset c^\lambda(\widehat{B})$ holds. Further to show that this inclusion is strict, consider the sequence $x = (x_k)$ defined by

$$x_k = \sum_{j=0}^k d_{kj} \quad (28)$$

for all $k \in \mathbb{N}$. Then

$$rx_k + sx_{k-1} + tx_{k-2} = \sum_{j=k-1}^k (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}}. \quad (29)$$

Then one can easily see that $\widehat{W}x = e \in c \setminus c_0$, where $e = (1, 1, 1, \dots)$. Thus, this sequence x is in $c^\lambda(\widehat{B})$ but not in $c_0^\lambda(\widehat{B})$. Hence, the inclusion $c_0^\lambda(\widehat{B}) \subset c^\lambda(\widehat{B})$ is strict and this completes the proof of the theorem. $\square$

Theorem 6. If $r + s + t = 0$ then the inclusion $c \subset c_0^\lambda(\widehat{B})$ strictly holds.

Proof. Suppose that $r + s + t = 0$ and $x \in c$. Then $\widehat{B}x = (rx_k + sx_{k-1} + tx_{k-2}) \in c_0$ and hence $\widehat{B}x \in c_0^\lambda$ since the inclusion $c_0 \subset c_0^\lambda$ holds [35]. This shows that $x \in c_0^\lambda(\widehat{B})$; that is, $c \subset c_0^\lambda(\widehat{B})$. Further consider the sequence $y = (y_k)$ defined by $y_k = \ln(k+3)$ for all $k \in \mathbb{N}$. Then it is trivial that $y \notin c$. On the other hand, it can easily be seen that $\widehat{B}y \in c_0$. Hence, $y \in c_0^\lambda(\widehat{B})$. Thus the sequence $y \in c_0^\lambda(\widehat{B}) \setminus c$. Hence, the inclusion $c \subset c_0^\lambda(\widehat{B})$ is strict. This completes the proof. $\square$

Lemma 7. $A \in (\ell_\infty : c_0)$ if and only if $\lim_{n \rightarrow \infty} \sum_k |a_{nk}| = 0$.

Theorem 8. Let $z = (z_k)$ be defined by

$$z_k = \left| \frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})}{\lambda_k - \lambda_{k-1}} \right| \quad (30)$$

for all $k \in \mathbb{N}$. Then the inclusion $\ell_\infty \subset c_0^\lambda(\widehat{B})$ strictly holds if and only if $z \in c_0^\lambda$.

Proof. Let ℓ_∞ be a subset of $c_0^\lambda(\widehat{B})$. Then we obtain that $\widehat{W}x \in c_0$ for every $x \in \ell_\infty$ and the matrix $\widehat{W} = (w_{nk}^\lambda)$ is in the class $(\ell_\infty : c_0)$. By using Lemma 7 it follows that

$$\lim_{n \rightarrow \infty} \sum_k |w_{nk}^\lambda| = 0. \quad (31)$$

Now, by taking into account the definition of matrix $\widehat{W} = (w_{nk}^\lambda)$ given by (17), we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} \sum_k |w_{nk}^\lambda| &= \frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})| \\ &\quad + \left| \frac{r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1})}{\lambda_n} \right| + |r| \frac{(\lambda_n - \lambda_{n-1})}{\lambda_n}. \end{aligned} \quad (32)$$

By Lemma 7, we have that

$$\lim_{n \rightarrow \infty} |r| \frac{(\lambda_n - \lambda_{n-1})}{\lambda_n} = 0, \quad (33)$$

$$\lim_{n \rightarrow \infty} \left| \frac{r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1})}{\lambda_n} \right| = 0, \quad (34)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \\ \times \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})| &= 0. \end{aligned} \quad (35)$$

Now, we have for every $n \geq 1$ that

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})| \\ = \frac{\lambda_{n-2}}{\lambda_n} \left[\frac{1}{\lambda_{n-2}} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1}) z_k \right] \end{aligned} \quad (36)$$

and since $\lim_{n \rightarrow \infty} (\lambda_{n-2}/\lambda_n) = 1$ by (33) and (34), we obtain that by (35)

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_{n-2}} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1}) z_k = 0 \quad (37)$$

which shows that $z = (z_k) \in c_0^\lambda$ where the sequence $z = (z_k)$ is defined by

$$z_k = \left| \frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})}{\lambda_k - \lambda_{k-1}} \right| \quad (38)$$

for all $k \in \mathbb{N}$.

