Research Article

On Iterated Entire Functions with (p, q) th Order

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We prove some results relating to the comparative growth properties of iterated entire functions using (p, q) th order ((p, q) th lower order).

1. Introduction, Definitions, and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$, the maximum modulus $M(r, f)$ and Nevanlinna's characteristic function $T(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are, respectively, defined as $\mu(r, f) = \max(|a_n| r^n)$, $M(r, f) = \max_{|z|=r} |f(z)|$ and $T(r, f) = (1/2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ where $\log^+ x = \max(\log x, 0)$ for all $x \geq 0$. We do not explain the standard definitions and notations in the theory of entire function as those are available in [1]. In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log(\log^{[k-1]} x) \quad \text{for } k = 1, 2, 3, \dots; \\ \log^{[0]} x &= x, \\ \exp^{[k]} x &= \exp(\exp^{[k-1]} x) \quad \text{for } k = 1, 2, 3, \dots; \\ \exp^{[0]} x &= x. \end{aligned} \quad (1)$$

To start our paper we just recall the following definitions.

Definition 1. The order ρ_f and lower order λ_f of an entire function f are defined as follows:

$$\begin{aligned} \rho_f &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}, \\ \lambda_f &= \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}. \end{aligned} \quad (2)$$

Definition 2 (see [2]). Let l be an integer ≥ 2 . The generalised order $\rho_f^{[l]}$ and generalised lower order $\lambda_f^{[l]}$ of an entire function f are defined as

$$\begin{aligned} \rho_f^{[l]} &= \limsup_{r \rightarrow \infty} \frac{\log^{[l]} M(r, f)}{\log r}, \\ \lambda_f^{[l]} &= \liminf_{r \rightarrow \infty} \frac{\log^{[l]} M(r, f)}{\log r}. \end{aligned} \quad (3)$$

When $l = 2$, Definition 2 coincides with Definition 1.

Definition 3. A function $\rho_f^{[l]}(r)$ is called a generalised proximate order of a meromorphic function f relative to $T(r, f)$ if

(i) $\rho_f^{[l]}(r)$ is nonnegative and continuous for $r \geq r_0$,

- (ii) $\rho_f^{[l]}(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho_f^{[l]'}(r+0)$ and $\rho_f^{[l]'}(r-0)$ exist,
- (iii) $\lim_{r \rightarrow \infty} \rho_f^{[l]}(r) = \rho_f^{[l]} < \infty$,
- (iv) $\lim_{r \rightarrow \infty} \rho_f^{[l]'}(r) \prod_{i=0}^{l-1} \log^{[i]} r = 0$,
- (v) $\limsup_{r \rightarrow \infty} (\log^{[l-2]} T(r, f) / r^{\rho_f^{[l]}(r)}) = 1$.

The existence of such a proximate order is proved by Lahiri [3].

Similarly one can define the generalised lower proximate order of f in the following way.

Definition 4. A function $\lambda_f^{[l]}(r)$ is defined as a generalised lower proximate order of a meromorphic function f relative to $T(r, f)$ if

- (i) $\lambda_f^{[l]}(r)$ is nonnegative and continuous for $r \geq r_0$,
- (ii) $\lambda_f^{[l]}(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f^{[l]'}(r+0)$ and $\lambda_f^{[l]'}(r-0)$ exist,
- (iii) $\lim_{r \rightarrow \infty} \lambda_f^{[l]}(r) = \lambda_f^{[l]} < \infty$,
- (iv) $\lim_{r \rightarrow \infty} \lambda_f^{[l]'}(r) \prod_{i=0}^{l-1} \log^{[i]} r = 0$,
- (v) $\liminf_{r \rightarrow \infty} (\log^{[l-2]} T(r, f) / r^{\lambda_f^{[l]}(r)}) = 1$.

Definitions 3 and 4 are both valid for entire f .

Juneja et al. [4] defined the (p, q) th order and (p, q) th lower order of an entire function f , respectively, as follows:

$$\begin{aligned} \rho_f(p, q) &= \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r}, \\ \lambda_f(p, q) &= \liminf_{r \rightarrow \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r}, \end{aligned} \quad (4)$$

where p, q are positive integers with $p \geq q$.

For $p = 2$ and $q = 1$, we respectively denote $\rho_f(2, 1)$ and $\lambda_f(2, 1)$ by ρ_f and λ_f .

Since for $0 \leq r < R$, {cf. [5]}

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f). \quad (5)$$

It is easy to see that

$$\begin{aligned} \rho_f &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f)}{\log r}, \\ \lambda_f &= \liminf_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f)}{\log r}, \\ \rho_f^{[l]} &= \limsup_{r \rightarrow \infty} \frac{\log^{[l]} \mu(r, f)}{\log r}, \\ \lambda_f^{[l]} &= \liminf_{r \rightarrow \infty} \frac{\log^{[l]} \mu(r, f)}{\log r}, \\ \rho_f(p, q) &= \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r}, \\ \lambda_f(p, q) &= \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r}. \end{aligned} \quad (6)$$

According to Lahiri and Banerjee [6] if $f(z)$ and $g(z)$ are entire functions, then the iteration of f with respect to g is defined as follows:

$$\begin{aligned} f(z) &= f_1(z) \\ f(g(z)) &= f(g_1(z)) = f_2(z) \\ f(g(f(z))) &= f(g(f_1(z))) = f(g_2(z)) = f_3(z) \\ &\vdots \\ f(g(f \cdots (f(z) \text{ or } g(z)) \cdots)) &= f_n(z), \text{ according to the} \\ &\text{fact that } n \text{ is odd or even, and so} \\ g(z) &= g_1(z) \\ g(f(z)) &= g(f_1(z)) = g_2(z) \\ g(f(g(z))) &= g(f(g_1(z))) = g(f_2(z)) = g_3(z) \\ &\vdots \\ g(f(g_{n-2}(z))) &= g(f_{n-1}(z)) = g_n(z). \end{aligned} \quad (7)$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

In this paper we would like to investigate some growth properties of iterated entire functions on the basis of their maximum terms, (p, q) th order and (p, q) th lower order where p, q are positive integers with $p \geq q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 5 (see [7]). *If f and g are any two entire functions then for all sufficiently large values of r ,*

$$M(r, f \circ g) \leq M(M(r, g), f). \quad (9)$$

Lemma 6. Let f and g be any two entire functions such that $\rho_f(p, q) < \infty$ and $\rho_g(a, b) < \infty$ where a, b, p , and q are any four positive integers with $a \geq b$ and $p \geq q$. Then for any even number n and for all sufficiently large values of r ,

$$\begin{aligned}
 \text{(i)} \quad & \log^{[p+((n-2)/2)(a-q)+((n-2)/2)(p-b)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } q < a, \quad b < p, \\
 \text{(ii)} \quad & \log^{[p+((n-2)/2)(a-q)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \\
 & \quad \times \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } p = b \geq q, \quad a > q, \quad n > 2, \\
 \text{(iii)} \quad & \log^{[p+((n-2)/2)(p-b)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } p \geq b, \quad q = a, \\
 \text{(iv)} \quad & \log^{[p]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon)^{n/2} (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \\
 & \quad \times \log^{[q]} M(r, g) \quad \text{when } a = b = p = q, \\
 \text{(v)} \quad & \log^{[p+(n-2)(p-q)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } p = a > q = b, \\
 \text{(vi)} \quad & \log^{[p+a-q]} \mu(r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \log^{[a]} M(r, g) + O(1) \\
 & \quad \text{when } p < b, \quad q < a, \quad b - p = a - q, \\
 \text{(vii)} \quad & \log^{[p+a+((n-2)/2)(a-b)-q]} \mu(r, f_n) \\
 & \leq \log^{[a]} M(r, g) + O(1) \quad \text{when } p < b, q < a, \\
 & \quad b - p < a - q, \\
 \text{(viii)} \quad & \log^{[p]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon)^{n/2} \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } p > b, \quad q > a, \quad q - a = p - b, \\
 \text{(ix)} \quad & \log^{[p+((n-2)/2)(p+a-b-q)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \\
 & \quad \text{when } p > b, \quad q > a, \quad q - a < p - b,
 \end{aligned}$$

