Research Article

Inferior Mean of Measures on Curves and Subspaces

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We obtain new sharp upper bounds of the inferior mean for positive harmonic functions defined by finite boundary measures that lie on curves or subspaces of the boundary of the half-space.

“Dedicated to L. D. Ivanov and my mother, Faina M. Movshovich”

1. Introduction

Let Ω be the unit ball, S^n , or the upper half-space, R_+^{n+1} , in R^{n+1} , and let $\partial\Omega$ be the boundary of Ω . The notion of the *inferior mean*, $\text{IM}(u)$, of a positive function u is due to Maurice Heins.

Definition 1. Let u be a positive function defined on Ω and let $A_\delta = \{q \in \Omega : \text{dist}(q, \partial\Omega) < \delta\}$. The inferior mean of u is

$$\text{IM}(u) = \lim_{\delta \rightarrow 0} \inf_{\Gamma \subset A_\delta} \int_{\Gamma} u(q) d\Gamma, \quad (1)$$

where $d\Gamma$ is the volume measure of a continuous, piecewise differentiable, orientable surface Γ that separates boundaries of A_δ .

The study of the inferior mean began with the following unpublished result of Heins, which is obtained in connection with E_p spaces.

If a positive function u has a subharmonic logarithm on the annulus $\{R < |z| < 1\}$ and Γ are rectifiable Jordan curves in $\{r < |z| < 1\}$ separating 0 from ∞ , then

$$\liminf_{r \rightarrow 1} \int_{\Gamma} u(z) |dz| = \lim_{r \rightarrow 1} \int_{|z|=r} u(z) |dz|. \quad (2)$$

In [1], Wu showed that, for a positive harmonic function u defined on the unit disc, the same result holds, if the

boundary measure μ of u is absolutely continuous with respect to the arclength measure of the unit circle; while, for an arbitrary positive harmonic function u in the disc, $(2/\pi)\mu(\partial\Omega) \leq \text{IM}(u) \leq \mu(\partial\Omega)$. In [2], these results of Wu were extended to positive functions u in the harmonic Hardy space $h^1(\Omega)$, which is the space of harmonic functions defined on Ω by finite (positive) Borel boundary measures μ :

$$\frac{\omega_{n+1}}{\pi\omega_n} \mu(\partial\Omega) \leq \text{IM}(u) \leq \mu(\partial\Omega), \quad (3)$$

where ω_k is the area of the unit sphere S^k in R^{k+1} . Equality on the right holds for functions whose boundary measures are absolutely continuous with respect to the Lebesgue measure of $\partial\Omega$.

This paper refines the upper bound in (3) for $u \in h^1(\Omega)$ defined on the upper half-space $\Omega = R_+^{n+1}$ by a finite positive Borel boundary measure μ that lies either on a subspace or on a smooth curve $\gamma \subset \partial\Omega$. With $\mu(\gamma) = \mu(\partial\Omega)$, we will prove the following theorems (throughout smooth means C^k -differentiable, $k \geq 1$).

In Theorem 5 we show that if μ lies on a subspace $\gamma = R^m$, $m \in \{0, \dots, n\}$, then

$$\text{IM}(u) \leq \frac{\omega_{n-m+1}}{\pi\omega_{n-m}} \mu(\gamma). \quad (4)$$

In Theorem 7 we show that equality in (4) holds for boundary measures μ that are absolutely continuous with respect to the Lebesgue measure of γ .

In Theorem 9 we show that if μ lies on a smooth curve γ , then (4) is true with $m = 1$.

In Theorem 10 we show that equality in (4) holds for the boundary measure μ that is the arclength measure of γ .

2. Tubular Coordinates

In this section, we recall a few facts about spherical transformations, obtain the volume elements of hypersurfaces in tubular coordinates, and evaluate three constants occurring in the integration over the tubes. In all sections, depending on the context, we use the same notation for points and position vectors, vectors and line segments, and so forth.

2.1. Spherical Coordinates. Let $q \in R_+^{k+1} = R_+ \times R^k$. In Cartesian coordinates $q = (x_0, \mathbf{x})$, one has $x_0 \in R_+$ and $\mathbf{x} = (x_1, \dots, x_k) \in R^k$. In local spherical coordinates with the origin at $p \in R^k$, spherical radius $r = |q - p|$, the first spherical angle $\phi_1 \in [0, \pi/2]$ is measured from R_+ to $q - p$, and the remaining spherical angles lie in R^k with the range: $\phi_i \in [0, \pi]$, $1 < i < k$, $\phi_k \in [0, 2\pi]$. A k -tuple $(\phi_1, \dots, \phi_k)$ will be denoted by ϕ .

Define Λ_i by $\Lambda_0 = 1$, $\Lambda_i = \prod_{j=1}^i \sin \phi_j$. The spherical transformation from Cartesian to spherical coordinates is given by the identities:

$$x_i = r \Lambda_i \cos \phi_{i+1}, \quad i = 0, \dots, k-1, \quad x_k = r \Lambda_k. \quad (5)$$

We denote by S^k the unit sphere in R^{k+1} and by $S_+^k = S^k \cap R_+^{k+1}$ its upper-half in R^{k+1} . The volume element of a sphere of constant radius r (see e.g., [3, Section 676]) is

$$r^k dS^k = I_k d\phi_1 \cdots d\phi_k, \quad (6)$$

where

$$I_k = \frac{D(x_0, \dots, x_k)}{D(r, \phi_1, \dots, \phi_k)} = \frac{D(x_0, \mathbf{x})}{D(r, \phi)} = r^k \prod_{i=0}^k \Lambda_i. \quad (7)$$

We recall recurrence relations for the area, ω_k , and area element of S^k :

$$\omega_{k+1} = \frac{2\pi}{k} \omega_{k-1}, \quad \omega_0 = 2, \quad \omega_1 = 2\pi, \quad (8)$$

$$\begin{aligned} dS^k(\phi_1, \dots, \phi_k) \\ = \sin^{k-1} \phi_1 d\phi_1 dS^{k-1}(\phi_2, \dots, \phi_k). \end{aligned} \quad (9)$$

2.2. Tubular (Cylindrical) Coordinates. In tubular coordinates, points $q = (r, \phi, \mathbf{z})$ have $(r, \phi) = (x_0, \mathbf{x}) \in R_+^{n-m+1}$ with $r = \text{dist}(q, R^m)$, $\phi = (\phi_1, \dots, \phi_{n-m})$, and $\mathbf{z} \in R^m$. The volume element of a tubular surface T_r^m about R^m of constant radius r is given by (6) and (7):

$$dT_r^m(q) = r^{n-m} dS^{n-m}(\phi) d\mathbf{z} = I_{n-m} d\phi d\mathbf{z}. \quad (10)$$

The next result is a corollary to the surface measure lemma in [2].

