Erratum

Erratum to “Pointwise Analog of the Stečkin Approximation Theorem”

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Received 23 July 2014; Accepted 14 October 2014; Published 14 December 2014

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We present the proper definitions of the pointwise best approximation of f which implies the correctness of the proof of Lemma 5 as well as the other results of the paper.

In the paper entitled “Pointwise Analog of the Stečkin Approximation Theorem,” there was given an incorrect proof of Lemma 5 based on improper definitions of the pointwise best approximation of f . Here, we define the mentioned $E_n(f, x; \delta)_{X^p}$ and $E_n^\circ(f, x; \delta)_{X^p}$ in the following way:

$$\begin{aligned} E_n(f, x; \delta)_{X^p} &:= \inf_{0 \leq k \leq n} \|f - T_k\|_{X^p, x, \delta} \\ &= \begin{cases} \inf_{0 \leq k \leq n} \sup_{0 < h \leq \delta} \left\{ \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - T_k(t)|^p dt \right\}^{1/p} & \text{when } p < \infty \\ \inf_{0 \leq k \leq n} \sup_{0 < h \leq \delta} \left\{ \sup_{0 < |t| \leq h} |f(x+t) - T_k(x+t)| \right\} & \text{when } p = \infty, \end{cases} \\ E_n^\circ(f, x; \delta)_{X^p} &:= \inf_{0 \leq k \leq n} \|f - T_k\|_{X^p, x, \delta}^\circ \\ &= \begin{cases} \inf_{0 \leq k \leq n} \left\{ \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(t) - T_k(t)|^p dt \right\}^{1/p} & \text{when } p < \infty \\ \inf_{0 \leq k \leq n} \left\{ \sup_{0 < |t| \leq \delta} |f(x+t) - T_k(x+t)| \right\} & \text{when } p = \infty, \end{cases} \end{aligned}$$

(16)

where T_k is the polynomial of the best approximation of f in the usual norm $\|\cdot\|_{X^p, x, \pi} = \|\cdot\|_{X^p}$ and $\delta > 0$. Under the above definitions, we correct Lemma 5 as follows.

Lemma 5. *If T_n is the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p}$, then it is also the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p, x, \delta}$ for any $\delta \in [0, \pi]$.*

Proof. From the inequalities

$$\begin{aligned} \|E_n(f, \cdot, \delta)_{X^p}\|_{X^p} &\geq \|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} = \|f - T_{n, \delta}\|_{X^p, \cdot, \delta}^\circ \\ &= \left\{ \int_{-\pi}^{\pi} \left[\frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(t) - T_{n, \delta}(t)|^p dt \right] dx \right\}^{1/p} \\ &= \left\{ \int_{-\pi}^{\pi} \left[\frac{1}{2\delta} \int_{-\delta}^{\delta} |f(x+t) - T_{n, \delta}(x+t)|^p dt \right] dx \right\}^{1/p} \\ &= \left\{ \frac{1}{2\delta} \int_{-\delta}^{\delta} \left[\int_{-\pi}^{\pi} |f(x+t) - T_{n, \delta}(x+t)|^p dx \right] dt \right\}^{1/p} \\ &= \left\{ \frac{1}{2\delta} \int_{-\delta}^{\delta} \left[\int_{-\pi}^{\pi} |f(x) - T_{n, \delta}(x)|^p dx \right] dt \right\}^{1/p} \\ &= \|f - T_{n, \delta}\|_{X^p} \geq \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \\ \|E_n^\circ(f, \cdot, \delta)_{X^p}\|_{X^p} &\leq \|f - T_n\|_{X^p, \cdot, \delta}^\circ \\ &= \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \end{aligned}$$

(22)

where $T_{n,\delta}$ and T_n are the trigonometric polynomials of the degree at most n of the best approximation of $f \in X^p$ with respect to the norms $\|\cdot\|_{X^p,x,\delta}^\circ$ and $\|\cdot\|_{X^p}$, respectively, we obtain relation

$$\|f - T_{n,\delta}\|_{X^p} = \|f - T_n\|_{X^p} = E_n(f)_{X^p}, \quad (23)$$

whence $T_{n,\delta} = T_n$ for any $\delta \in [0, \pi]$ by uniqueness of the trigonometric polynomial of the degree at most n of the best approximation of $f \in X^p$ with respect to the norm $\|\cdot\|_{X^p}$ (see, e.g., [1] p. 96). We can also observe that for such T_n and any $h \in [0, \delta]$

$$\begin{aligned} \|f - T_n\|_{X^p,x,h}^\circ &= E_n^\circ(f, x, h)_{X^p} \\ &\leq E_n(f, x, \delta)_{X^p} \leq \|f - T_n\|_{X^p,x,\delta}. \end{aligned} \quad (24)$$

Hence

$$E_n(f, x, \delta)_{X^p} = \|f - T_n\|_{X^p,x,\delta} \quad (25)$$

and our proof is complete. $\square$

Now, the proof is correct as well as other lemmas and all other results of the paper. Mainly, we have the following.

Theorem 2. *If $f \in X^p$, then, for any positive integer $m \leq n$ and all real x ,*

$$\begin{aligned} &|\sigma_{n,m}f(x) - f(x)| \\ &\leq K \sum_{v=0}^n \frac{F_{n-m+v,m}(f, x)_{X^p} + F_{n-m+v,v}(f, x)_{X^p}}{m+v+1} \\ &\quad + E_{2n}(f, x; 0)_{X^p}, \end{aligned} \quad (19)$$

where $F_{n,m}(f, x)_{X^p} := (1/(m+1)) \sum_{k=0}^m E_n(f, x; \pi/(k+1))_{X^p}$, which immediately yields the known result of Stečkin [2], by the obvious relation $\|E_n(f, x; \delta)\|_C \leq E_n(f)_C$, where $X^p = C$ when $p = \infty$ and $\delta > 0$.

Remark. By our previous definitions (see (16)), with T instead of T_k , the polynomial T was any trigonometric polynomial of the degree at most n and therefore the polynomial considered in the proof of Lemma 5 in the above mentioned paper (see (16)) was essentially dependent on variable x , that is, $T_{n,\delta}(t) = T_{n,\delta,x}(t)$. Thus, inequality (22) from the proof of Lemma 5 in the above mentioned paper (see (16)) was not true because change of order of norms was not possible in general.

References

- [1] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser, Basel, Switzerland, 1971.
- [2] S. B. Stečkin, "On the approximation of periodic functions by de la Vallée Poussin sums," *Analysis Mathematica*, vol. 4, no. 1, pp. 61–74, 1978.