Conversely, we suppose that $z = (z_k) \in c_0^\lambda$. Then we have (37). Further, for every $n \geq 2$, we derive that

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})| \\ = \frac{1}{\lambda_n} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1}) z_k \leq \frac{1}{\lambda_{n-2}} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1}) z_k. \end{aligned} \quad (39)$$

Then, (37) and (39) together imply that (35) holds. On the other hand, we have for every $n \geq 2$ that

$$\begin{aligned} \left| \frac{r\lambda_{n-2} + s(\lambda_{n-1} - \lambda_0) + t(\lambda_n - \lambda_1)}{\lambda_n} \right| \\ = \left| \frac{1}{\lambda_n} \sum_{k=0}^{n-2} r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1}) \right| \\ \leq \frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})|. \end{aligned} \quad (40)$$

Therefore, it follows by (35)

$$\lim_{n \rightarrow \infty} \frac{r\lambda_{n-2} + s(\lambda_{n-1} - \lambda_0) + t(\lambda_n - \lambda_1)}{\lambda_n} = 0. \quad (41)$$

Particularly if (i) $r = 0, s = -1, t = 1$, then we obtain $\lim_{n \rightarrow \infty} [(\lambda_n - \lambda_{n-1})/\lambda_n] = 0$, which shows that (33) holds. (ii) $r = -1, s = 1, t = 0$, and then we obtain $\lim_{n \rightarrow \infty} [(\lambda_{n-1} - \lambda_{n-2})/\lambda_n] = 0$, which shows that (34) holds. Thus, we deduce by relation (32) that (31) holds. This leads us with Lemma 7 to the consequence that $\widehat{W} \in (\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty \subset c_0^\lambda(\widehat{B})$ holds. Finally, it is obvious that the sequence y , defined in the proof of Theorem 6, is in $c_0^\lambda(\widehat{B})$ but not in ℓ_∞ , so the inclusion $\ell_\infty \subset c_0^\lambda(\widehat{B})$ is strict. $\square$

Theorem 9. The inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{B})$ strictly holds.

Proof. To prove the validity of the inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{B})$, it suffices to show that, for every $x \in \ell_\infty^\lambda(\widehat{B})$, there exists a positive real number K such that $\|x\|_{\ell_\infty^\lambda(\widehat{B})} \leq K\|x\|_{\ell_\infty}$. Let $x \in \ell_\infty$. Then, we have

$$\begin{aligned} \|x\|_{\ell_\infty^\lambda(\widehat{B})} &= \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) (rx_k + sx_{k-1} + tx_{k-2}) \right| \\ &\leq \sup_{0 \leq k \leq n} |rx_k + sx_{k-1} + tx_{k-2}| \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \right| \\ &\leq \max\{|r| + |s| + |t|\} \|x\|_{\ell_\infty} \end{aligned} \quad (42)$$

so that $x \in \ell_\infty^\lambda(\widehat{B})$ and hence $\ell_\infty \subset \ell_\infty^\lambda(\widehat{B})$. Furthermore, we consider $x = (x_k)$ defined by $x_k = \sum_{j=0}^k d_{kj}$ for all $k \in \mathbb{N}$. Then we have $\widehat{W}x = e$. Thus, we deduce that $\|x\|_{\ell_\infty^\lambda(\widehat{B})} = \|\widehat{W}x\|_\infty = 1$. Hence $x = (x_k) \in \ell_\infty^\lambda(\widehat{B})$.

On the other hand, we know from Theorem 8 that the inclusion $\ell_\infty \subset \ell_0^\lambda(\widehat{B})$ is strict. Since $\ell_0^\lambda(\widehat{B}) \subset \ell_\infty^\lambda(\widehat{B})$, the inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{B})$ strictly holds. $\square$

4. The α -, β -, and γ -Duals of the Spaces $\mu^\lambda(\widehat{B})$ of Nonabsolute Type

In this section, we determine the α -, β -, and γ -duals of the generalized difference sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$ of nonabsolute type.

We will firstly give the definition of α -, β -, and γ -duals of sequence spaces and secondly we give the lemmas due to [39] which are needed in proving the theorems given in Sections 4 and 5.

For the sequence spaces λ and μ , define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \forall x \in \lambda\}. \quad (43)$$

With the notation of (43), the α -, β -, and γ -duals λ^α , λ^β , and λ^γ of a sequence space λ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs), \quad \lambda^\gamma = S(\lambda, bs). \quad (44)$$

Lemma 10. (i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty. \quad (45)$$

(ii) $A = (a_{nk}) \in (\ell_p : \ell_1)$ if and only if

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} \right|^q < \infty. \quad (46)$$

Lemma 11. $A \in (c_0 : c)$ if and only if

$$\lim_{n \rightarrow \infty} a_{nk} = \alpha_k, \quad \text{for each fixed } k \in \mathbb{N}, \quad (47)$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \quad (48)$$

Lemma 12. $A \in (c : c)$ if and only if (47) and (48) hold, and

$$\lim_{n \rightarrow \infty} \sum_k a_{nk} \text{ exists.} \quad (49)$$

Lemma 13. $A \in (\ell_\infty : c)$ if and only if (47) holds and

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk}| = \sum_k |\alpha_k|. \quad (50)$$

Lemma 14. $A = (a_{nk}) \in (\ell_p : c)$ if and only if (47) holds and

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty. \quad (51)$$

Lemma 15. $A \in (c : \ell_\infty) = (c_0 : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (48) holds.