(10)

and for any odd number $n(\neq 1)$ and for all sufficiently large values of r ,

$$\begin{aligned}
 \text{(x)} \quad & \log^{[p+((n-1)/2)(a-q)+((n-3)/2)(p-b)]} \mu(r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \\
 & \quad \text{when } q < a, \quad b < p, \\
 \text{(xi)} \quad & \log^{[(p-1)+((n-1)/2)(a-q)]} T(r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \\
 & \quad \text{when } p = b \geq q, \quad a > q, \\
 \text{(xii)} \quad & \log^{[p+((n-3)/2)(p-b)]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \\
 & \quad \times \log^{[b]} M(r, f) + O(1) \\
 & \quad \text{when } p \geq b, \quad q = a, \\
 \text{(xiii)} \quad & \log^{[p]} \mu(r, f_n) \\
 & \leq [(\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon)]^{(n-1)/2} \\
 & \quad \times \log^{[b]} M(r, f) \\
 & \quad \text{when } a = b = p = q, \\
 \text{(xiv)} \quad & \log^{[p+(n-2)(p-q)]} \mu(r, f_n) \\
 & \leq (\rho_g(p, q) + \varepsilon) \log^{[q]} M(r, f) + O(1) \\
 & \quad \text{when } p = a > q = b, \\
 \text{(xv)} \quad & \log^{[p+a-q]} \mu(r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon)^{(n-1)/2} \log^{[b]} M(r, f) + O(1) \\
 & \quad \text{when } p < b, \quad q < a, \quad b - p = a - q, \\
 \text{(xvi)} \quad & \log^{[p+a+((n-3)/2)(a-b)-q]} \mu(r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \\
 & \quad \text{when } p < b, \quad q < a, \quad b - p < a - q, \\
 \text{(xvii)} \quad & \log^{[p]} \mu(r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon)^{(n-1)/2} \log^{[p]} M(r, f) + O(1) \\
 & \quad \text{when } p > b, \quad q > a, \quad q - a = p - b, \\
 \text{(xviii)} \quad & \log^{[p+((n-1)/2)(p+a-b-q)]} \mu(r, f_n) \\
 & \leq \log^{[p]} M(r, f) + O(1) \quad \text{when } p > b, q > a, \\
 & \quad q - a < p - b,
 \end{aligned}$$

(11)

where $\varepsilon > 0$ is any arbitrary number.

Proof. Let us consider n to be an even number.

Then in view of Lemma 5 and the inequality $\mu(r, f) \leq M(r, f)$ {cf. [5]} we get for all sufficiently large values of r that

$$\mu(r, f_n) \leq M(r, f_n),$$

$$\text{i.e., } \log^{[p]} \mu(r, f_n) \leq \log^{[p]} M(M(r, g_{n-1}), f), \quad (12)$$

$$\text{i.e., } \log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g_{n-1}).$$

Case I. Let $q < a$ and $b < p$. We obtain from (12) for all sufficiently large values of r that

$$\log^{[p+a-q]} \mu(r, f_n) \leq \log^{[a]} M(r, g_{n-1}) + O(1)$$

$$\text{i.e., } \log^{[p+a-q]} \mu(r, f_n) \leq \log^{[a]} M(M(r, f_{n-2}), g) + O(1)$$

$$\text{i.e., } \log^{[p+a-q]} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon)$$

$$\times \log^{[b]} M(r, f_{n-2}) + O(1),$$

$$\text{i.e., } \log^{[p+a-q+p-b]} \mu(r, f_n) \leq \log^{[p]} M(M(r, g_{n-3}), f) + O(1)$$

$$\vdots$$

$$(13)$$

Therefore,

$$\begin{aligned} & \log^{[p+((n-2)/2)(a-q)+((n-2)/2)(p-b)]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \end{aligned} \quad (14)$$

when n is even.

Similarly,

$$\begin{aligned} & \log^{[p+((n-1)/2)(a-q)+((n-3)/2)(p-b)]} \mu(r, f_n) \\ & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \end{aligned} \quad (15)$$

when n is odd, $n \neq 1$.

This establishes the first and the tenth part of the lemma, respectively.

Case II. Let $p = b \geq q$ and $a > q$. Now we get from (13) for all sufficiently large values of r that

$$\log^{[p+a-q]} \mu(r, f_n) \leq (\rho_g(a, b) + \varepsilon) \log^{[p]} M(r, f_{n-2}) + O(1),$$

$$\text{i.e., } \log^{[p+a-q]} \mu(r, f_n)$$

$$\leq (\rho_g(a, b) + \varepsilon) \log^{[p]} M(M(r, g_{n-3}), f) + O(1)$$

$$\vdots$$

$$(16)$$

Thus,

$$\begin{aligned} & \log^{[p+((n-2)/2)(a-q)]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \log^{[q]} M(r, g) + O(1) \end{aligned}$$

when n is even, $n > 2$.

$$(17)$$

Analogously,

$$\begin{aligned} & \log^{[p+((n-1)/2)(a-q)]} \mu(r, f_n) \\ & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \end{aligned} \quad (18)$$

when n is odd, $n \neq 1$,

from which the second and the eleventh part of the lemma follow.

Case III. Let $p \geq b$ and $q = a$. Therefore it follows from (12) for all sufficiently large values of r that

$$\log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[a]} M(r, g_{n-1}),$$

$$\text{i.e., } \log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[a]} M(M(r, f_{n-2}), g),$$

$$\text{i.e., } \log^{[p]} \mu(r, f_n)$$

$$\leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f_{n-2}),$$

$$\text{i.e., } \log^{[p+p-b]} \mu(r, f_n) \leq \log^{[p]} M(M(r, g_{n-3}), f) + O(1)$$

$$\vdots$$

$$(19)$$

Therefore,

$$\begin{aligned} & \log^{[p+((n-2)/2)(p-b)]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \end{aligned} \quad (20)$$

when n is even.

Similarly,

$$\begin{aligned} & \log^{[p+((n-3)/2)(p-b)]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \end{aligned}$$

when n is odd, $n \neq 1$.

$$(21)$$

This proves the third and the twelfth part of the lemma, respectively.

Case IV. Let $a = b = p = q$. Now we have from (12) for all sufficiently large values of r that

$$\begin{aligned}
 \log^{[p]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) \log^{[a]} M(r, g_{n-1}), \\
 \text{i.e., } \log^{[p]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) \log^{[a]} M(M(r, f_{n-2}), g), \\
 \text{i.e., } \log^{[p]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \\
 &\quad \times \log^{[b]} M(M(r, g_{n-3}), f), \\
 \text{i.e., } \log^{[p]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \\
 &\quad \times \log^{[p]} M(M(r, g_{n-3}), f) \\
 &\quad \vdots
 \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
 \log^{[p]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon)^{n/2} \\
 &\quad \times (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \log^{[q]} M(r, g) \quad (23) \\
 &\quad \text{when } n \text{ is even.}
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 \log^{[p]} \mu(r, f_n) &\leq [(\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon)]^{(n-1)/2} \\
 &\quad \times \log^{[b]} M(r, f) \quad (24) \\
 &\quad \text{when } n \text{ is odd, } n \neq 1.
 \end{aligned}$$

This establishes the fourth and the thirteenth part of the lemma, respectively.

Case V. Let $p = a > q = b$. We obtain from (12) for all sufficiently large values of r that

$$\begin{aligned}
 \log^{[p+p-q]} \mu(r, f_n) &\leq \log^{[p]} M(r, g_{n-1}) + O(1) \\
 \text{i.e., } \log^{[p+p-q]} \mu(r, f_n) &\leq \log^{[p]} M(M(r, f_{n-2}), g) + O(1), \\
 \text{i.e., } \log^{[p+p-q]} \mu(r, f_n) &\leq (\rho_g(p, q) + \varepsilon) \log^{[q]} M(M(r, g_{n-3}), f) + O(1), \\
 \text{i.e., } \log^{[p+2(p-q)]} \mu(r, f_n) &\leq \log^{[p]} M(M(r, g_{n-3}), f) + O(1) \\
 &\quad \vdots
 \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned}
 \log^{[p+(n-2)(p-q)]} \mu(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \quad \text{when } n \text{ is even.} \\
 &\quad (26)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \log^{[p+(n-2)(p-q)]} \mu(r, f_n) &\leq (\rho_g(p, q) + \varepsilon) \log^{[q]} M(r, f) + O(1) \quad (27) \\
 &\quad \text{when } n \text{ is odd, } n \neq 1,
 \end{aligned}$$

from which the fifth and the fourteenth part of the lemma follow.

Case VI. Let $p < b, q < a, b - p = a - q$. We get from (13) for all sufficiently large values of r that

$$\begin{aligned}
 \log^{[p+a-q]} \mu(r, f_n) &\leq (\rho_g(a, b) + \varepsilon) \log^{[b-p]} [\log^{[p]} M(r, f_{n-2})] + O(1) \\
 \text{i.e., } \log^{[p+a-q]} \mu(r, f_n) &\leq (\rho_g(a, b) + \varepsilon) \\
 &\quad \times \log^{[b-p]} [(\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g_{n-3})] \\
 &\quad + O(1), \\
 \text{i.e., } \log^{[p+a-q]} \mu(r, f_n) &\leq (\rho_g(a, b) + \varepsilon) \log^{[a]} M(M(r, f_{n-4}), g) + O(1) \\
 &\quad \vdots
 \end{aligned} \tag{28}$$

Thus,

$$\begin{aligned}
 \log^{[p+a-q]} \mu(r, f_n) &\leq (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \\
 &\quad \times \log^{[a]} M(r, g) + O(1) \quad (29) \\
 &\quad \text{when } n \text{ is even.}
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 \log^{[p+a-q]} \mu(r, f_n) &\leq (\rho_g(a, b) + \varepsilon)^{(n-1)/2} \\
 &\quad \times \log^{[b]} M(r, f) + O(1) \quad (30) \\
 &\quad \text{when } n \text{ is odd, } n \neq 1,
 \end{aligned}$$

and this proves the sixth and the fifteenth part of the lemma, respectively.