Lemma 2. *If a hypersurface Γ is locally given by a C^1 -differentiable function $r = r(\phi, \mathbf{z})$, then its volume element admits this form:*

$$d\Gamma(q) = \sqrt{1 + \sum_{k=1}^{n-m} \frac{r_{\phi_k}^2}{r^2 \Lambda_{k-1}^2} + \sum_{k=1}^m r_{z_k}^2} dT_r^m(q). \quad (11)$$

Proof. By Cramer's rule (see e.g., [2]),

$$\begin{aligned} d\Gamma = & \left(\sum_{k=0}^{n-m} \left| \frac{D(\dots, \widehat{x}_k, \dots, \mathbf{z})}{D(\phi, \mathbf{z})} \right|^2 \right. \\ & \left. + \sum_{k=1}^m \left| \frac{D(x_0, \mathbf{x}, \dots, \widehat{z}_k, \dots)}{D(\phi, \mathbf{z})} \right|^2 \right)^{1/2} d\phi d\mathbf{z}, \end{aligned} \quad (12)$$

where $\widehat{x}_k$ and $\widehat{z}_k$ indicate that columns

$$\begin{aligned} & \left(\frac{\partial x_k}{\partial \phi_1}, \dots, \frac{\partial x_k}{\partial \phi_{n-m}}, \frac{\partial x_k}{\partial z_1}, \dots, \frac{\partial x_k}{\partial z_m} \right), \\ & \left(\frac{\partial z_k}{\partial \phi_1}, \dots, \frac{\partial z_k}{\partial \phi_{n-m}}, \frac{\partial z_k}{\partial z_1}, \dots, \frac{\partial z_k}{\partial z_m} \right) \\ & = \left(0, \dots, 0, \frac{\partial z_k}{\partial z_k}, 0, \dots, 0 \right) \end{aligned} \quad (13)$$

are omitted. The surface measure lemma in [2] computes the first sum in (12) as

$$\begin{aligned} \sum_{k=0}^{n-m} \left| \frac{D(\dots, \widehat{x}_k, \dots, \mathbf{z})}{D(\phi, \mathbf{z})} \right|^2 &= \sum_{k=0}^{n-m} \left| \frac{D(\dots, \widehat{x}_k, \dots)}{D(\phi)} \right|^2 \\ &= I_{n-m}^2 \left(1 + \sum_{k=1}^{n-m} \frac{r_{\phi_k}^2}{r^2 \Lambda_{k-1}^2} \right). \end{aligned} \quad (14)$$

When the chain rule is applied to each Jacobian in the second sum of (12), the definition (7) for I_{n-m} allows us to write this sum as

$$\begin{aligned} \sum_{k=1}^m \left| \frac{D(x_0, \mathbf{x}, \dots, \widehat{z}_k, \dots)}{D(\phi, \mathbf{z})} \right|^2 &= \sum_{k=1}^m \left| \frac{D(x_0, \mathbf{x})}{D(r, \phi)} \frac{D(r, \phi)}{D(\phi, z_k)} \right|^2 \\ &= I_{n-m}^2 \sum_{k=1}^m r_{z_k}^2. \end{aligned} \quad (15)$$

□

2.3. A Remark on Integration in Tubular Coordinates. For $\delta > 0$, let $A_\delta = \{q \in R_+^{n+1} : \text{dist}(q, R^n) < \delta\}$, where R^n is the boundary of R_+^{n+1} . Consider a continuous, piecewise smooth, orientable hypersurface $\Gamma \subset A_\delta$ separating boundaries of A_δ . We assume that each smooth component of Γ has a piecewise smooth boundary. Denote by Ψ the following set:

$$\begin{aligned} \Psi = & \left\{ (\phi, \mathbf{z}) : \phi_1 \in \left[0, \frac{\pi}{2}\right], \phi_i \in [0, \pi], \right. \\ & \left. 1 < i < n-m, \phi_{n-m} \in [0, 2\pi], \mathbf{z} \in R^m \right\}. \end{aligned} \quad (16)$$

Then, Ψ is a subspace of coordinate space $(r, \phi, \mathbf{z})$ in R_+^{n+1} and can be regarded as an n -dim manifold without boundary. A projection $\Gamma \rightarrow \Psi$ given by $(r, \phi, \mathbf{z}) \rightarrow (1, \phi, \mathbf{z}) \rightarrow (\phi, \mathbf{z})$ is continuous, onto, and smooth on each smooth component of Γ . Under this projection by Sard's theorem [4], the image of points $q \in \Gamma$, where $r(\phi, \mathbf{z})$ is defined in some neighborhood of q , has full measure in Ψ . This will justify integration over Γ in coordinates $(r(\phi, \mathbf{z}), \phi, \mathbf{z})$ in Theorems 7 and 10.

2.4. Constants $g_{m,n}$, $G_{m,n}$, and $S_{m,n}$. The first set, $\{g_{m,n}\}$, is defined for $m = 1, \dots, n$. These are Eulerian integrals (see e.g., [3, Section 534]):

$$\begin{aligned} g_{m,n} &= \int_0^\infty g(v) dv = \int_0^\infty \frac{v^{m-1} dv}{(1+v^2)^{(n+1)/2}} \\ &= \frac{\omega_n}{\omega_{m-1}\omega_{n-m}}. \end{aligned} \quad (17)$$

The second set, $\{G_{m,n}\}$, is defined for $m = 0, \dots, n$, where $G_{0,n} = 1$ and

$$\begin{aligned} G_{m,n} &= \int_{R^m} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}} = \omega_{m-1} g_{m,n} \\ &= \frac{\omega_n}{\omega_{n-m}}, \quad (m \geq 1). \end{aligned} \quad (18)$$

The next lemma is used in Theorems 7 and 10 with Y small or 0.

Lemma 3. Let $\rho > 0$, $Y \subset R^m$, and $B(0, \rho) = \{\mathbf{v} \in R^m : |\mathbf{v}| \leq \rho\}$. Then,

$$\begin{aligned} \int_{B(0,\rho) \setminus Y} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}} &> G_{m,n} - \frac{34}{\rho} - |Y|, \\ m &= 1, \dots, n. \end{aligned} \quad (19)$$

Proof. We split the integral into three parts and then use $\int_\rho^\infty g(v) dv \leq 1/\rho$ and $\max_k \omega_k = \omega_6 < 34$ in the middle integral:

$$\begin{aligned} &\int_{R^m} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}} - \int_{R^m \setminus B(0,\rho)} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}} \\ &- \int_{B(0,\rho) \cap Y} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}}. \end{aligned} \quad (20)$$