Lemma 16. $A = (a_{nk}) \in (\ell_p : \ell_\infty)$ if and only if (51) holds.

Now we consider the following sets:

$$f_1^\lambda = \left\{ a = (a_n) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} f_{nk}^\lambda \right| < \infty \right\},$$

$$f_2^\lambda = \left\{ a = (a_n) \in w : \sum_{j=k}^\infty d_{jk} a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$f_3^\lambda = \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\widehat{g}_k(n)|^q < \infty \right\},$$

$$f_4^\lambda = \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n \right| < \infty \right\},$$

$$f_5^\lambda = \left\{ a = (a_n) \in w : \lim_{n \rightarrow \infty} \sum_{k=0}^n \left[\sum_{j=0}^k d_{kj} \right] a_k \text{ exists} \right\},$$

$$f_6^\lambda = \left\{ a = (a_n) \in w : \sup_{N \in \mathcal{F}} \sum_{k=0}^\infty \left| \sum_{n \in N} f_{nk}^\lambda \right|^q < \infty \right\},$$

$$f_7^\lambda = \left\{ a = (a_n) \in w : \lim_{n \rightarrow \infty} \sum_k |v_{nk}^\lambda| = \sum_k \left| \lim_{n \rightarrow \infty} v_{nk}^\lambda \right| \right\}, \quad (52)$$

where the matrices $F^\lambda = (f_{nk}^\lambda)$ and $V^\lambda = (v_{nk}^\lambda)$ are defined as follows:

$$f_{nk}^\lambda = \begin{cases} \left(\frac{d_{nk} \lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{d_{n,k+1} \lambda_k}{\lambda_{k+1} - \lambda_k} \right) a_n, & (k < n), \\ \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n, & (k = n), \\ 0, & (k > n), \end{cases} \quad (53)$$

$$v_{nk}^\lambda = \begin{cases} \widehat{g}_k(n), & (k < n), \\ \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n, & (k = n), \\ 0, & (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$ and the $\widehat{g}_k(n)$ is defined by

$$\widehat{g}_k(n) = \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^n d_{jk} a_j - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^n d_{j,k+1} a_j \right),$$

for $k < n$.

(54)

Theorem 17. (i) $\{c_0^\lambda(\widehat{B})\}^\alpha = \{c^\lambda(\widehat{B})\}^\alpha = \{\ell_\infty^\lambda(\widehat{B})\}^\alpha = f_1^\lambda$.
(ii) $\{\ell_p^\lambda(\widehat{B})\}^\alpha = f_6^\lambda$.

Proof. We prove the theorem for the space $c_0^\lambda(\widehat{B})$. Let $a = (a_n) \in w$. Then, we obtain the equality

$$a_n x_n = \sum_{k=0}^n d_{nk} \sum_{j=k-1}^k (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} a_n y_i = (F^\lambda y)_n, \quad (55)$$

$\forall n \in \mathbb{N}$,

by relation (22). Thus we observe by (55) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0^\lambda(\widehat{B})$ if and only if $F^\lambda y \in \ell_1$ whenever $y = (y_k) \in c_0$. This means that the sequence $a = (a_n) \in \{c_0^\lambda(\widehat{B})\}^\alpha$ if and only if $F^\lambda \in (c_0 : \ell_1)$. Therefore we obtain by Lemma 10 with F^λ instead of A that $a = (a_n) \in \{c_0^\lambda(\widehat{B})\}^\alpha$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} f_{nk}^\lambda \right| < \infty \quad (56)$$

which leads us to the consequence that $\{c_0^\lambda(\widehat{B})\}^\alpha = f_1^\lambda$. This completes the proof. $\square$

Theorem 18. (i) $\{c_0^\lambda(\widehat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda$ (with $q = 1$).

(ii) $\{c^\lambda(\widehat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \cap f_5^\lambda$ (with $q = 1$).

(iii) $\{\ell_\infty^\lambda(\widehat{B})\}^\beta = f_2^\lambda \cap f_4^\lambda \cap f_7^\lambda$.

(iv) $\{\ell_p^\lambda(\widehat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda$.