Case VII. Let $p < b, q < a, b - p < a - q$. Therefore it follows from (28) for all sufficiently large values of r that

$$\begin{aligned}
 \log^{[p+a-q+a-q-b+p]} \mu(r, f_n) &\leq \log^{[a]} M(M(r, f_{n-4}), g) \\
 &\quad + O(1) \quad (31) \\
 &\quad \vdots
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \log^{[p+a-q+(n-2)/2(a-q-b+q)]} \mu(r, f_n) &\leq \log^{[a]} M(r, g) \\
 &\quad + O(1) \quad (32) \\
 &\quad \text{when } n \text{ is even.}
 \end{aligned}$$

Similarly,

$$\begin{aligned} & \log^{[p+a-q+((n-3)/2)(a-q-b+q)]} \mu(r, f_n) \\ & \leq (\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f) + O(1) \quad (33) \\ & \text{when } n \text{ is odd, } n \neq 1. \end{aligned}$$

This establishes the seventh and the sixteenth part of the lemma respectively.

Case VIII. Let $p > b$, $q > a$, $q - a = p - b$. We obtain from (12) for all sufficiently large values of r that

$$\begin{aligned} & \log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g_{n-1}), \\ \text{i.e., } & \log^{[p]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \log^{[q-a]} \log^{[a]} M(r, g_{n-1}), \\ \text{i.e., } & \log^{[p]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \\ & \quad \times \log^{[q-a]} [\log^{[a]} M(M(r, f_{n-2}), g)], \\ \text{i.e., } & \log^{[p]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \\ & \quad \times \log^{[q-a]} [(\rho_g(a, b) + \varepsilon) \log^{[b]} M(r, f_{n-2})], \\ \text{i.e., } & \log^{[p]} \mu(r, f_n) \\ & \leq (\rho_f(p, q) + \varepsilon) \log^{[p]} M(M(r, g_{n-3}), f) + O(1) \\ & \vdots \end{aligned} \quad (34)$$

Thus,

$$\begin{aligned} \log^{[p]} \mu(r, f_n) & \leq (\rho_f(p, q) + \varepsilon)^{n/2} \log^{[q]} M(r, g) \\ & + O(1) \quad \text{when } n \text{ is even.} \end{aligned} \quad (35)$$

Analogously,

$$\begin{aligned} \log^{[p]} \mu(r, f_n) & \leq (\rho_f(p, q) + \varepsilon)^{(n-1)/2} \log^{[p]} M(r, f) \\ & + O(1) \quad \text{when } n \text{ is odd, } n \neq 1, \end{aligned} \quad (36)$$

from which the eighth and the seventeenth part of the lemma follow.

Case IX. Let $p > b$, $q > a$, $q - a < p - b$. We get from (34) for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} \mu(r, f_n) & \leq (\rho_f(p, q) + \varepsilon) \\ & \quad \times \log^{[q-a+b]} M(r, f_{n-2}) + O(1), \\ \text{i.e., } \log^{[p+a-b-q]} \mu(r, f_n) & \quad (37) \\ & \leq \log^{[p]} M(M(r, g_{n-3}), f) + O(1) \\ & \vdots \end{aligned}$$

Therefore,

$$\begin{aligned} \log^{[p+((n-2)/2)(p+a-b-q)]} \mu(r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \quad \text{when } n \text{ is even.} \end{aligned} \quad (38)$$

Similarly,

$$\begin{aligned} \log^{[p+((n-1)/2)(p+a-b-q)]} \mu(r, f_n) & \leq \log^{[p]} M(r, f) + O(1) \\ & \quad \text{when } n \text{ is odd, } n \neq 1. \end{aligned} \quad (39)$$

This proves the ninth and the eighteenth part of the lemma, respectively.

Thus the lemma follows. $\square$

Lemma 7 (see [8]). *Let g be an entire function. Then for any $\delta(> 0)$ the function $r^{\lambda_g^{[l]} + \delta - \lambda_g^{[l]}(r)}$ is an increasing function of r .*

Lemma 8 (see [8]). *Let g be an entire function. Then for any $\delta(> 0)$ the function $r^{\rho_g^{[l]} + \delta - \rho_g^{[l]}(r)}$ is an increasing function of r .*

3. Theorems

In this section we present the main results of the paper.

Theorem 9. *If f and g are any two entire functions such that $\rho_f(p, q)$ and $\rho_g^{[l]}$ are both finite where p, q, l are positive integers with $p \geq q$ and $l \geq 2$, then for any even number n and $\beta > 1$,*

$$\begin{aligned} \text{(i)} \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} & \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g^{[l]}} \\ & \quad \text{if } p > 1, \quad q = l > 2, \\ \text{(ii)} \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log \mu(r, g)} & \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\lambda_g} \\ & \quad \text{if } p > 1, \quad q = l = 2, \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} & \leq [\rho_f(p, q)]^{n/2} \cdot [2\beta]^{\lambda_g^{[l]}} \\ & \quad \text{if } p > 1, \quad q > l > 2, \quad q - l = p - 1, \end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+l-1-q)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\
& \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g^{[l]}} \quad \text{if } p > 1, \quad q > l > 2, \\
& \quad q - l < p - 1, \\
\text{(v)} \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+1-q)]} \mu(r, f_n)}{\log \mu(r, g)} \\
& \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\lambda_g} \quad \text{if } p > 1, \quad q > l = 2, \\
& \quad q - l < p - 1, \\
\text{(vi)} \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}} \\
& \quad \text{if } q < l - 1, \quad 1 < p, \\
\text{(vii)} \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}} \\
& \quad \text{if } p = q = 1, \quad l - 1 > q, \quad n > 2, \\
\text{(viii)} \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}} \\
& \quad \text{if } p = l, \quad l - 1 > q = 1.
\end{aligned} \tag{40}$$

Proof. Putting $R = \beta r$ in the inequality $M(r, f) \leq R/(R - r)\mu(R, f)$ {cf. [2]} and in view of the inequality $T(r, g) \leq \log^+ M(r, g)$ we get that

$$\log \mu(\beta r, g) + O(1) \geq \log M(r, g) \geq T(r, g), \tag{41}$$

$$\text{i.e., } \log^{[l-1]} \mu(r, g) + O(1) \geq \log^{[l-2]} T\left(\frac{r}{\beta}, g\right). \tag{42}$$

Let $l > 2$.

Since $\liminf_{r \rightarrow \infty} (\log^{[l-2]} T(r, g)/r^{\lambda_g^{[l]}(r)}) = 1$, for given ε ($0 < \varepsilon < 1$) we get for a sequence of values of r tending to infinity that

$$\log^{[l-2]} T(r, g) < (1 + \varepsilon) r^{\lambda_g^{[l]}(r)} \tag{43}$$

and for all large positive numbers of r ,

$$\log^{[l-2]} T(r, g) > (1 - \varepsilon) r^{\lambda_g^{[l]}(r)}. \tag{44}$$

Since $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of r tending to infinity we get for any $\delta(> 0)$ that

$$\begin{aligned}
& \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)} \\
& \leq \frac{\log^{[l-2]} T(2r, g) + O(1)}{\log^{[l-2]} T(r/\beta, g)} \\
& \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\lambda_g^{[l]} + \delta}}{(2r)^{\lambda_g^{[l]} + \delta - \lambda_g^{[l]}(2r)}} \cdot \frac{1}{(r/\beta)^{\lambda_g^{[l]}(r/\beta)}} + O(1) \\
& \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2\beta r/\beta)^{\lambda_g^{[l]} + \delta}}{(2\beta r/\beta)^{\lambda_g^{[l]} + \delta - \lambda_g^{[l]}(2\beta r/\beta)}} \cdot \frac{1}{(r/\beta)^{\lambda_g^{[l]}(r/\beta)}} + O(1) \\
& \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\lambda_g^{[l]} + \delta}
\end{aligned} \tag{45}$$

because $r^{\lambda_g^{[l]} + \delta - \lambda_g^{[l]}(r)}$ is an increasing function of r by Lemma 7.

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are both arbitrary, we get from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)} \leq (2\beta)^{\lambda_g^{[l]}}. \tag{46}$$

Again let $l = 2$.