□

The third set, $\{S_{m,n}\}$, is also defined for $m = 0, \dots, n$. These are the integrals of $\cos \phi_1$ over S_+^k (see (22) below). In the half-space $\Omega = R_+^{n+1}$, it is customary to denote $\text{dist}(q, \partial\Omega)$ by y . If $q = (y, \mathbf{x}, \mathbf{z}) = (r, \phi, \mathbf{z})$ is given in tubular coordinates of Section 2.2, then

$$\begin{aligned} r &= \text{dist}(q, R^m) = \sqrt{y^2 + |\mathbf{x}|^2}, \\ \text{where } y &= \begin{cases} r \cos \phi_1, & m < n, \\ r, & m = n. \end{cases} \end{aligned} \quad (21)$$

With the help of (9) and (8), we evaluate the constants:

$$S_{m,n} = \int_{S_+^{n-m}} \frac{y}{r} dS^{n-m}(\phi) = \frac{\omega_{n-m+1}}{2\pi}. \quad (22)$$

3. Poisson Kernel, Invariants, and Herglotz Representation of Harmonic Functions

3.1. Poisson Integral Invariants $P_{m,n}$. The Poisson kernel, $P(q, p)$, is a function of $q = (y, \mathbf{x}, \mathbf{z}) \in \Omega = R_+^{n+1}$ and $p \in \partial\Omega = R^n$:

$$P(q, p) = \frac{2 \text{dist}(q, \partial\Omega)}{\omega_n |q - p|^{n+1}} = \frac{2y}{\omega_n |q - p|^{n+1}}. \quad (23)$$

Lemma 4. Let $m \in \{0, \dots, n\}$. The integrals of P over half-tubes $T_r^m \cap \Omega$ do not depend on r , denoted by $P_{m,n}$; they are evaluated as

$$\begin{aligned} P_{m,n} &= \int_{T_r^m \cap \Omega} P(q, p) dT_r^m(q) = \frac{2}{\omega_n} S_{m,n} G_{m,n} \\ &= \frac{\omega_{n-m+1}}{\pi \omega_{n-m}}. \end{aligned} \quad (24)$$

Proof. Let $q = (r, \phi, \mathbf{z}) = (y, \mathbf{x}, \mathbf{z}) \in R_+ \times R^{n-m} \times R^m$ and $p = (0, \mathbf{0}, \mathbf{z}_p)$. Define $\mathbf{v} = (\mathbf{z} - \mathbf{z}_p)/r$. Then, (21) implies that $|q - p|^2 = r^2(1 + |\mathbf{v}|^2)$.

The invariants are computed with the help of (10), (18), and (22):

$$\begin{aligned} P_{m,n} &= \int_{T_r^m \cap \Omega} \frac{2y dT_r^m(q)}{\omega_n |q - p|^{n+1}} \\ &= \int_{R^m} \int_{S_+^{n-m}} \frac{2y r^{n-m} dS^{n-m}(\phi) dz}{\omega_n [r^2 + |\mathbf{z} - \mathbf{z}_p|^2]^{(n+1)/2}} \\ &= \frac{2}{\omega_n} \int_{S_+^{n-m}} \frac{y}{r} dS^{n-m}(\phi) \\ &\quad \cdot \int_{R^m} \frac{d\mathbf{v}}{(1+|\mathbf{v}|^2)^{(n+1)/2}} = \frac{2}{\omega_n} S_{m,n} G_{m,n}. \end{aligned} \quad (25)$$

□

Herglotz Representation (see [5–7]). A positive harmonic function $u \in h^1(\Omega)$ is defined by a finite positive Borel measure μ on $\partial\Omega$. It can be expressed as the Poisson-Stieltjes integral of μ :

$$u(q) = \int_{\partial\Omega} P(q, p) d\mu(p) = \int_{R^n} \frac{2y}{\omega_n |q - p|^{n+1}} d\mu(p). \quad (26)$$

3.2. Integration over Hyperplanes with Holes. Let $p_0 \in \partial\Omega$ and $H = \{q = (y, \mathbf{x}, \mathbf{z}) : y = r^2, 0 < r < 1\}$. Then, (see e.g., [2, Section 5])

$$\int_{H \cap \{|q - p_0| > r\}} P(q, p_0) dH(q) = O(r). \quad (27)$$

Now, consider a subset of H with holes above the support of μ :

$$H_r^\mu = \{q \in \Omega : y = r^2, \text{dist}(q, \text{supp}(\mu)) \geq r\}, \quad (28)$$

$$0 < r < 1.$$

Fubini's Theorem and (27) yield

$$\int_{H_r^\mu} u dH = O(r) \mu(\partial\Omega) = O(r). \quad (29)$$

4. Measures on Subspaces

Here, we show that if u is defined by a boundary measure μ lying on a subspace R^m , then $\text{IM}(u)$ is bounded by the integrals of u over the half-tubes $T_r^m \cap \Omega$. If μ is absolutely continuous with respect to the volume measure of R^m , these integrals approach $\text{IM}(u)$ as $r \rightarrow 0$.

4.1. Inequality (4) for μ on R^m

Theorem 5. Let Ω be the upper half-space R_+^{n+1} . Let a harmonic function u be defined on Ω by a finite positive Borel boundary measure μ that lies on a subspace $R^m \subset \partial\Omega$. Then

$$\text{IM}(u) \leq P_{m,n}\mu(R^m). \quad (30)$$

Proof. Fix $\delta > 0$ and let $r < \delta$. Consider hypersurfaces $\Gamma_r = \Gamma_1 \cup \Gamma_2$ separating the two boundaries of $A_\delta = \{q \in \Omega : \text{dist}(q, \partial\Omega) < \delta\}$:

$$\begin{aligned} \Gamma_1 &= \{q \in \Omega : y = r^2, \text{dist}(q, R^m) > r\}, \\ \Gamma_2 &= \{q \in \Omega : y \geq r^2, \text{dist}(q, R^m) = r\}. \end{aligned} \quad (31)$$

Then, $\Gamma_1 \subset H_r^\mu$ defined by (28) and $\Gamma_2 \subset T_r^m$ defined in Section 2.2. Now, (30) follows from (1), Fubini's Theorem, Lemma 4, and (29):

$$\begin{aligned} \text{IM}(u) &\leq \lim_{r \rightarrow 0} \left(\int_{T_r^m \cap \Omega} u(q) dT_r^m(q) + \int_{H_r^\mu} u(q) dH(q) \right) \\ &= \lim_{r \rightarrow 0} \left(\int_{\partial\Omega} \int_{T_r^m \cap \Omega} P(q, p) dT_r^m(q) d\mu(p) + O(r) \right) \\ &= P_{m,n}\mu(R^m). \end{aligned} \quad (32)$$

□

Next, we prove that $\text{IM}(u) \geq P_{m,n}\mu(R^m)$ when μ is absolutely continuous with respect to the Lebesgue measure of R^m . But first, we recall necessary facts about the derivative of such μ .