Proof. Consider the equality

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \left[\sum_{j=0}^k d_{kj} \sum_{i=j-1}^i (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] a_k \\ &= \sum_{k=0}^n (\lambda_k y_k - \lambda_{k-1} y_{k-1}) \left[\frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^n d_{jk} a_j \right] \\ &= \sum_{k=0}^{n-1} \lambda_k \left[\frac{\sum_{j=k}^n d_{jk} a_j}{\lambda_k - \lambda_{k-1}} - \frac{\sum_{j=k+1}^n d_{j,k+1} a_j}{\lambda_{k+1} - \lambda_k} \right] y_k \\ &\quad + \frac{1}{r} \frac{a_n \lambda_n}{(\lambda_n - \lambda_{n-1})} y_n \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{n-1} \widehat{g}_k(n) y_k + \frac{1}{r} \frac{a_n \lambda_n}{(\lambda_n - \lambda_{n-1})} y_n \\ &= (V^\lambda y)_n; \quad (n \in \mathbb{N}). \end{aligned} \quad (57)$$

Then we deduce by (57) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0^\lambda(\widehat{B})$ if and only if $V^\lambda y \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \{c_0^\lambda(\widehat{B})\}^\beta$ if and only if $V^\lambda \in (c_0 : c)$. Therefore, by using Lemma 11, we obtain

$$\begin{aligned} &\sum_{j=k}^{\infty} d_{jk} a_j \quad \text{exists for each } k \in \mathbb{N}, \\ &\sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\widehat{g}_k(n)| < \infty, \quad (58) \\ &\sup_{k \in \mathbb{N}} \left| \frac{1}{r} \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})} a_k \right| < \infty. \end{aligned}$$

Hence, we conclude that $\{c_0^\lambda(\widehat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda$. $\square$

Finally, we ended this section with the following theorem which determines the γ -duals of sequence spaces $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$.

Theorem 19. (i) $\{c_0^\lambda(\widehat{B})\}^\gamma = \{c^\lambda(\widehat{B})\}^\gamma = \{\ell_\infty^\lambda(\widehat{B})\}^\gamma = f_3^\lambda \cap f_4^\lambda$ (with $q = 1$).
(ii) $\{\ell_p^\lambda(\widehat{B})\}^\gamma = f_3^\lambda \cap f_4^\lambda$.

5. Some Matrix Transformations Related to the Spaces $\mu^\lambda(\widehat{B})$ of Nonabsolute Type for $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$

In this final section, we state some results on the characterization of several classes of matrix mappings on the $c_0^\lambda(\widehat{B})$, $c^\lambda(\widehat{B})$, $\ell_\infty^\lambda(\widehat{B})$, and $\ell_p^\lambda(\widehat{B})$. We will write throughout for brevity that

$$\begin{aligned} &\widehat{g}_{nk}(m) \\ &= \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^m d_{jk} a_{nj} - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^m d_{j,k+1} a_{nj} \right) \\ &\quad \text{for } k < m, \\ &\widehat{g}_{nk} = \lambda_k \left(\frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{\infty} d_{jk} a_{nj} - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^{\infty} d_{j,k+1} a_{nj} \right) \end{aligned} \quad (59)$$

for all $k, m, n \in \mathbb{N}$. We assume that the series are convergent in the last equation.

We will begin with lemmas which are needed in the proof of our theorems.

Lemma 20. *The matrix mappings between the BK-spaces are continuous.*

Lemma 21. $A = (a_{nk}) \in (c : \ell_p)$ if and only if

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} a_{nk} \right|^p < \infty. \quad (60)$$

Lemma 22. $A = (a_{nk}) \in (c : c_0)$ if and only if (48) holds and

$$\begin{aligned} \lim_{n \rightarrow \infty} a_{nk} &= 0, \quad \text{for each fixed } k \in \mathbb{N}, \\ \lim_{n \rightarrow \infty} \sum_k a_{nk} &= 0. \end{aligned} \quad (61)$$

Lemma 23. $A \in (c_0 : c_0)$ if and only if (48) and (61) hold.

Now, we may give our matrix transformations.

Theorem 24. *Let $A = (a_{nk})$ be an infinite matrix and $1 < p < \infty$. Then, $A \in (c^\lambda(\widehat{B}) : \ell_p)$ if and only if*

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} \widehat{g}_{nk} \right|^p < \infty, \quad (62)$$

$$\sum_{j=k}^{\infty} d_{jk} a_{nj} \quad \text{exists for all fixed } k, n \in \mathbb{N}, \quad (63)$$

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m-1} |\widehat{g}_{nk}(m)| < \infty, \quad (64)$$

$$\lim_{k \rightarrow \infty} \frac{1}{r} \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})} a_{nk} = a_n \quad \text{exists for each fixed } n \in \mathbb{N}, \quad (65)$$

$$\left\{ \sum_{j=k}^{\infty} d_{jk} a_{nj} \right\} \in cs; \quad (n \in \mathbb{N}), \quad (66)$$

$$(a_n) \in \ell_p. \quad (67)$$

Proof. If conditions (62)–(67) hold and $x = (x_k)$ is any sequence in the space $c^\lambda(\widehat{B})$ then by using Theorem 18 we have that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\widehat{B})\}^\beta$ for all $n \in \mathbb{N}$. Hence, A -transform of x exists. Furthermore, since the associated sequence $y = (y_k)$ is in the space c , we may write $\lim_k y_k = l$ for some suitable l . Also, the matrix $\widehat{A} = (\widehat{g}_{nk})$ is in the class $(c : \ell_p)$ by Lemma 21 and condition (62), where $1 < p < \infty$.