Since $\liminf_{r \rightarrow \infty} T(r, g)/r^{\lambda_g(r)} = 1$, in view of condition (v) of Definition 4 it follows for a sequence of values of r tending to infinity and for given ε ($0 < \varepsilon < 1$) that

$$T(r, g) < (1 + \varepsilon) r^{\lambda_g(r)} \tag{47}$$

and for all sufficiently large values of r ,

$$T(r, g) > (1 - \varepsilon) r^{\lambda_g(r)}. \tag{48}$$

As $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of r tending to infinity we get for any $\delta(> 0)$ that

$$\begin{aligned}
\frac{\log M(r, g)}{T(r/\beta, g)} & \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\lambda_g + \delta}}{(2r)^{\lambda_g + \delta - \lambda_g(2r)}} \cdot \frac{1}{(r/\beta)^{\lambda_g(r/\beta)}} \\
& \quad + O(1), \\
\text{i.e., } \frac{\log M(r, g)}{T(r/\beta, g)} & \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\lambda_g + \delta}
\end{aligned} \tag{49}$$

because $r^{\lambda_g + \delta - \lambda_g(r)}$ is an increasing function of r by Lemma 7.

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are both arbitrary, we get from (49) that

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r/\beta, g)} \leq 3 \cdot (2\beta)^{\lambda_g}. \tag{50}$$

Case I. Let $p > 1$ and $q = l > 2$. Then from the third part of Lemma 6, we obtain for all sufficiently large values of r that

$$\log^{[(n/2)(p-1)+1]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \times \log^{[l-1]} M(r, g) + O(1). \quad (51)$$

Since $\varepsilon(> 0)$ we get from (42) and above that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq \rho_f(p, q) \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (52)$$

Case II. Let $p > 1$, $q > l > 2$ and $q - l = p - 1$. Then from the eighth part of Lemma 6, we obtain for all sufficiently large values of r that

$$\log^{[p]} \mu(r, f_n) \leq (\rho_f(p, q) + \varepsilon)^{n/2} \log^{[l-1]} M(r, g) + O(1). \quad (53)$$

Since $\varepsilon(> 0)$ we get from (42) and above that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq [\rho_f(p, q)]^{n/2} \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (54)$$

Case III. Let $p > 1$, $q > l$ and $q - l < p - 1$. Then from the ninth part of Lemma 6, we obtain for all sufficiently large values of r that

$$\begin{aligned} \log^{[p+((n-2)/2)(p+l-1-q)]} \mu(r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \log^{[l-1]} M(r, g) + O(1). \end{aligned} \quad (55)$$

Since $\varepsilon(> 0)$ we get from (42) and above that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+l-1-q)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq \rho_f(p, q) \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (56)$$

Case IV. Let $q < l - 1$ and $1 < p$. Then from the first part of Lemma 6 and (42), we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[l-q-1]} \log^{[(n-2)/2)(l-q)+(n/2)(p-1)+1]} \mu(r, f_n) \\ \leq \log^{[l-q-1]} \{(\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1)\}, \\ \text{i.e., } \log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n) \leq \log^{[l-1]} M(r, g) + O(1), \\ \text{i.e., } \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq \frac{\log^{[l-1]} M(r, g) + O(1)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (57)$$

Therefore we get from above that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (58)$$

Case V. Let $p = q = 1$, $l - 1 > q$ and $n > 2$. Then from the second part of Lemma 6 and (42) we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[l-2]} \log^{[(n-2)/2)(l-1)+1]} \mu(r, f_n) \\ \leq \log^{[l-2]} \{(\rho_f(1, 1) + \varepsilon) (\rho_g^{[l]} + \varepsilon) \log M(r, g)\}, \\ \text{i.e., } \log^{[(n/2)(l-1)]} \mu(r, f_n) \leq \log^{[l-1]} M(r, g) + O(1), \\ \text{i.e., } \frac{\log^{[(n/2)(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \frac{\log^{[l-1]} M(r, g) + O(1)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (59)$$

Therefore we get from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \quad (60)$$

Case VI. Let $p = l$ and $l - 1 > q = 1$. Then from (42) and the fifth part of Lemma 6, we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[l-2]} \log^{[(n-1)(l-1)+1]} \mu(r, f_n) \\ \leq \log^{[l-2]} \{(\rho_f(p, q) + \varepsilon) \log M(r, g) + O(1)\} \\ \text{i.e., } \log^{[n(l-1)]} \mu(r, f_n) \leq \log^{[l-1]} M(r, g) + O(1) \\ \text{i.e., } \frac{\log^{[n(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \frac{\log^{[l-1]} M(r, g) + O(1)}{\log^{[l-2]} T(r/\beta, g)}. \end{aligned} \quad (61)$$

Therefore we get from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)}. \quad (62)$$

Now from (52) of Case I and (46), it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g^{[l]}}. \quad (63)$$

This proves the first part of the theorem.

Again for $l = 2$, we obtain in view of (50) and (52) of Case I that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log \mu(r, g)} \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\lambda_g}. \quad (64)$$

Thus the second part of the theorem follows.

Again from (54) of Case II and (46), it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [\rho_f(p, q)]^{n/2} \cdot [2\beta]^{\lambda_g^{[l]}}. \quad (65)$$

This proves the third part of the theorem.

Similarly, from (56) of Case III and (46), it follows that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+l-1-q)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\ \leq \rho_f(p, q) \cdot [2\beta]^{\lambda_g^{[l]}}. \end{aligned} \quad (66)$$

This proves the fourth part of the theorem.

Again for $l = 2$, we obtain in view of (50) and (56) of Case III that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+1-q)]} \mu(r, f_n)}{\log \mu(r, g)} \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\lambda_g}. \quad (67)$$

Thus the fifth part of the theorem follows.

Also from (58) of Case IV and (46), it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}}. \quad (68)$$

Hence the sixth part of the theorem is established.

Similarly, from (60) of Case V and (46), it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}}. \quad (69)$$

This concludes the seventh part of the theorem.

Also from (62) of Case VI and (46), we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\lambda_g^{[l]}}. \quad (70)$$

Thus the eighth part of the theorem is proved. $\square$

Corollary 10. Under the same conditions of Theorem 9, if $l = 2$ then

$$(i) \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+2]} \mu(r, f_n)}{\log^{[3]} \mu(r, g)} \leq 1 \quad \text{if } p > 1, q = l,$$

$$(ii) \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+1-q)+1]} \mu(r, f_n)}{\log^{[q+1]} \mu(r, g)} \leq 1$$

$$\text{if } p > 1, \quad q > l, \quad q - l < p - 1. \quad (71)$$

Proof. In view of (41) and from (49) we have for a sequence of values of r tending to infinity that

$$\log M(r, g) \leq \left\{ \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot 4^{\lambda_g+\delta} \right\} \cdot T\left(\frac{r}{\beta}, g\right),$$

$$\text{i.e., } \log^{[q+1]} M(r, g) \leq \log^{[q]} T\left(\frac{r}{\beta}, g\right) + O(1), \quad (72)$$

$$\text{i.e., } \log^{[q+1]} M(r, g) \leq \log^{[q+1]} \mu(r, g) + O(1).$$

Case I. Let $p > 1$ and $q = l$. Then from the third part of Lemma 6 we obtain for all sufficiently large values of r that

$$\begin{aligned} \log^{[(n/2)(p-1)+1]} \mu(r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1), \end{aligned} \quad (73)$$

$$\text{i.e., } \log^{[(n/2)(p-1)+2]} M(r, f_n) \leq \log^{[3]} M(r, g) + O(1).$$

Now combining (72) and (73) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[(n/2)(p-1)+2]} \mu(r, f_n) &\leq \log^{[3]} \mu(r, g) + O(1), \\ \text{i.e., } \frac{\log^{[(n/2)(p-1)+2]} \mu(r, f_n)}{\log^{[3]} \mu(r, g)} &\leq 1 + \frac{O(1)}{\log^{[3]} \mu(r, g)}. \end{aligned} \quad (74)$$

So from above we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+2]} \mu(r, f_n)}{\log^{[3]} \mu(r, g)} \leq 1. \quad (75)$$

Thus the first part of the corollary follows.

Case II. Let $p > 1, q > l$ and $q - l < p - 1$. Then from the ninth part of Lemma 6 we obtain for all sufficiently large values of r that

$$\begin{aligned} \log^{[p+((n-2)/2)(p+1-q)+1]} \mu(r, f_n) \\ \leq \log^{[q+1]} M(r, g) + O(1). \end{aligned} \quad (76)$$

Therefore combining (72) and (76) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p+((n-2)/2)(p+1-q)+1]} \mu(r, f_n) &\leq \log^{[q+1]} \mu(r, g) + O(1) \\ \text{i.e., } \frac{\log^{[p+((n-2)/2)(p+1-q)+1]} \mu(r, f_n)}{\log^{[q+1]} \mu(r, g)} &\leq 1 + \frac{O(1)}{\log^{[q+1]} \mu(r, g)}. \end{aligned} \quad (77)$$

So from above we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+1-q)+1]} \mu(r, f_n)}{\log^{[q+1]} \mu(r, g)} \leq 1. \quad (78)$$

Thus the second part of the corollary follows. $\square$

In the line of Theorem 9, we may state the following theorem without its proof.