4.2. Approximation of $u^* = f = d\mu/dz$ by Step Functions. Let $\mathbf{z} \in R^m$. Denote by $Q(\mathbf{z}, h)$ a cube of side $2h$ and by $B(\mathbf{z}, h)$ a ball of radius h both centered at $\mathbf{z}$. Let $V_m = |B(0, 1)|$. When a cube $Q(0, \rho)$ is broken into N^m equal cubes Q_i of side $l = 2\rho/N$, the smaller cubes of side $l - 2\sigma$ at the centers of Q_i are denoted by Δ_i .

Lemma 6. Let $f \in L^1(R^m)$, $f \geq 0$, and a positive $\varepsilon \ll \|f\|_1$ is given. For sufficiently large ρ and N , there exist positive numbers $\{f_i\}_{i=1}^{N^m}$, σ , and K and sets $F, H \subset F$ such that

$$f|_{F \cap Q_i} \geq f_i, \quad (33)$$

$$|B(\mathbf{z}, h) \cap F| \geq |B(\mathbf{z}, h)| \left(1 - \frac{\sigma^{m+1}}{V_m} \right), \quad (34)$$

$$\mathbf{z} \in H, \quad h \leq \frac{1}{K},$$

$$\sum_{i=1}^{N^m} \int_{H \cap \Delta_i} f_i d\mathbf{z} \geq \|f\|_1 - 5\varepsilon. \quad (35)$$

Proof. If the support set, E , of f is unbounded, there exists $\rho > 0$ such that the integral over the bounded set, $E_\rho = E \cap Q(0, \rho)$, satisfies

$$\int_{E_\rho} f(\mathbf{z}) d\mathbf{z} > \|f\|_1 - \varepsilon. \quad (36)$$

Let $\varepsilon_j \downarrow 0$, $\varepsilon_1 < \varepsilon$. By Luzin's theorem, there exists a sequence of closed sets $F_{j-1} \subset F_j \subset E_\rho$ approaching E_ρ , so that $|F_j| \geq |E_\rho| - \varepsilon_j$ and the function f is (uniformly) continuous on F_j . Let $F = F_j$ for j such that

$$\int_F f(\mathbf{z}) d\mathbf{z} \geq \int_{E_\rho} f(\mathbf{z}) d\mathbf{z} - \varepsilon \stackrel{(36)}{\geq} \|f\|_1 - 2\varepsilon. \quad (37)$$

Now, find $\xi = \xi(\varepsilon, \rho, F)$ so that, for all $\mathbf{z}, \mathbf{w} \in F$, if $|\mathbf{z} - \mathbf{w}| < \xi$, then

$$|f(\mathbf{z}) - f(\mathbf{w})| < \frac{\varepsilon}{(2\rho)^m}. \quad (38)$$

We break $Q(0, \rho)$ into N^m equal cubes Q_i with diameters less than ξ taking $N \geq 2\rho\sqrt{m}/\xi$. Then, (38) implies that, for $\mathbf{z} \in Q_i$,

$$f_i = \min_{\mathbf{w} \in F \cap Q_i} f(\mathbf{w}) > f(\mathbf{z}) - \frac{\varepsilon}{(2\rho)^m}. \quad (39)$$

We observe next that, since $|Q_i \setminus \Delta_i| \rightarrow 0$ as $\sigma \rightarrow 0$, there exists $\sigma > 0$ small enough to yield

$$\sum_{i=1}^{N^m} \int_{Q_i \setminus \Delta_i} f(\mathbf{z}) d\mathbf{z} < \varepsilon. \quad (40)$$

Let D be the set of points of density for F . Lebesgue's Theorem states that $|D| = |F|$. Therefore, for $k = 1, 2, \dots$, the sets

$$\begin{aligned} D_k &= \left\{ \mathbf{z} \in D : |B(\mathbf{z}, h) \cap F| \geq |B(\mathbf{z}, h)| \left(1 - \frac{\sigma^{m+1}}{V_m} \right), h \leq \frac{1}{k} \right\} \end{aligned} \quad (41)$$

approach D as $k \rightarrow \infty$. Then, (41) and (37) imply that there exists K such that, for all $k \geq K$,

$$\int_{D_k} f d\mathbf{z} \geq \int_D f d\mathbf{z} - \varepsilon = \int_F f d\mathbf{z} - \varepsilon \stackrel{(37)}{\geq} \|f\|_1 - 3\varepsilon. \quad (42)$$

This allows us to take $H = D_K$ in (34). Next, we apply (39) to each term in the sum:

$$\begin{aligned} \sum_i \int_{H \cap \Delta_i} f_i d\mathbf{z} &> \sum_i \int_{H \cap \Delta_i} f(\mathbf{z}) - \frac{\varepsilon}{(2\rho)^m} d\mathbf{z} \\ &= \sum_i \int_{H \cap Q_i} f(\mathbf{z}) d\mathbf{z} - \sum_i \int_{H \cap (Q_i \setminus \Delta_i)} f(\mathbf{z}) d\mathbf{z} \\ &\quad - \sum_i \int_{H \cap \Delta_i} \frac{\varepsilon}{(2\rho)^m} d\mathbf{z}. \end{aligned} \quad (43)$$

It can be seen that $\cup_i (H \cap Q_i) = H$ and $H \cap (Q_i \setminus \Delta_i) \subset Q_i \setminus \Delta_i$, and also $\cup_i (H \cap \Delta_i) \subset \cup_i Q_i = Q(0, \rho)$. Hence,

$$\begin{aligned} \sum_i \int_{H \cap \Delta_i} f_i d\mathbf{z} &> \int_H f(\mathbf{z}) d\mathbf{z} - \sum_i \int_{Q_i \setminus \Delta_i} f(\mathbf{z}) d\mathbf{z} \\ &\quad - \frac{\varepsilon}{(2\rho)^m} |Q(0, \rho)|. \end{aligned} \quad (44)$$

The last inequality becomes (35) when (42) (with $H = D_K$) and (40) are applied. $\square$

4.3. Equality in (4) for Absolutely Continuous μ

Theorem 7. Let Ω be the upper half-space R_+^{n+1} . Let a harmonic function u be defined on Ω by a finite positive Borel boundary measure μ that lies on a subspace $R^m \subset \partial\Omega$ and is absolutely continuous with respect to the Lebesgue measure of R^m . Then,

$$IM(u) = P_{m,n} \mu(R^m). \quad (45)$$

Proof. By hypothesis, $d\mu(p) = f(\mathbf{z}_p) d\mathbf{z}_p$, where $f \in L^1(R^m)$, $f \geq 0$. Given $\varepsilon > 0$, we obtain constants $\rho, N, \{f_i\}_{i=1}^N, K, \sigma$, sets F, H , and cubes $Q_i \supset \Delta_i$ of sides $l = 2\rho/N, l - 2\sigma$ from Lemma 6. We may assume that $\sigma < \varepsilon$. Let $\delta = \sigma^4/K^2$. In view of Theorem 5, it is enough to show that $\int_\Gamma u > P_{m,n} \|f\|_1 - O(\varepsilon)$ for any continuous, piecewise smooth, orientable surface Γ in A_δ that separates boundaries of A_δ . Consider

$$\cup \Gamma_i = \cup \Gamma \cap \left\{ q = (r, \phi, \mathbf{z}) : \mathbf{z} \in \Delta_i \cap H, r \leq \sqrt{\delta} = \frac{\sigma^2}{K} \right\}. \quad (46)$$