Now, we may consider the m th partial sum of the series $\sum_k a_{nk} x_k$ which is derived by using relation (22):

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^{m-1} \widehat{g}_{nk}(m) y_k + \frac{1}{r} \frac{\lambda_m}{(\lambda_m - \lambda_{m-1})} a_{nm} y_m; \quad (68)$$

$$\forall n, m \in \mathbb{N}.$$

Then, since $y \in c$ and $\widehat{A} \in (c : \ell_p)$, $\widehat{A}y$ exists and so the series $\sum_k \widehat{g}_{nk} y_k$ converges for every $n \in \mathbb{N}$. Also, it follows by (63)

that we have $\lim_{m \rightarrow \infty} \widehat{g}_{nk}(m) = \widehat{g}_{nk}$. Therefore, if we pass to limit in (68) as $m \rightarrow \infty$, then we obtain by (65) that

$$\sum_k a_{nk} x_k = \sum_k \widehat{g}_{nk} y_k + l a_n, \quad \forall n \in \mathbb{N} \quad (69)$$

which can be written as follows:

$$(Ax)_n = (\widehat{A}y)_n + l a_n, \quad \forall n \in \mathbb{N}. \quad (70)$$

This yields by taking ℓ_p -norm that

$$\|Ax\|_{\ell_p} \leq \|\widehat{A}y\|_{\ell_p} + |l| \|(a_n)\|_{\ell_p} < \infty. \quad (71)$$

Consequently, we have that $Ax \in \ell_p$; that is, $A \in (c^\lambda(\widehat{B}) : \ell_p)$.

Conversely, if $A \in (c^\lambda(\widehat{B}) : \ell_p)$, we have $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\widehat{B})\}^\beta$ for all $n \in \mathbb{N}$ which implies with Theorem 18 that conditions (63), (64), and (66) are necessary.

On the other hand, since $c^\lambda(\widehat{B})$ and ℓ_p are BK-spaces, we have by Lemma 20 that there is a constant $M > 0$ such that

$$\|Ax\|_p \leq M \|x\|_{c^\lambda(\widehat{B})} \quad (72)$$

holds for all $x \in c^\lambda(\widehat{B})$. Let $F \in \mathcal{F}$. Then, the sequence $z = \sum_{k \in F} b^{(k)}(\lambda)$ is in $c^\lambda(\widehat{B})$, where the sequences $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n \in \mathbb{N}}$ are defined by (27) for every fixed $k \in \mathbb{N}$.

Since $\widehat{W}b^{(k)}(\lambda) = e^{(k)}$ for each fixed $k \in \mathbb{N}$, we have

$$\|z\|_{c^\lambda(\widehat{B})} = \|\widehat{W}(z)\|_{\ell_\infty} = \left\| \sum_{k \in F} \widehat{W}b^{(k)}(\lambda) \right\|_{\ell_\infty} = \left\| \sum_{k \in F} e^{(k)} \right\|_{\ell_\infty} = 1. \quad (73)$$

Furthermore, for every $n \in \mathbb{N}$, we obtain by (27) that

$$A_n(z) = \sum_{k \in F} A_n(b^{(k)}(\lambda)) = \sum_{k \in F} \sum_j a_{nj} b_j^{(k)}(\lambda) = \sum_{k \in F} \widehat{g}_{nk}. \quad (74)$$

Hence, since inequality (72) is satisfied for the sequence $z \in c^\lambda(\widehat{B})$, we have for any $F \in \mathcal{F}$ that

$$\left(\sum_n \left| \sum_{k \in F} \widehat{g}_{nk} \right|^p \right)^{1/p} \leq M \quad (75)$$

which shows the necessity of (62). Thus, it follows by Lemma 21 that $\widehat{A} = (\widehat{g}_{nk}) \in (c : \ell_p)$.

Moreover, we consider the sequence $x = (x_k)$ defined by (22) for every $k \in \mathbb{N}$ and suppose that $y = (y_k) \in c \setminus c_0$. Then, since $x \in c^\lambda(\widehat{B})$ such that $y = \widehat{W}x$ by (18), the transforms Ax and $\widehat{A}y$ exist. Hence, the series $\sum_k a_{nk} x_k$ and $\sum_k \widehat{g}_{nk} y_k$ are convergent for every $n \in \mathbb{N}$. So we infer that

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \widehat{g}_{nk}(m) y_k = \sum_k \widehat{g}_{nk} y_k; \quad (n \in \mathbb{N}). \quad (76)$$

Consequently, we obtain from (68) that

$$\lim_{m \rightarrow \infty} \frac{1}{r} \frac{\lambda_m}{(\lambda_m - \lambda_{m-1})} a_{nm} y_m \quad \text{exists for each fixed } n \in \mathbb{N}, \quad (77)$$

Hence, we deduce that

$$\lim_{m \rightarrow \infty} \frac{1}{r} \frac{\lambda_m}{(\lambda_m - \lambda_{m-1})} a_{nm} \quad \text{exists for each fixed } n \in \mathbb{N}. \quad (78)$$

which leads us to the necessity of (65) and so the relation (68) holds, where $l = \lim_k y_k$.