Theorem 11. Let f and g be any two entire functions such that $\rho_g(a, b)$ and $\rho_f^{[l]}$ are both finite where a, b, l are positive integers with $a \geq b$ and $l \geq 2$. Then for any $\beta > 1$,

$$\begin{aligned}
 (i) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-1)/2)(a-1)+((n-2)/2)((l-b)-2)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \\
 & \leq [2\beta]^{\lambda_f^{[l]}} \quad \text{if } a > 1, \quad b < l-1, \\
 (ii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-2)/2)((l-b)-2)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq [2\beta]^{\lambda_f^{[l]}} \\
 & \quad \text{if } a = 1, \quad b < l-1, \\
 (iii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)-b+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq [2\beta]^{\lambda_f^{[l]}} \\
 & \quad \text{if } l = a > 1, \quad l-1 > b = 1, \\
 (iv) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-1)/2)(a-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq \rho_g(a, b) \cdot [2\beta]^{\lambda_f^{[l]}} \\
 & \quad \text{if } l = b > 2, \quad a > 1, \\
 (v) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)/2+2]} \mu(r, f_n)}{\log \mu(r, f)} \leq 3 \cdot \rho_g(2, 2) \cdot [2\beta]^{\lambda_f} \\
 & \quad \text{if } a = b = l = 2, \\
 (vi) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+a-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \\
 & \leq [\rho_g(a, b)]^{(n-1)/2} \cdot [2\beta]^{\lambda_f^{[l]}} \quad \text{if } a > 1, \quad b > l, \\
 & \quad b-l = a-1, \\
 (vii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+a+((n-3)/2)(a-b)-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \\
 & \leq \rho_g(a, b) \cdot [2\beta]^{\lambda_f^{[l]}} \quad \text{if } a > 1, \quad b > l > 2, \\
 & \quad b-l < a-1, \\
 (viii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[a+((n-3)/2)(a-b)+1]} \mu(r, f_n)}{\log \mu(r, f)} \\
 & \leq 3 \cdot \rho_g(a, b) \cdot [2\beta]^{\lambda_f} \quad \text{if } a > 1, \quad b > l = 2, \\
 & \quad b-l < a-1
 \end{aligned} \tag{79}$$

when n is odd and $n \neq 1$.

Corollary 12. Under the same conditions of Theorem 11, if $l = 2$ then

$$\begin{aligned}
 (i) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)/2+2]} \mu(r, f_n)}{\log \mu(r, f)} \leq 1 \quad \text{if } a = b = l, \\
 (ii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[a+((n-3)/2)(a-b)+2]} \mu(r, f_n)}{\log^{[b+1]} \mu(r, f)} \leq 1 \\
 & \quad \text{if } a > 1, \quad b > l, \quad b-l < a-1.
 \end{aligned} \tag{80}$$

The proof of the Corollary 12 is omitted as it can be carried out in the line of Theorem 11 and Corollary 10.

Theorem 13. Let f and g be any two entire functions such that $\rho_f(p, q)$ and $\rho_g^{[l]}$ are both finite where p, q, l are positive integers with $p \geq q$ and $l \geq 2$. Then for any even number n ,

$$\begin{aligned}
 (i) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\rho_g^{[l]}} \\
 & \quad \text{if } p > 1, \quad q = l > 2, \\
 (ii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log \mu(r, g)} \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\rho_g} \\
 & \quad \text{if } p > 1, \quad q = l = 2, \\
 (iii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [\rho_f(p, q)]^{n/2} \cdot [2\beta]^{\rho_g^{[l]}} \\
 & \quad \text{if } p > 1, \quad q > l > 2, \quad q-l = p-1, \\
 (iv) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+l-1-q)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \\
 & \leq \rho_f(p, q) \cdot [2\beta]^{\rho_g^{[l]}} \quad \text{if } p > 1, \quad q > l > 2, \\
 & \quad q-l < p-1, \\
 (v) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+1-q)]} \mu(r, f_n)}{\log \mu(r, g)} \\
 & \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\rho_g} \quad \text{if } p > 1, \quad q > l = 2, \\
 & \quad q-l < p-1, \\
 (vi) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\rho_g^{[l]}} \\
 & \quad \text{if } q < l-1, \quad 1 < p,
 \end{aligned}$$

$$(vii) \liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\rho_g^{[l]}}$$

$$\text{if } p = q = 1, \quad l - 1 > q, \quad n > 2,$$

$$(viii) \liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\rho_g^{[l]}}$$

$$\text{if } p = l, \quad l - 1 > q = 1.$$

(81)

Proof.

Case I. Let $l > 2$. As $\limsup_{r \rightarrow \infty} (\log^{[l-2]} T(r, f) / r^{\rho_f^{[l]}(r)}) = 1$, for given ε ($0 < \varepsilon < 1$) we obtain for all sufficiently large values of r that

$$\log^{[l-2]} T(r, g) < (1 + \varepsilon) r^{\rho_g^{[l]}(r)} \quad (82)$$

and for a sequence of values of r tending to infinity,

$$\log^{[l-2]} T(r, g) > (1 - \varepsilon) r^{\rho_g^{[l]}(r)}. \quad (83)$$

Since $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of r tending to infinity we get for any $\delta(> 0)$ that

$$\begin{aligned} & \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)} \\ & \leq \frac{\log^{[l-2]} T(2r, g) + O(1)}{\log^{[l-2]} T(r/\beta, g)} \\ & \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\rho_g^{[l]} + \delta}}{(2r)^{\rho_g^{[l]} + \delta - \rho_g^{[l]}(2r)}} \cdot \frac{1}{(r/\beta)^{\rho_g^{[l]}(r/\beta)}} + O(1) \\ & \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2\beta r/\beta)^{\rho_g^{[l]} + \delta}}{(2\beta r/\beta)^{\rho_g^{[l]} + \delta - \rho_g^{[l]}(2\beta r/\beta)}} \cdot \frac{1}{(r/\beta)^{\rho_g^{[l]}(r/\beta)}} \\ & \quad + O(1) \\ & \leq \frac{(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\rho_g^{[l]} + \delta} \end{aligned} \quad (84)$$

because $r^{\rho_g^{[l]} + \delta - \rho_g^{[l]}(r)}$ is an increasing function of r by Lemma 8.

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are both arbitrary, we get from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r/\beta, g)} \leq (2\beta)^{\rho_g^{[l]}}. \quad (85)$$

Case II. Let $l = 2$. Since $\limsup_{r \rightarrow \infty} (T(r, g) / r^{\rho_g(r)}) = 1$, in view of condition (v) of Definition 3 it follows for all sufficiently large values of r and for a given ε ($0 < \varepsilon < 1$) that

$$T(r, g) < (1 + \varepsilon) r^{\rho_g(r)} \quad (86)$$

and for a sequence values of r tending to infinity

$$T(r, g) > (1 - \varepsilon) r^{\rho_g(r)}. \quad (87)$$

As $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of r tending to infinity we get for any $\delta(> 0)$ that

$$\begin{aligned} & \frac{\log M(r, g)}{T(r/\beta, g)} \\ & \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\rho_g + \delta}}{(2r)^{\rho_g + \delta - \rho_g(2r)}} \cdot \frac{1}{(r/\beta)^{\rho_g(r/\beta)}} + O(1) \\ & \text{i.e., } \frac{\log M(r, g)}{T(r/\beta, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot (2\beta)^{\rho_g + \delta} \end{aligned} \quad (88)$$

because $r^{\rho_g + \delta - \rho_g(r)}$ is an increasing function of r by Lemma 8.

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are both arbitrary, we get from above that

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r/\beta, g)} \leq 3 \cdot (2\beta)^{\rho_g}. \quad (89)$$

Therefore from (52) and (85) it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq \rho_f(p, q) \cdot [2\beta]^{\rho_g^{[l]}}. \quad (90)$$

This proves the first part of the theorem.

Again for $l = 2$, we obtain in view of (52) and (89) that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(p-1)+1]} \mu(r, f_n)}{\log \mu(r, g)} \leq 3 \cdot \rho_f(p, q) \cdot [2\beta]^{\rho_g}. \quad (91)$$

Thus the second part of the theorem is established.

Similarly, from (58) and (85) we get that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n/2)(l-q)+(n/2)(p-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, g)} \leq [2\beta]^{\rho_g^{[l]}}. \quad (92)$$

Thus the seventh part of the theorem follows.