Since the range of ϕ_1 on Γ is $[0, \pi/2)$, the range of ϕ_1 on Γ_i is $[0, \arccos \sqrt{\delta}]$. When $\mathbf{z} \in \Delta_i \cap H$ is fixed, the ball $B(\mathbf{z}, r/\sigma) \subset Q_i$ because $r/\sigma \leq \sigma/K$. When $p = (0, \mathbf{0}, \mathbf{z}_p)$ with $\mathbf{z}_p \in B(\mathbf{z}, \frac{r}{\sigma})$, we have $f(\mathbf{z}_p) > f_i$ by (33). Inequality (34) implies that $|B(\mathbf{z}, r/\sigma) \cap F| \geq |B(\mathbf{z}, r/\sigma)|(1 - \sigma^{m+1}/V_m)$; that is,

$$\left| B\left(\mathbf{z}, \frac{r}{\sigma}\right) \setminus F \right| \leq \left| B\left(\mathbf{z}, \frac{r}{\sigma}\right) \right| \frac{\sigma^{m+1}}{V_m} = r^m \sigma. \quad (47)$$

Let $q = (r, \phi, \mathbf{z}) \in \Gamma_i$; then $|q - p| = \sqrt{r^2 + |\mathbf{z} - \mathbf{z}_p|^2}$, so we may write

$$\begin{aligned} \int_\Gamma u(q) d\Gamma(q) &= \int_\Gamma \int_{R^m} \frac{2y d\mu(\mathbf{z}_p) d\Gamma(q)}{\omega_n (r^2 + |\mathbf{z} - \mathbf{z}_p|^2)^{(n+1)/2}} \\ &> \frac{2}{\omega_n} \sum_{i=1}^N \int_{\Gamma_i} \frac{y/r d\Gamma(q)}{r^{n-m}} f_i \\ &\quad \times \int_{B(\mathbf{z}, r/\sigma) \cap F} \frac{d\mathbf{z}_p / r^m}{(1 + |\mathbf{z} - \mathbf{z}_p|^2 / r^2)^{(n+1)/2}}. \end{aligned} \quad (48)$$

The change of variables $\mathbf{v} = (\mathbf{z}_p - \mathbf{z})/r$ defines a map $T : B(\mathbf{z}, r/\sigma) \rightarrow B(0, 1/\sigma)$ and a set $Y = T(B(\mathbf{z}, r/\sigma) \setminus F)$ whose size $|Y| \leq r^m \sigma(1/r^m) = \sigma$ by (47). Now, Lemma 3 estimates the factor to the right of f_i in (48) as follows:

$$\begin{aligned} &\int_{B(\mathbf{z}, r/\sigma) \cap F} \frac{d\mathbf{z}_p / r^m}{(1 + |\mathbf{z}_p - \mathbf{z}|^2 / r^2)^{(n+1)/2}} \\ &= \int_{B(0, 1/\sigma) \setminus Y} \frac{d\mathbf{v}}{(1 + |\mathbf{v}|^2)^{(n+1)/2}} > G_{m,n} - 35\sigma. \end{aligned} \quad (49)$$

If Γ is given by $r = r(\phi, \mathbf{z})$, we have $d\Gamma(r, \phi, \mathbf{z}) \geq r^{n-m} dS^{n-m}(\phi) d\mathbf{z}$ by Lemma 2. Then, (35) and (22) estimate the sum:

$$\begin{aligned} &\sum_i \int_{\Gamma_i} \frac{y d\Gamma(q)}{r r^{n-m}} f_i \\ &> \int_{S_+^{n-m} \cap \{\cos \phi_1 \geq \sqrt{\delta}\}} \frac{y}{r} dS^{n-m}(\phi) \sum_i \int_{\Delta_i \cap H} f_i d\mathbf{z} \\ &> \left(\int_{S_+^{n-m}} \frac{y}{r} dS^{n-m} - \eta \right) (\|f\|_1 - 5\varepsilon) \\ &= (S_{m,n} - \eta) (\|f\|_1 - 5\varepsilon), \end{aligned} \quad (50)$$

where $\eta = 0$ if $m = n$, and $\eta = \omega_{n-m-1} [1 - (1 - \delta)^{(n-m)/2}] / (n - m) = O(\delta) = o(\varepsilon)$ otherwise. Finally, (48)–(50) and Lemma 4 yield the desired inequality:

$$\begin{aligned} \int_\Gamma u d\Gamma &> \frac{2}{\omega_n} (S_{m,n} - \eta) (\|f\|_1 - 5\varepsilon) (G_{m,n} - 35\sigma) \\ &= P_{m,n} \|f\|_1 - \zeta, \end{aligned} \quad (51)$$

with $\zeta = (2/\omega_n)(5\varepsilon S_{m,n} + \eta \|f\|_1 - 5\eta\varepsilon)(G_{m,n} - 35\sigma) = O(\varepsilon)$.

Since ε, Γ were arbitrary and $\|f\|_1 = \mu(R^m)$, the inferior mean satisfies

$$IM(u) = \lim_{\delta \rightarrow 0} \inf_{\Gamma \subset A_\delta} \int_\Gamma u(q) d\Gamma(q) \geq P_{m,n} \mu(R^m). \quad (52)$$

The opposite inequality of Theorem 5 gives us equality in (52). $\square$

5. Measures on Curves

In this section, we show that if u is defined by a boundary measure μ lying on a smooth curve, γ , then $IM(u)$ is bounded by the integrals of $u(q)$ over half-tubes $\{q \in \Omega : \text{dist}(q, \gamma) = r\}$ about γ . If μ is the arclength measure of γ , these integrals approach $IM(u)$ as $r \rightarrow 0$. We assume that the dimension n of $\partial\Omega = R^n$ is greater than 1. The case $n = 1$, where $\gamma = R^1$ or $\gamma = [a, b]$, is covered by Theorems 5 and 7.

5.1. On Linear Approximation of C^1 -Curves. Let $\gamma : [0, 1] \rightarrow R^n$ be a simple C^1 -curve parameterized by arclength. Let τ be γ 's injectivity radius (radius of the largest embedded normal tube around γ).