Finally, since $Ax \in \ell_p$ and $\widehat{A}y \in \ell_p$, the necessity of (67) is immediate by (68). This completes the proof. $\square$

Theorem 25. $A = (a_{nk}) \in (c^\lambda(\widehat{B}) : \ell_\infty)$ if and only if (65) and (66) hold, and

$$\sup_{n \in \mathbb{N}} \sum_k |\widehat{g}_{nk}| < \infty, \quad (79)$$

$$(a_n) \in \ell_\infty. \quad (80)$$

Proof. This is an immediate consequence of Lemma 15 and Theorem 24. $\square$

Theorem 26. (i) $A = (a_{nk}) \in (c_0^\lambda(\widehat{B}) : \ell_p)$ if and only if (62), (63), and (64) hold and

$$\sup_k \left| \frac{1}{r} \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})} a_{nk} \right| < \infty, \quad (n \in \mathbb{N}). \quad (81)$$

(ii) $A \in (c^\lambda(\widehat{B}) : \ell_\infty)$ if and only if (64), (79), and (81) hold.

Proof. This may be obtained by proceedings as in Theorem 24, above. So, we omit the detail. $\square$

Theorem 27. $A = (a_{nk}) \in (c^\lambda(\widehat{B}) : c)$ if and only if (65), (66), and (79) hold and

$$\lim_n a_n = a, \quad (82)$$

$$\lim_n \widehat{g}_{nk} = \alpha_k, \quad \text{for each fixed } k \in \mathbb{N}, \quad (83)$$

$$\lim_n \sum_k \widehat{g}_{nk} = \alpha. \quad (84)$$

Proof. Suppose that A satisfies conditions (65), (66), (79), (82), (83), and (84), and let $x = (x_k)$ be a sequence in the space $c^\lambda(\widehat{B})$. Since (79) implies (64), we have by Theorem 18 that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\widehat{B})\}^\beta$ for all $n \in \mathbb{N}$. Hence, Ax exists. We also observe from (79) and (83) that

$$\sum_{j=0}^k |\alpha_j| \leq \sup_{n \in \mathbb{N}} \sum_j |\widehat{g}_{nj}| < \infty \quad (85)$$

holds for every $k \in \mathbb{N}$. So $(\alpha_k) \in \ell_1$ and hence the series $\sum_k \alpha_k (y_k - l)$ converges, where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ by relation (18) such that $\lim_k y_k = l$. Also, it is obvious by combining Lemma 12 with conditions (79), (83), and (84) that the matrix $\widehat{A} = (\widehat{g}_{nk})$ is in the class $(c : c)$.

Now, by following the similar way used in the proof of Theorem 24, we obtain that relation (69) holds, which can be written as follows:

$$\sum_k a_{nk} x_k = \sum_k \widehat{g}_{nk} (y_k - l) + l \sum_k \widehat{g}_{nk} + l a_n, \quad \forall n \in \mathbb{N}. \quad (86)$$

If we pass the limit in (86) as $n \rightarrow \infty$ we have that

$$\lim_n (Ax)_n = \sum_k \alpha_k (y_k - l) + l (\alpha + a), \quad (87)$$

which shows that $Ax \in c$; that is, $A \in (c^\lambda(\widehat{B}) : c)$.

Conversely, suppose that $A \in (c^\lambda(\widehat{B}) : c)$. Since the inclusion $c \subset \ell_\infty$ holds, $A \in (c^\lambda(\widehat{B}) : \ell_\infty)$. Therefore, the necessity of conditions (65), (66), and (79) is obvious from Theorem 25. Furthermore, consider the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n \in \mathbb{N}} \in c^\lambda(\widehat{B})$ defined by (27) for every fixed $k \in \mathbb{N}$. Then, one can see that $Ab^{(k)}(\lambda) = \{\widehat{g}_{nk}\}_{n \in \mathbb{N}}$ and hence $\{\widehat{g}_{nk}\}_{n \in \mathbb{N}} \in c$ for every $k \in \mathbb{N}$ which shows the necessity of (83). Let $z = \sum_k b^{(k)}(\lambda)$. Then, since the linear transformation $T : c^\lambda(\widehat{B}) \rightarrow c$, defined as in the proof of Theorem 3 by analogy, is continuous and $\widehat{W}b^{(k)}(\lambda) = e^{(k)}$ for each fixed $k \in \mathbb{N}$, we obtain that