Now, in the line of Theorem 9 and using the similar manner, one may easily prove the remaining parts of Theorem 13. $\square$

Theorem 14. Let f and g be any two entire functions such that $\rho_g(a, b)$ and $\rho_f^{[l]}$ are both finite where a, b, l are positive integers with $a \geq b$ and $l \geq 2$. Then

$$\begin{aligned}
 (i) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-1)/2)(a-1)+((n-2)/2)(l-b)-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \\
 & \leq [2\beta]^{\rho_f^{[l]}} \quad \text{if } a > 1, \quad b < l-1, \\
 (ii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-2)/2)(l-b)-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq [2\beta]^{\rho_f^{[l]}} \\
 & \quad \text{if } a = 1, \quad b < l-1, \\
 (iii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[n(l-1)-b+1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq [2\beta]^{\rho_f^{[l]}} \\
 & \quad \text{if } l = a > 1, \quad l-1 > b = 1, \\
 (iv) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+((n-1)/2)(a-1)]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq \rho_g(a, b) \cdot [2\beta]^{\rho_f^{[l]}} \\
 & \quad \text{if } l = b > 2, \quad a > 1, \\
 (v) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)/2+2]} \mu(r, f_n)}{\log \mu(r, f)} \leq 3 \cdot \rho_g(2, 2) \cdot [2\beta]^{\rho_f} \\
 & \quad \text{if } a = b = l = 2, \\
 (vi) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[a-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \leq [\rho_g(a, b)]^{(n-1)/2} \cdot [2\beta]^{\rho_f^{[l]}} \\
 & \quad \text{if } a > 1, \quad b > l, \quad b-l = a-1, \\
 (vii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[l+a+((n-3)/2)(a-b)-1]} \mu(r, f_n)}{\log^{[l-1]} \mu(r, f)} \\
 & \leq \rho_g(a, b) \cdot [2\beta]^{\rho_f^{[l]}} \quad \text{if } a > 1, \quad b > l > 2, \\
 & \quad b-l < a-1, \\
 (viii) \quad & \liminf_{r \rightarrow \infty} \frac{\log^{[a+((n-3)/2)(a-b)+1]} \mu(r, f_n)}{\log \mu(r, f)} \\
 & \leq 3 \cdot \rho_g(a, b) \cdot [2\beta]^{\rho_f} \quad \text{if } a > 1, \quad b > l = 2, \\
 & \quad b-l < a-1
 \end{aligned} \tag{93}$$

when n is odd and $n \neq 1$.

The proof of Theorem 14 is omitted as it can be carried out in the line of Theorem 13.

Theorem 15. Let f and g be any two entire functions such that $\rho_g(a, b) < \lambda_f(p, q) \leq \rho_f(p, q) < \infty$ where a, b, p, q are positive integers with $a \geq b$ and $p \geq q$. Then

$$\begin{aligned}
 (i) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[(p-1)+((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{if } p > b, \quad q = a, \\
 (ii) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \quad \text{if } a = b = p = q \\
 & \quad \text{or } p > b, \quad q > a, \quad q-a = p-b, \\
 (iii) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+a-b-q)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{if } p > b, \quad q > a, \quad q-a < p-b, \\
 (iv) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[A]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{where } A = p + \frac{n-2}{2}(a-q) + \frac{n-2}{2}(p-b) \\
 & \quad \quad + a - q - 1 \quad \text{when } q < a \text{ and } b < p, \\
 (v) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(a-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{if } p = b \geq q, \quad a > q, \quad n > 2, \\
 (vi) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{if } p = a > q = b, \\
 (vii) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+2a-2q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{if } p < b, \quad q < a, \quad b-p = a-q, \\
 (viii) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[B]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0 \\
 & \quad \text{where } B = (p-q) + (a+b) + \frac{n}{2}(a-b) - (q+2) + 1 \\
 & \quad \text{when } p < b, \quad q < a \text{ and } b-p < a-q,
 \end{aligned} \tag{94}$$

where n is any even number.

Proof. Since $\rho_g(m, n) < \lambda_f(p, q)$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_g(a, b) + \varepsilon < \lambda_f(p, q) - \varepsilon. \tag{95}$$

Now for all sufficiently large values of r ,

$$\begin{aligned}\log^{[a]} M(\exp^{[b-1]} r, g) &\leq (\rho_g(a, b) + \varepsilon) \log^{[b]} \exp^{[b-1]} r, \\ \text{i.e., } \log^{[a]} M(\exp^{[b-1]} r, g) &\leq (\rho_g(a, b) + \varepsilon) \log r, \\ \text{i.e., } \log^{[a]} M(\exp^{[b-1]} r, g) &\leq \log r^{\rho_g(a, b) + \varepsilon},\end{aligned}\quad (96)$$

$$\text{i.e., } \log^{[a-1]} M(\exp^{[b-1]} r, g) \leq r^{\rho_g(a, b) + \varepsilon}. \quad (97)$$

Again for all sufficiently large values of r , we obtain that

$$\begin{aligned}\log^{[p]} \mu(\exp^{[q-1]} r, f) &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r, \\ \text{i.e., } \log^{[p]} \mu(\exp^{[q-1]} r, f) &\geq (\lambda_f(p, q) - \varepsilon) \log r,\end{aligned}\quad (98)$$

$$\text{i.e., } \log^{[p]} \mu(\exp^{[q-1]} r, f) \geq \log r^{(\lambda_f(p, q) - \varepsilon)}, \quad (99)$$

$$\text{i.e., } \log^{[p-1]} \mu(\exp^{[q-1]} r, f) \geq r^{(\lambda_f(p, q) - \varepsilon)}.$$

Now the following cases may arise.

Case I. Let $p > b$ and $q = a$. Then we have, from the third part of Lemma 6, for all sufficiently large values of r that

$$\begin{aligned}\log^{[(p-1) + ((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \log^{[a-1]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (100)$$

Now from (97) and (100) we have for all sufficiently large values of r that

$$\begin{aligned}\log^{[(p-1) + ((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) r^{\rho_g(a, b) + \varepsilon}.\end{aligned}\quad (101)$$

Case II. Let $a = b = p = q$. Now we obtain from the fourth part of Lemma 6 for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon)^{n/2} (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \\ \times \log^{[a-1]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (102)$$

Then from (97) and (102), we obtain for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon)^{n/2} (\rho_g(a, b) + \varepsilon)^{(n-2)/2} r^{\rho_g(a, b) + \varepsilon}.\end{aligned}\quad (103)$$

Case III. Let $p > b$, $q > a$ and $q - a = p - b$. Then we get from Lemma 6(viii) for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon)^{n/2} \log^{[a-1]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (104)$$

Now from (97) and (104), we get for all sufficiently large values of r that

$$\log^{[p]} \mu(\exp^{[b-1]} r, f_n) \leq (\rho_f(p, q) + \varepsilon)^{n/2} r^{\rho_g(a, b) + \varepsilon}. \quad (105)$$

Case IV. Let $p > b$, $q > a$ and $q - a < p - b$. Now we obtain from Lemma 6(ix) for all sufficiently large values of r that

$$\begin{aligned}\log^{[p + ((n-2)/2)(p+a-b-q)]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \log^{[a-1]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (106)$$

Then from (97) and (106), we have for all sufficiently large values of r that

$$\begin{aligned}\log^{[p + ((n-2)/2)(p+a-b-q)]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) r^{\rho_g(a, b) + \varepsilon}.\end{aligned}\quad (107)$$

Further from (96), it follows for all sufficiently large values of r that

$$\begin{aligned}\exp^{[a-q]} \log^{[a]} \mu(\exp^{[b-1]} r, g) &\leq \exp^{[a-q]} \log r^{\rho_g(a, b) + \varepsilon}, \\ \text{i.e., } \exp^{[a-q]} \log^{[a]} \mu(\exp^{[b-1]} r, g) &\leq \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon}.\end{aligned}\quad (108)$$

Case V. Let $q < a$ and $b < p$. Then for all sufficiently large values of r we get in view of the first part of Lemma 6 that

$$\begin{aligned}\log^{[(p-1) + ((n-2)/2)(a-q) + ((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \exp^{[a-q]} \log^{[a]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (109)$$

Now from (108) and (109), we have for all sufficiently large values of r that

$$\begin{aligned}\log^{[p + ((n-2)/2)(a-q) + ((n-2)/2)(p-b)]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon},\end{aligned}$$

$$\begin{aligned}\text{i.e., } \log^{[p + ((n-2)/2)(a-q) + ((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon},\end{aligned}$$

$$\begin{aligned}\text{i.e., } \log^{[p + ((n-2)/2)(a-q) + ((n-2)/2)(p-b)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon},\end{aligned}$$

$$\begin{aligned}\text{i.e., } \log^{[p + ((n-2)/2)(a-q) + ((n-2)/2)(p-b)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\ \leq r^{\rho_g(a, b) + \varepsilon}.\end{aligned}\quad (110)$$

Case VI. Let $p = b \geq q$, $a > q$ and $n > 2$. Now for all sufficiently large values of r we get in view of the second part of Lemma 6 that

$$\begin{aligned}\log^{[p + ((n-2)/2)(a-q)]} \mu(\exp^{[b-1]} r, f_n) \\ \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \\ \times \exp^{[a-q]} \log^{[a]} M(\exp^{[b-1]} r, g).\end{aligned}\quad (111)$$