Lemma 8. *For any $\varepsilon > 0$, there exists $\nu < \min\{\tau, \varepsilon\}$ so that, for $|s - t| < \nu$, one has the following:*

$$|\gamma(s) - \gamma(t)| \leq |s - t| \leq \frac{|\gamma(s) - \gamma(t)|}{(1 - \varepsilon)}, \quad (53)$$

$$\max_{\xi \in (t, s)} \text{dist}(\gamma(\xi), \gamma(t) - \gamma(s)) < \varepsilon |s - t|. \quad (54)$$

For any $q \in R_+ \times R^n$, there exists t_q such that

$$r_q = \text{dist}(q, \gamma) = |q - \gamma(t_q)|. \quad (55)$$

If $t_q \neq 0, 1$ and q is sufficiently close to γ , then

$$|q - \gamma(s)| < \sqrt{r_q^2 + (s - t_q)^2} (1 + \varepsilon). \quad (56)$$

Proof. Since γ' is continuous, there exists ν such that, for $s, t \in [0, 1] : |s - t| < \nu$, there exist n numbers $\beta_i \in (t, s)$, $i = 1, \dots, n$, so that

$$\begin{aligned} \gamma_i(s) - \gamma_i(t) &= (s - t) \gamma'_i(\beta_i), \\ |\gamma'_i(\beta_i) - \gamma'_i(t)| &< \frac{\varepsilon}{3\sqrt{n}}. \end{aligned} \quad (57)$$

This implies (53) because we have $\text{length}(\gamma|_{[t, s]}) = |s - t|$. Denote by $\mathbf{d}$ the unit vector $\gamma'(t)$. Using (57), we write vectors $\mathbf{w}, \mathbf{u}$ below as

$$\begin{aligned} \mathbf{w} &= \gamma(s) - \gamma(t) = (s - t)(\mathbf{d} + \mathbf{e}), \\ \mathbf{u} &= \gamma(\xi) - \gamma(t) = (\xi - t)(\mathbf{d} + \mathbf{h}), \end{aligned} \quad (58)$$

where $|\mathbf{e}|, |\mathbf{h}| < \varepsilon/3$. Then, (54) follows from the estimates of $\angle \mathbf{w}, \mathbf{u} = A$:

$$\begin{aligned} \sin^2 A &= 1 - \frac{(\mathbf{w} \cdot \mathbf{u})^2}{(\mathbf{w} \cdot \mathbf{w})(\mathbf{u} \cdot \mathbf{u})} \\ &= \left| (\mathbf{e} \cdot \mathbf{e} + \mathbf{h} \cdot \mathbf{h} + 2(\mathbf{d} \cdot \mathbf{e})(\mathbf{d} \cdot \mathbf{h}) \right. \\ &\quad \left. - (\mathbf{d} \cdot \mathbf{e})^2 - (\mathbf{d} \cdot \mathbf{h})^2 - 2(\mathbf{e} \cdot \mathbf{h}) + \sigma \right) \\ &\quad \times ((1 + 2\mathbf{d} \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{e})(1 + 2\mathbf{d} \cdot \mathbf{h} + \mathbf{h} \cdot \mathbf{h}))^{-1} \Big| \\ &< \varepsilon^2. \end{aligned} \quad (59)$$

One can check that $\sigma < \varepsilon^3$. Let q, r_q , and t_q be as in (55) and $\mathbf{r}_q = q - \gamma(t_q)$. If $t_q \neq 0, 1$, then $\mathbf{r}_q \perp \mathbf{d}$. Now, (56) is obtained from

$$\begin{aligned} |q - \gamma(s)|^2 &= |q - \gamma(t_q) + \gamma(t_q) - \gamma(s)|^2 = |\mathbf{r}_q - \mathbf{w}|^2 \\ &= \mathbf{r}_q \cdot \mathbf{r}_q + (s - t_q)^2 (1 + 2\mathbf{d} \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{e}) \\ &\quad - 2(s - t_q) \mathbf{r}_q \cdot \mathbf{e}. \end{aligned} \quad (60)$$

□

5.2. Inequality (4) for Measures on Curves. Since measures μ are finite, we may assume that curves are bounded (see (36)).

Theorem 9. *Let Ω be the upper half-space R_+^{n+1} and let $\gamma : [0, 1] \rightarrow \partial\Omega$ be a simple C^1 -curve. Let a harmonic function u be defined on Ω by a finite positive Borel boundary measure μ that lies on γ . Then,*

$$IM(u) \leq P_{1,n}\mu(\gamma). \quad (61)$$

Proof. We assume that γ is parameterized by the arclength parameter s . Thus, $|\gamma'| = 1$. Let $0 < \varepsilon < 10^{-4}$. Consider a curve γ with injectivity radius τ and $\nu = \nu(\gamma, \varepsilon, \tau)$ as in Lemma 8. We will show that, for $\delta = \nu^2 \varepsilon^{1/2}$, there exists a surface $\Gamma \subset A_\delta$ such that

$$\int_\Gamma u(q) d\Gamma < P_{1,n}\mu(\gamma) + O(\varepsilon^{1/2}). \quad (62)$$

Let $[0, 1] = \cup_{i=1}^N [(i-1)/N, i/N]$ and let $\cup_{i=1}^N l_i$ be a polygonal approximation of γ with line segments l_i connecting $\gamma((i-1)/N)$ with $\gamma(i/N)$. We take

$$N = \left\lceil \left\lceil \frac{2}{\nu^2} \right\rceil \right\rceil + 1, \quad l = \frac{1}{N}, \quad r = l\varepsilon^{1/2}. \quad (63)$$

Note that $\sqrt{l} < \nu$ and $r < \delta$. It follows from (53) and (54) that

$$\max_s \text{dist} \left(\gamma(s), \bigcup_{i=1}^N l_i \right) < l\varepsilon \ll r \ll l(1 - \varepsilon) < |l_i| \leq l. \quad (64)$$

Let l_0 and l_{N+1} be segments of length l extending l_1 and l_N . Denote by L_i lines containing l_i . We build a Quonset hut $\Gamma_T = \cup \Gamma_i$ above γ with the axis along $\cup l_i$ planted in the hyperplane $H = \{q \in \Omega : y = (r/2)^2\}$ (See Figure 1). It consists of $N + 2$ straight cylindrical huts $\Gamma_i \subset T_r^1(L_i)$ glued together where $T_r^1(L_i)$ meets $T_r^1(L_{i+1})$ and two spherical doors Γ_{-1} and Γ_{N+2} . The part of H outside the hut is called Γ_H and our surface Γ is $\Gamma_T \cup \Gamma_H$. By (64), $\Gamma_H \subset H_{r/2}^\mu$ (subset of H is defined in (28)). Thus by (29),

$$\int_{\Gamma_H} u dH < \int_{H_{r/2}^\mu} u dH = O\left(\frac{r}{2}\right) = o(\varepsilon). \quad (65)$$