$$(\widehat{W}z)_n = \sum_k (\widehat{W}b^{(k)}(\lambda))_n = \sum_k \delta_{nk} = 1, \quad \text{for each } n \in \mathbb{N} \quad (88)$$

which shows that $(\widehat{W}z)_n = e \in c$ and hence $z \in c^\lambda(\widehat{B})$. On the other hand, since $c^\lambda(\widehat{B})$ and c are the BK -spaces, Lemma 20 implies the continuity of the matrix mapping $A : c^\lambda(\widehat{B}) \rightarrow c$. Thus, we have for every $n \in \mathbb{N}$ that

$$(Az)_n = \sum_k (Ab^{(k)}(\lambda))_n = \sum_k \widehat{g}_{nk}. \quad (89)$$

This shows the necessity of (84).

Now, it follows by (79), (83), and (84) with Lemma 12 that $\widehat{A} = (\widehat{g}_{nk}) \in (c : c)$. So by (65), (66) and relation (70) hold for all $x \in c^\lambda(\widehat{B})$ and $y \in c$.

Finally, the necessity of (82) is immediate by (70) since $Ax \in c$ and $\widehat{A}x \in c$. This completes the proof. $\square$

Theorem 28. $A = (a_{nk}) \in (c^\lambda(\widehat{B}) : c_0)$ if and only if (65), (66), and (79) hold and

$$\lim_n a_n = 0,$$

$$\lim_n \widehat{g}_{nk} = 0, \quad \text{for each fixed } k \in \mathbb{N}, \quad (90)$$

$$\lim_n \sum_k \widehat{g}_{nk} = 0.$$

Proof. This is obtained in the similar way used in the proof of Theorem 27 with Lemma 22 instead of Lemma 12 and so we omit the detail. $\square$

Theorem 29. $A = (a_{nk}) \in (c_0^\lambda(\widehat{B}) : c)$ if and only if (63), (79), (81), and (83) hold.

Proof. This is an immediate consequence of Lemma 11 and Theorems 18 and 26 (ii). $\square$

Theorem 30. $A = (a_{nk}) \in (c_0^\lambda(\widehat{B}) : c_0)$ if and only if (63), (79), and (83) hold.

Proof. This is an immediate consequence of Lemma 23 and Theorems 18 and 29. $\square$

Theorem 31. $A = (a_{nk}) \in (\ell_p^\lambda(\widehat{B}) : \ell_\infty)$ if and only if (64) and (81) hold and

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q < \infty, \quad (n \in \mathbb{N}). \quad (91)$$

Proof. Suppose that $A = (a_{nk})$ satisfies conditions (64), (81), and (91) and take any $x = (x_k) \in \ell_p^\lambda(\widehat{B})$. Then, it is obvious that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_p^\lambda(\widehat{B})\}^\beta$ for each $n \in \mathbb{N}$. Hence, $(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$ is convergent for all $x \in \ell_p^\lambda(\widehat{B})$; that is, Ax exists. Also, it is obvious by combining Lemma 16 with condition (91) that $\widehat{A} = (\widehat{g}_{nk}) \in (\ell_p : \ell_\infty)$.

Now, we will show that $Ax \in \ell_\infty$ for all $x \in \ell_p^\lambda(\widehat{B})$. By using relation (68) we have that

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \widehat{g}_{nk} y_k. \quad (92)$$

By applying Hölder's inequality to (92) we obtain that

$$|(Ax)_n| = \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| \leq \sum_{k=0}^{\infty} |\widehat{g}_{nk}| |y_k| \leq \|y\|_{\ell_p} \left(\sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q \right)^{1/q} \quad (93)$$

which gives us by taking supremum over $n \in \mathbb{N}$ in this last inequality that

$$\sup_{n \in \mathbb{N}} |(Ax)_n| \leq \|y\|_{\ell_p} \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q \right)^{1/q} < \infty \quad (94)$$

which implies that $Ax \in \ell_\infty$; that is, $A \in (\ell_p^\lambda(\widehat{B}) : \ell_\infty)$.