Then from (108) and (111), we have for all sufficiently large values of r that

$$\begin{aligned}
 & \log^{[p+((n-2)/2)(a-q)]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) (\rho_g(a, b) + \varepsilon) \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon}, \\
 & \text{i.e., } \log^{[p+((n-2)/2)(a-q)+1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon}, \\
 & \text{i.e., } \log^{[p+((n-2)/2)(a-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon}, \\
 & \text{i.e., } \log^{[p+((n-2)/2)(a-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \leq r^{\rho_g(a, b) + \varepsilon}.
 \end{aligned} \tag{112}$$

Case VII. Let $p = a > q = b$. Then for all sufficiently large values of r we get in view of the fifth part of Lemma 6 that

$$\begin{aligned}
 & \log^{[p+(n-2)(p-q)]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \exp^{[a-q]} \log^{[a]} M(\exp^{[b-1]} r, g).
 \end{aligned} \tag{113}$$

Now from (108) and (113), we have for all sufficiently large values of r that

$$\begin{aligned}
 & \log^{[p+(n-2)(p-q)]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq (\rho_f(p, q) + \varepsilon) \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+(n-2)(p-q)+1]} \mu(\exp^{[b-1]} r, f_n) \leq \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \leq r^{\rho_g(a, b) + \varepsilon}.
 \end{aligned} \tag{114}$$

Case VIII. Let $p < b, q < a$ and $b - p = a - q$. Now for all sufficiently large values of r we get in view of the sixth part of Lemma 6 that

$$\begin{aligned}
 & \log^{[p+a-q]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \exp^{[a-q]} \log^{[a]} M(\exp^{[b-1]} r, g).
 \end{aligned} \tag{115}$$

Then from (108) and (104), we have for all sufficiently large values of r that

$$\begin{aligned}
 & \log^{[p+a-q]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq (\rho_g(a, b) + \varepsilon)^{(n-2)/2} \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+a-q+1]} \mu(\exp^{[b-1]} r, f_n) \leq \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+a-q+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+2a-2q-1]} \mu(\exp^{[b-1]} r, f_n) \leq r^{\rho_g(a, b) + \varepsilon}.
 \end{aligned} \tag{116}$$

Case IX. Let $p < b, q < a$ and $b - p < a - q$. Then for all sufficiently large values of r we get in view of the seventh part of Lemma 6 that

$$\begin{aligned}
 & \log^{[p+a+((n-2)/2)(a-b)-q]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \exp^{[a-q]} \log^{[a]} M(\exp^{[b-1]} r, g).
 \end{aligned} \tag{117}$$

Now from (108) and (117), we have for all sufficiently large values of r that

$$\begin{aligned}
 & \log^{[p+a+((n-2)/2)(a-b)-q]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \exp^{[a-q-1]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+a+((n-2)/2)(a-b)-q+1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+a+((n-2)/2)(a-b)-q+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a, b) + \varepsilon} \\
 & \text{i.e., } \log^{[p+a+((n-2)/2)(a-b)-q+a-q-1]} \mu(\exp^{[b-1]} r, f_n) \\
 & \leq r^{\rho_g(a, b) + \varepsilon}.
 \end{aligned} \tag{118}$$

Now combining (101) of Case I and (99), we get for all sufficiently large values of r that

$$\begin{aligned}
 & \frac{\log^{[p+((n-2)/2)(p-b)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\
 & \leq \frac{(\rho_f(p, q) + \varepsilon) r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}.
 \end{aligned} \tag{119}$$

Now in view of (95), it follows from (119) that

$$\begin{aligned}
 & \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p-b)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0, \\
 & \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p-b)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0.
 \end{aligned} \tag{120}$$

This proves the first part of the theorem.

Similarly combining (95), (103) of Case II, and (99), we obtain for all sufficiently large values of r that

$$\begin{aligned}
 & \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\
 & \leq \frac{(\rho_f(p, q) + \varepsilon)^{n/2} (\rho_g(a, b) + \varepsilon)^{(n-2)/2} r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \\
 & \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0, \\
 & \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0.
 \end{aligned} \tag{121}$$

Further combining (105) of Case III and (99) and in view of (95) it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &\leq \frac{(\rho_f(p, q) + \varepsilon)^{n/2} r^{(\rho_g(a, b) + \varepsilon)}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0, \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0. \end{aligned} \quad (122)$$

Thus the second part of the theorem follows from (121) and (122).

Again combining (107) of Case IV and (99), we get for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+((n-2)/2)(p+a-b-q)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\ \leq \frac{(\rho_f(p, q) + \varepsilon) r^{(\rho_g(a, b) + \varepsilon)}}{r^{(\lambda_f(p, q) - \varepsilon)}}. \end{aligned} \quad (123)$$

Therefore in view of (95) we get from (123) that

$$\lim_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+a-b-q)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0. \quad (124)$$

This proves the third part of the theorem.

Similarly combining (110) of Case V and (99), we obtain in view of (95) for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+((n-2)/2)(a-q)+((n-2)/2)(p-b)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\ \leq \frac{r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \end{aligned} \quad (125)$$

$$\text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[A]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} = 0$$

$$\text{where } A = p + \frac{n-2}{2}(a-q) + \frac{n-2}{2}(p-b) + a - q - 1. \quad (126)$$

This establishes the fourth part of the theorem.

Analogously, in view of (95), (112) of Case VI, and (99) it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+((n-2)/2)(a-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &\leq \frac{r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(a-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0. \end{aligned} \quad (127)$$

Thus the fifth part of the theorem follows from above.

Again combining (95), (114) of Case VII, and (99) we obtain for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &\leq \frac{r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \\ \limsup_{r \rightarrow \infty} \frac{\log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0, \quad (128) \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p+(n-2)(p-q)+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0, \end{aligned}$$

which is the sixth part of the theorem.

Further in view of (95), (116) of Case VIII, and (99) we get for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+2a-2q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &\leq \frac{r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}}, \quad (129) \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p+2a-2q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0. \end{aligned}$$

This proves the seventh part of the theorem.

Similarly, combining (95), (118) of Case IX, and (99) it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[p+a+((n-2)/2)(a-b)-q+a-q-1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\ \leq \frac{r^{\rho_g(a, b) + \varepsilon}}{r^{(\lambda_f(p, q) - \varepsilon)}} \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[(p-q)+(a+b)+(n/2)(a-b)-(q+1)]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\ = 0. \end{aligned} \quad (130)$$

This establishes the eighth part of the theorem. $\square$

Remark 16. The condition $\rho_g(a, b) < \lambda_f(p, q)$ in Theorem 15 is essential as we see in the following example.

Example 17. Let $f = g = \exp z$ and $p = a = n = 2$, $q = b = 1$. Then $\rho_g(a, b) = \lambda_f(p, q) = \rho_f(p, q) = 1$. Now

$$\begin{aligned} \log \mu(r, f \circ g) &\geq \log \mu\left(\frac{r}{2}, f \circ g\right) + O(1) \geq T\left(\frac{r}{2}, f \circ g\right) \\ &\quad + O(1) \\ &= T\left(\frac{r}{2}, \exp^{[2]} z\right) + O(1) \sim \frac{\exp(r/2)}{(2\pi^3(r/2))^{1/2}} \\ &\quad + O(1) \quad (r \rightarrow \infty), \end{aligned}$$

$$\text{i.e., } \log^{[2]} \mu(r, f \circ g) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1),$$

$$\log \mu(r, f) \leq \log M(r, f) = \log M(r, \exp z) = r. \quad (132)$$

Then

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \\
 &= \lim_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r, f)} \\
 &\geq \lim_{r \rightarrow \infty} \frac{(r/2) - (1/2) \log r + O(1)}{r} \\
 &= 1 \neq 0,
 \end{aligned} \tag{133}$$

which is contrary to Theorem 15.

Remark 18. Theorem 15 is still valid with “limit inferior” instead of “limit” if we replace the condition “ $\rho_g(a, b) < \lambda_f(p, q) \leq \rho_f(p, q) < \infty$ ” by “ $\lambda_g(a, b) < \lambda_f(p, q) \leq \rho_f(p, q) < \infty$ ”.

Remark 19. Considering $f = g = \exp z$ and $p = a = n = 2, q = b = 1$, one can easily verify that the condition $\lambda_g(a, b) < \lambda_f(p, q)$ in Remark 18 is essential.

In the line of Theorem 15, we may state the following theorem without its proof.

Theorem 20. Let f and g be any two entire functions such that $\rho_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty$ where a, b, p, q are positive integers with $a \geq b$ and $p \geq q$. Then

$$\begin{aligned}
 \text{(i)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[(p-1)+((n-3)/2)(p-b)+1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p > b, \quad q = a, \\
 \text{(ii)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \quad \text{if } a = b = p = q \\
 & \quad \text{or } p > b, \quad q > a, \quad q - a = p - b, \\
 \text{(iii)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+((n-1)/2)(p+a-b-q)]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p > b, \quad q > a, \quad q - a < p - b, \\
 \text{(iv)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[C]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{where } C = 2p + \frac{n-1}{2}(a-q) + \frac{n-2}{2}(p-b) - b - 1 \\
 & \quad \text{when } q < a \text{ and } b < p, \\
 \text{(v)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[2p+((n-1)/2)(a-q)-b-1]} M(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p = b \geq q, \quad a > q,
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[2p+(n-2)(p-q)-b-1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p = a > q = b,
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[2p+a-q-b-1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p < b, \quad q < a, \quad b - p = a - q,
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad & \lim_{r \rightarrow \infty} \frac{\log^{[2p+a+((n-3)/2)(a-b)-q-b-1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a-1]} \mu(\exp^{[b-1]} r, g)} = 0 \\
 & \quad \text{if } p < b, \quad q < a, \quad b - p < a - q
 \end{aligned} \tag{134}$$

when n is odd and $n \neq 1$.