We apply Fubini's theorem to $\int_{\Gamma_T} u$ and estimate $\int_{\Gamma_T} P(q, \gamma(s))$ for a fixed $s \in [(i-1)/N, i/N]$. The main contribution to $\int_{\Gamma_T} P$ will come from $\Gamma_i \cup \Gamma_{i\pm 1}$.

$$r = l\varepsilon^{1/2}, |l_i| \approx l, \Delta l < 2r\varepsilon, \text{length}(\Gamma_i) \leq l + 2\Delta l, \sin \angle l_i, l_{i+1} < 2\varepsilon \text{ by (59).}$$

FIGURE 1

FIGURE 2

Integration over Γ_{i+1} (and similarly Γ_{i-1}) is replaced with that over its copy, Γ'_{i+1} , on $T_r^1(L_i)$ obtained by swiveling Γ_{i+1} to Γ_i until the two contours are aligned horizontally (see Figure 1). Under this rotation, the distance of $q_{i+1} \in \Gamma_{i+1}$ to $\partial\Omega$ is preserved and the distance to its image $q \in \Gamma'_{i+1}$ is

$$|q_{i+1} - q| < (l + 2\Delta l) \sin \angle l_i, \quad l_{i+1} + 2\Delta l < 3l\varepsilon. \quad (66)$$

If p is the projection of $\gamma(s)$ on l_i , then, by (64), $|p - \gamma(s)| < l\varepsilon$. So

$$\begin{aligned} |q_{i+1} - \gamma(s)| &> |q - p| - |p - \gamma(s)| - |q_{i+1} - q| \\ &> |q - p| (1 - 4\varepsilon^{1/2}). \end{aligned} \quad (67)$$

That gives us $P(q_{i+1}, p) < P(q, p)/(1 - 4\varepsilon^{1/2})^{n+1}$ and now, with the assumption that $\gamma(s)$ is closer to Γ_{i+1} , Lemma 4 yields the main estimate:

$$\begin{aligned} \int_{\Gamma_i \cup \Gamma_{i+1}} P d\Gamma_T &< \int_{\Gamma_i} P(q, \gamma(s)) d\Gamma_i + 2 \int_{\Gamma_{i+1}} P(q_{i+1}, \gamma(s)) d\Gamma_{i+1} \\ &< \frac{1}{(1 - 4\varepsilon^{1/2})^{n+1}} \int_{T_r^1(L_i)} P(q, p) dT_r^1(q) \\ &= P_{1,n} + O(\varepsilon^{1/2}). \end{aligned} \quad (68)$$

Next, we integrate over $\Gamma_{i\pm k}$, $k \geq 2$ as far as $|q - \gamma(s)| < \sqrt{l} < \nu$ (where (53)–(59) of Lemma 8 are valid). Let s, ξ, t be as in Figure 2.

By (59), the angle $\angle \gamma(s)\gamma(\xi)\gamma(t) > \pi - 2\varepsilon$, so $\gamma(s)$ is closer to $\gamma(\xi)$ than to the projection of $q \in \Gamma_{i\pm k}$ on $l_{i\pm k}$. But $|\gamma(s) -$

$|\gamma(\xi)| > |s - \xi|(1 - \varepsilon)$ by (53). Hence, by the triangle inequality, we have a lower bound for

$$\begin{aligned} |\gamma(s) - q| &> |\gamma(s) - \gamma(\xi)| - r \\ &> |s - \xi|(1 - \varepsilon) - r \geq \frac{k-1}{N}(1 - \varepsilon) - r \\ &= (k-1)l \left(1 - \varepsilon - \frac{\sqrt{\varepsilon}}{k-1}\right) > (k-1)\frac{l}{2}. \end{aligned} \quad (69)$$

Recall that $n \geq 2, r = l\varepsilon^{1/2}, y = \text{dist}(q, \partial\Omega) \leq r, |\Gamma_k| < 2lr^{n-1}\omega_{n-1}$ and $|\Gamma_T| < 2r^{n-1}\omega_{n-1}$. These, combined with (69), estimate the integrals of P over the two remaining parts of Γ_T as follows:

$$\begin{aligned} \sum_{k \geq 2} \int_{\Gamma_{i\pm k}} \frac{2y/\omega_n d\Gamma_T}{|q - \gamma(s)|^{n+1}} \\ < \frac{2}{\omega_n} \sum_{k \geq 2} \frac{2r2l^{n-1}\omega_{n-1}}{[(k-1)l/2]^{n+1}} = O(\varepsilon^{n/2}), \end{aligned} \quad (70)$$

$$\begin{aligned} \int_{\Gamma_T \cap \{|q - \gamma(s)| \geq \sqrt{l}\}} \frac{2y/\omega_n d\Gamma_T}{|q - \gamma(s)|^{n+1}} \\ < \frac{2r2l^{n-1}\omega_{n-1}}{\omega_n l^{(n+1)/2}} = O(l^{(n-1)/2} \varepsilon^{n/2}). \end{aligned}$$

Finally, from (65), (68), (70), and Fubini's theorem, we can obtain the estimate (62):

$$\begin{aligned} \int_{\Gamma_H \cup \Gamma_T} u d\Gamma &= o(\varepsilon) + \int_0^1 \int_{\Gamma_T} P(q, \gamma(s)) d\Gamma_T(q) d\mu(\gamma(s)) \\ &= o(\varepsilon) + \int_0^1 P_{1,n} + O(\varepsilon^{1/2}) d\mu(\gamma(s)) \\ &= P_{1,n}\mu(\gamma) + O(\varepsilon^{1/2}). \end{aligned} \quad (71)$$

□

5.3. Equality in (4) for Arclength Measures

Theorem 10. Let Ω be the upper half-space R_+^{n+1} and let $\gamma : [0, 1] \rightarrow \partial\Omega$ be a simple C^1 -curve. Let a harmonic function

FIGURE 3

u be defined on Ω by the boundary measure μ , that is, the arclength measure of γ . Then,

$$IM(u) = P_{1,n}\mu(\gamma). \quad (72)$$

Proof. We assume that γ is parameterized by the arclength parameter s . Thus, $\mu(\gamma) = |\gamma| = 1$ and $d\mu(\gamma(s)) = ds$. Let $0 < \varepsilon < 10^{-4}$. Consider a curve γ with injectivity radius τ and $\nu = \nu(\gamma, \varepsilon, \tau)$ as in Lemma 8. We fix an integer $N > 1/\nu$, and, for $i = 1, \dots, N$, define

$$\sigma = \frac{\varepsilon}{2N}, \quad \delta = \varepsilon^4, \quad \Delta_i = \left[\frac{i-1}{N} + \sigma, \frac{i}{N} - \sigma \right], \quad (73)$$

$$\gamma_i = \gamma|_{\Delta_i}.$$

Note that $\sigma \ll 1/N < \nu < \tau$. Consider segments l_i connecting endpoints of γ_i ; then, by (54) and (73),

$$\sum_i |l_i| > \sum_i |\Delta_i| (1 - \varepsilon) = (1 - \varepsilon)^2. \quad (74)$$