Conversely, suppose that $A = (a_{nk}) \in (\ell_p^\lambda(\widehat{B}) : \ell_\infty)$. Thus, Ax exists and bounded for all $x \in \ell_p^\lambda(\widehat{B})$. Also, $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_p^\lambda(\widehat{B})\}^\beta$ for all $n \in \mathbb{N}$ which implies the necessity of (64) and (81). If we define the sequences $\widehat{g}_n$ such that $\widehat{g}_n = \{\widehat{g}_{nk}\}_{k \in \mathbb{N}}$ then we have that

$$\|\widehat{g}_n\|_{\ell_q} < \infty. \quad (95)$$

So, bearing in mind (81) one can easily obtain relation (92) by using relation (68). On the other hand, the sequences $a_n = \{a_{nk}\}_{k \in \mathbb{N}}$ define the continuous linear functionals on the space $\ell_p^\lambda(\widehat{B})$ as follows:

$$f_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}). \quad (96)$$

Since the spaces $\ell_p^\lambda(\widehat{B})$ and ℓ_p^λ are linear isomorphic, we have that

$$\|f_n\| = \|\widehat{g}_n\|_{\ell_q}. \quad (97)$$

Also, since $\{f_n(x)\}_{n \in \mathbb{N}} \in \ell_\infty$, it is obvious that

$$\sup \|f_n(x)\| < \infty. \quad (98)$$

Therefore, by using the Banach-Steinhaus Theorem [38, see pages: 1-2] we obtain that

$$\sup_{n \in \mathbb{N}} \|f_n\| = \sup_{n \in \mathbb{N}} \|\widehat{g}_n\|_{\ell_q} = \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q \right)^{1/q} < \infty \quad (99)$$

which shows the necessity of (91). This step completes the proof. $\square$

Theorem 32. $A = (a_{nk}) \in (\ell_p^\lambda(\widehat{B}) : c)$ if and only if (64), (81), (83), and (91) hold.

Proof. Suppose that conditions (64), (81), (83), and (91) hold. Then, $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell_p^\lambda(\widehat{B})\}^\beta$ for each $n \in \mathbb{N}$. Hence, the series $\sum_{k=0}^{\infty} a_{nk} x_k$ is convergent for each $n \in \mathbb{N}$ and for all $x \in \ell_p^\lambda(\widehat{B})$ which implies that Ax exists. By using (83) we obtain that

$$|\widehat{g}_{nk}|^q \rightarrow |\alpha_k|^q; \quad (n \rightarrow \infty). \quad (100)$$

Since

$$\left(\sum_{k=0}^m |\widehat{g}_{nk}|^q \right)^{1/q} \leq \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q \right)^{1/q} = M \quad (101)$$

for all $m > 0$, in this situation, we see by passing to the limit in the last inequality as $n \rightarrow \infty$ that

$$\left(\sum_{k=0}^m |\alpha_k|^q \right)^{1/q} \leq \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{\infty} |\widehat{g}_{nk}|^q \right)^{1/q}. \quad (102)$$

Because (102) holds for all positive integers m , we have that

$$\left(\sum_{k=0}^{\infty} |\alpha_k|^q \right)^{1/q} < \infty. \quad (103)$$

We remember that $x = (x_k)$ and $y = (y_k)$ are associated sequences by relation (22) where $y = (y_k) \in \ell_p$ for $x = (x_k) \in \ell_p^\lambda(\widehat{B})$.

Let ε be any positive number. Then, there exists a number N such that

$$\left(\sum_{k=N}^{\infty} |y_k|^q \right)^{1/q} < \frac{\varepsilon}{4M}. \quad (104)$$

On the other hand, there is an integer N_1 such that

$$\left| \sum_{k=0}^N \{\widehat{g}_{nk} - \alpha_k\} y_k \right| \leq \frac{\varepsilon}{2} \quad (105)$$

whenever $n \geq N_1$. Therefore, we obtain

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \{\hat{g}_{nk} - \alpha_k\} y_k \right| \\ & \leq \left| \sum_{k=0}^N \{\hat{g}_{nk} - \alpha_k\} y_k \right| + \left| \sum_{k=N+1}^{\infty} \{\hat{g}_{nk} - \alpha_k\} y_k \right| \\ & \leq \frac{\varepsilon}{2} + \left(\sum_{k=N+1}^{\infty} [|\hat{g}_{nk}| + |\alpha_k|]^q \right)^{1/q} \left(\sum_{k=N+1}^{\infty} |y_k|^p \right)^{1/p} \\ & \leq \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon, \end{aligned} \quad (106)$$

which shows that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \hat{g}_{nk} y_k = \sum_{k=0}^{\infty} \alpha_k y_k. \quad (107)$$

Since

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \hat{g}_{nk} y_k = \sum_{k=0}^{\infty} \alpha_k y_k \quad (108)$$

we have that $Ax \in c$ which implies that $A \in (\ell_p^\lambda(\hat{B}) : c)$.

The necessity part can be proved by using the similar way in the proof of Theorem 27, so we omit the detail. $\square$

Theorem 33. $A = (a_{nk}) \in (\ell_p^\lambda(\hat{B}) : c_0)$ if and only if (64), (81), and (91) hold and

$$\lim_n \hat{g}_{nk} = 0, \quad \text{for each fixed } k \in \mathbb{N}. \quad (109)$$

Proof. This result can be proved similarly as the proof of Theorem 32. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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