The proof is omitted.

Remark 21. In Theorem 20 if we take the condition $\lambda_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty$ instead of $\rho_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty$, then also Theorem 20 remains true with “limit inferior” in place of “limit”.

Theorem 22. Let f and g be any two entire functions such that $\lambda_f(p, q) \leq \rho_f(p, q) < \infty$ and $\rho_g(a, b) < \infty$ where a, b, p, q are positive integers with $a \geq b$ and $p \geq q$. Then

$$\begin{aligned}
 \text{(i)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
 & \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} = 0 \quad \text{if } p > b, \quad q = a, \\
 \text{(ii)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \\
 & \quad \text{if } a = b = p = q \quad \text{or } p > b, \quad q > a, \\
 & \quad \quad \quad q - a = p - b, \\
 \text{(iii)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(p+a-b-q)+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
 & \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \quad \text{if } p > b, \quad q > a, \quad q - a < p - b, \\
 \text{(iv)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[D]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \\
 & \quad \text{where } D = p + \frac{n-2}{2}(a-q) + \frac{n-2}{2}(p-b) + a - q \\
 & \quad \text{when } q < a \text{ and } b < p,
 \end{aligned}$$

$$\begin{aligned}
\text{(v)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-2)/2)(a-q)+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \quad \text{if } p = b \geq q, \quad a > q, \quad n > 2, \\
\text{(vi)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+(n-2)(p-q)+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \quad \text{if } p = a > q = b, \\
\text{(vii)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+2a-2q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \\
& \quad \text{if } p < b, \quad q < a, \quad b - p = a - q, \\
\text{(viii)} \quad & \limsup_{r \rightarrow \infty} \frac{\log^{[p+a+((n-2)/2)(a-b)-q+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)} \quad \text{if } p < b, \quad q < a, \quad b - p < a - q,
\end{aligned} \tag{135}$$

where n is any even number.

Proof. Let $p > b$ and $q = a$.

Now from (101) we get for all sufficiently large values of r that

$$\begin{aligned}
& \log^{[p+((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n) \\
& \leq (\rho_g(a, b) + \varepsilon) \log r + O(1).
\end{aligned} \tag{136}$$

Therefore combining (98) and (136), we get for all sufficiently large values of r that

$$\begin{aligned}
& \frac{\log^{[p+((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& \leq \frac{(\rho_g(a, b) + \varepsilon) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r}, \\
\text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[p+((n-2)/2)(p-b)+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}.
\end{aligned} \tag{137}$$

This proves the first part of the theorem.

Further suppose that $a = b = p = q$. Then from (98) and (103), we obtain for all sufficiently large values of r that

$$\begin{aligned}
& \frac{\log^{[p+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_g(a, b) + \varepsilon) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r}, \\
\text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[p+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}.
\end{aligned} \tag{138}$$

Again let $p > b$, $q > a$ and $q - a = p - b$. Then also combining (98) and (105), it follows for all sufficiently large values of r that

$$\begin{aligned}
& \frac{\log^{[p+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_g(a, b) + \varepsilon) \log r + O(1)}{(\lambda_f(p, q) - \varepsilon) \log r} \\
\text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[p+1]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \leq \frac{\rho_g(a, b)}{\lambda_f(p, q)}.
\end{aligned} \tag{139}$$

Thus the second part of the theorem follows from (138) and (139).

Similarly, using the same technique of above one can easily prove the remaining parts of Theorem 22 from (107), (110), (112), (114), (116), and (118), respectively, and with the help of inequality (98). Hence their proofs are omitted. $\square$

Remark 23. The condition $\rho_g(a, b) < \infty$ in Theorem 22 is necessary which is evident from the following example.

Example 24. Let $f = \exp z$, $g = \exp^{[2]} z$ and $p = a = n = 2$, $q = b = 1$.

Then $\lambda_f(p, q) = \rho_f(p, q) = 1$ and $\rho_g(a, b) = \infty$.

Now

$$\begin{aligned}
& \log^{[3]} \mu(r, f \circ g) \geq \log^{[3]} M\left(\frac{r}{2}, f \circ g\right) + O(1), \\
\text{i.e., } \log^{[3]} \mu(r, f \circ g) & \geq \log^{[3]} \exp^{[3]} \left(\frac{r}{2}\right) + O(1),
\end{aligned} \tag{140}$$

$$\text{i.e., } \log^{[3]} \mu(r, f \circ g) = \left(\frac{r}{2}\right) + O(1),$$

$$\log^{[2]} \mu(r, f) \leq \log^{[2]} M(r, f) = \log r.$$

Therefore

$$\begin{aligned}
& \limsup_{r \rightarrow \infty} \frac{\log^{[p+(n-2)(p-q)+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& = \limsup_{r \rightarrow \infty} \frac{\log^{[3]} \mu(r, f \circ g)}{\log^{[2]} \mu(r, f)}, \\
\text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[p+(n-2)(p-q)+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} \\
& = \limsup_{r \rightarrow \infty} \frac{(r/2) + O(1)}{\log r}, \\
\text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[p+(n-2)(p-q)+a-q]} \mu(\exp^{[b-1]} r, f_n)}{\log^{[p]} \mu(\exp^{[q-1]} r, f)} = \infty,
\end{aligned} \tag{141}$$

which is contrary to Theorem 22.

Remark 25. Theorem 22 is still valid with “limit inferior” instead of “limit superior” if we replace $\rho_g(a, b)$ by $\lambda_g(a, b)$.

Remark 26. Considering $f = \exp z$, $g = \exp^{[2]}z$ and $p = a = n = 2$, $q = b = 1$, one can easily verify that the condition $\lambda_g(a, b) < \infty$ in Remark 25 is essential.

Theorem 27. Let f and g be any two entire functions such that $\lambda_g(a, b) \leq \rho_g(a, b) < \infty$ and $\rho_f(p, q) < \infty$ where a, b, p, q are positive integers with $a \geq b$ and $p \geq q$. Then

$$\begin{aligned}
 & \text{(i) } \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-3)/2)(p-b)+1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \\
 & \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \quad \text{if } p > b, \quad q = a, \\
 & \text{(ii) } \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \\
 & \quad \text{if } a = b = p = q \quad \text{or} \quad p > b, \quad q > a, \\
 & \quad \quad \quad q - a = p - b, \\
 & \text{(iii) } \limsup_{r \rightarrow \infty} \frac{\log^{[p+((n-1)/2)(p+a-b-q)+1]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \\
 & \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \quad \text{if } p > b, \quad q > a, \quad q - a < p - b, \\
 & \text{(iv) } \limsup_{r \rightarrow \infty} \frac{\log^{[E]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \\
 & \quad \text{where } E = 2p + \frac{n-1}{2}(a-q) + \frac{n-2}{2}(p-b) - b \\
 & \quad \text{if } q < a \text{ and } b < p, \\
 & \text{(v) } \limsup_{r \rightarrow \infty} \frac{\log^{[2p+((n-1)/2)(a-q)-b]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \\
 & \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \quad \text{if } p = b \geq q, \quad a > q, \\
 & \text{(vi) } \limsup_{r \rightarrow \infty} \frac{\log^{[2p+(n-2)(p-q)-b]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \\
 & = \frac{\rho_f(p, q)}{\lambda_g(a, b)} \quad \text{if } p = a > q = b, \\
 & \text{(vii) } \limsup_{r \rightarrow \infty} \frac{\log^{[2p+a-q-b]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \\
 & \quad \text{if } p < b, \quad q < a, \quad b - p = a - q, \\
 & \text{(viii) } \limsup_{r \rightarrow \infty} \frac{\log^{[2p+a+((n-3)/2)(a-b)-q-b]} \mu(\exp^{[q-1]} r, f_n)}{\log^{[a]} \mu(\exp^{[b-1]} r, g)} \\
 & \leq \frac{\rho_f(p, q)}{\lambda_g(a, b)} \quad \text{if } p < b, \quad q < a, \quad b - p < a - q
 \end{aligned}$$

(142)

when n is odd and $n \neq 1$.

The proof is omitted.

Remark 28. In Theorem 27 if we replace $\rho_f(p, q)$ by $\lambda_f(p, q)$ and the other conditions remaining the same, then also Theorem 20 remains true with “limit inferior” in place of “limit superior.”

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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