In view of Theorem 9, it suffices to show that, for any continuous piecewise smooth surface Γ in A_δ separating its boundaries,

$$\int_\Gamma u d\Gamma > P_{1,n} - O(\varepsilon). \quad (75)$$

The $\text{dist}(\Gamma, \gamma)$ can be quite small; therefore, we break l_i into equal segments l_{ij} and place their copies, u_{ij} , at points p_{ij} on γ_i so that l_{ij}, u_{ij} are parallel sides in rectangles (Figure 3). We require that

$$|l_{ij}| = |u_{ij}| \leq [\text{dist}(\Gamma, \gamma)]^2. \quad (76)$$

In local tubular coordinates where the short z -axes lie on u_{ij} , let

$$\Gamma_{ij} = \{q = (r, \phi, z) \in \Gamma : z \in u_{ij}, r = \text{dist}(q, u_{ij}) \leq \sqrt{\delta}\}. \quad (77)$$

Fix $q \in \Gamma_{ij}$. Recall that $r_q = \text{dist}(q, \gamma) = |q - t_q|$. Then, (76) yields

$$\begin{aligned} |u_{ij}|^{1/2} \leq r_q \leq \sigma^2 &\implies r_q - |u_{ij}| \geq r_q - r_q^2 > r_q(1 - \sigma^2), \\ r &\geq |q - p_{ij}| - |u_{ij}| \geq r_q - |u_{ij}| \geq r_q(1 - \sigma^2). \end{aligned} \quad (78)$$

Next, we claim that $\Gamma_i = \cup_j \Gamma_{ij}$ (and thus Γ_{ij}) do not overlap. Indeed,

$$\text{dist}(\gamma_i, \gamma_k) \geq 2\tau, \quad k \neq i \pm 1, \quad \text{dist}(\gamma_i, \gamma_{i+1}) \approx 2\sigma > \sigma, \quad (79)$$

$$\max_{q \in \Gamma_i} r_q = \max_j \max_{q \in \Gamma_{ij}} r_q \leq \sqrt{\delta} = \sigma^2 \quad (80)$$

prove the claim

$$\text{dist}(\Gamma_i, \Gamma_k) > \text{dist}(\gamma_i, \gamma_k) - \max_{q \in \Gamma_i} r_q - \max_{q \in \Gamma_k} r_q > \sigma - 2\sigma^2 > 0. \quad (81)$$

Now, the first estimate below is analogous to (49) of Theorem 7. It is obtained from (56), (80), the change of variables $v = (s - t_q)/r_q$, and Lemma 3 with $\rho = 1/\sigma$ and $Y = \emptyset$

$$\begin{aligned} \int_\gamma \frac{d\mu(\gamma(s))}{|q - \gamma(s)|^{n+1}} &> \int_{\Delta_i} \frac{ds}{|q - \gamma(s)|^{n+1}} \\ &> \int_{t_q - r_q/\sigma}^{t_q + r_q/\sigma} \frac{(1 + \varepsilon)^{-n-1} ds}{\left[r_q^2 + (s - t_q)^2\right]^{(n+1)/2}} \\ &= \frac{1}{r_q^n} \int_{-1/\sigma}^{1/\sigma} \frac{(1 + \varepsilon)^{-n-1} dv}{(1 + v^2)^{(n+1)/2}} \\ &> \frac{G_{1,n} - 34\sigma}{r_q^n (1 + \varepsilon)^{n+1}}. \end{aligned} \quad (82)$$

Since the range of ϕ_1 on Γ is $[0, \pi/2)$, the range of ϕ_1 on Γ_{ij} is $[0, \arccos \sqrt{\delta}]$. If Γ is given by $r = r(\phi, z)$, then $d\Gamma(r, \phi, z) \geq r^{n-1} dS^{n-1} dz$ by Lemma 2. The second estimate is analogous to (50) of Theorem 7. It is obtained from (22) and (74) as follows:

$$\begin{aligned} \int_\Gamma \frac{y d\Gamma}{r^n} &> \int_{\cup_{i,j} \Gamma_{ij}} \frac{y d\Gamma}{r^n} \\ &> \int_{S_+^{n-1} \cap \{\cos \phi_1 \geq \sqrt{\delta}\}} \frac{y r^{n-1} dS^{n-1}}{r^n / (1 - \sigma^2)^n} \int_{\cup_{i,j} u_{ij}} dz \\ &> (S_{1,n} - \eta) (1 - \sigma^2)^n (1 - \varepsilon)^2, \end{aligned} \quad (83)$$

where $\eta = \omega_{n-2} [1 - (1 - \delta)^{(n-1)/2}] / (n-1)$ and we estimated $\sum_{i,j} |u_{ij}| = \sum_i |l_i| > (1 - \varepsilon)^2$ using (74).

Finally, $\sigma, \eta, \delta < \varepsilon$, Lemma 4, and (82)-(83) yield the inequality (75):

$$\begin{aligned} \int_\Gamma u d\Gamma &= \int_\Gamma \frac{2y}{\omega_n} \left(\int_\gamma \frac{d\mu(\gamma(s))}{|q - \gamma(s)|^{n+1}} \right) d\Gamma \\ &> \int_\Gamma \frac{2y}{\omega_n} \frac{(G_{1,n} - 34\sigma)}{r_q^n (1 + \varepsilon)^{n+1}} d\Gamma \\ &> \frac{2}{\omega_n} \frac{(S_{1,n} - \eta) (G_{1,n} - 34\sigma) (1 - \sigma^2)^n (1 - \varepsilon)^2}{(1 + \varepsilon)^{n+1}} \\ &= P_{1,n} - O(\varepsilon). \end{aligned} \quad (84)$$

□

6. Conjectures

With methods of Theorems 9 and 10 and triangulation of m -dimensional manifolds, we conjecture that these theorems are true for $m \geq 1$.

Conjecture 1. *Let Ω be the upper half-space $\mathbb{R}_+^{n+1}$. Let a harmonic function u be defined on Ω by a finite positive Borel boundary measure μ that lies on an m -dimensional manifold $\gamma \subset \partial\Omega$, $m \in \{0, \dots, n\}$, and then*

$$IM(u) \leq P_{m,n}\mu(\gamma). \quad (85)$$

Moreover, equality in (85) holds, if μ is the volume measure of γ .

The next conjecture, if true, combines the results of Theorems 7 and 10.

Conjecture 2. *With the assumptions of Conjecture 1, equality in (85) holds, if μ is absolutely continuous with respect to the volume measure of γ .*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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