Research Article

Nonexplosion and Pathwise Uniqueness of Stochastic Differential Equation Driven by Continuous Semimartingale with Non-Lipschitz Coefficients

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We study a class of stochastic differential equations driven by semimartingale with non-Lipschitz coefficients. New sufficient conditions on the strong uniqueness and the nonexplosion are derived for d -dimensional stochastic differential equations on $\mathbb{R}^d$ ($d > 2$) with non-Lipschitz coefficients, which extend and improve Fei's results.

1. Introduction

Consider the following stochastic differential equations driven by a nonlinear integrator (or semimartingale) of the form

$$\begin{aligned} dX_t &= x_0 + \int_0^t F(X_s, ds) \\ &= x_0 + \int_0^t b(X_s, s) ds + \int_0^t M(X_s, ds), \end{aligned} \quad (1)$$

equivalently

$$dX_t = b(X_t, t) dt + M(X_t, dt), \quad X_0 = x_0, \quad (2)$$

where $(\Omega, \mathcal{F}, P)$ is a probability space, $(\mathcal{F}_t)_{t \geq 0}$ is σ fields, $b(t, \cdot, \cdot)$ is a d -dimensional process of finite variation and $\mathcal{F} \times \mathcal{B}(W^d)/\mathcal{B}(\mathbb{R}^d)$ adapted function, $M(x, t)$ is a d -dimensional continuous local martingale, and the pair $(a(x, y, t), b(x, t))$ is the local characteristic of $F(x, t)$ (for details, see Mao [1]).

Recently, many studies have focused on the strong uniqueness and the nonexplosion of stochastic differential equations with the coefficients satisfying the local Lipschitz condition (see Stroock and Varadhan [2] and Krylov [3]). However, the results on the strong uniqueness of stochastic differential equations are still very few when the coefficients satisfy non-Lipschitz condition, except the one-dimensional

case (for details, see Ikeda and Watanabe [4] and Revuz and Yor [5]). Let $X_t(x_0, \omega)$ be the solution of the stochastic differential equations (1). If the driving process $M(x, t)$ is only the Brownian motion and local characteristic is locally Lipschitzian, Protter proved that the solution admitted a continuous version $\tilde{X}_t(x_0, \omega)$ (see [6]). When the drift term is independent of t , satisfying local $L_q - L_p$ integrated condition, and the diffusion term is identity matrix, Krylov and Röckner proved the existence and path uniqueness for stochastic differential equations on a given area (see [7]). From the viewpoint of application, to what extent the condition on coefficients should be weakened is an important problem. Fang and Zhang discussed the pathwise uniqueness and the nonexplosion for a class of stochastic differential equations driven by Brownian motion with non-Lipschitz coefficients (see [8]). Using Zvonkin's transformation, Zhang studied the homeomorphic property of solutions of multidimensional stochastic differential equations with non-Lipschitz coefficients (see [9]). When the diffusion coefficient is uniformly nondegenerate and non-Lipschitz and drift coefficient is locally integrable, Zhang also proved the existence of a unique strong solution up to the explosion time for a stochastic differential equation. Moreover, two nonexplosion conditions are given (see [10]). Davie proved the uniqueness of solutions of stochastic differential equations when the drift term is bounded Borel function and the diffusion term is identity

matrix (see [11]). Using Gronwall lemma, Fei studied the existence and uniqueness of solutions for fuzzy random differential equations with non-Lipschitz coefficients and discussed the dependence of fuzzy random differential equations on initial values (see [12]). Luo studied the behaviors of small subsets under the flows generated by the ordinary and stochastic differential equations whose coefficients satisfy certain non-Lipschitz conditions (see [13]). Further, for the stochastic differential equations with non-Lipschitz coefficients driven by Brown motion, Lan proved the pathwise uniqueness and nonexplosion, deriving the new sufficient condition (see [14]). Note that there are very few results about the case of semimartingale. Recently, Fei proved that the solution of the stochastic differential equations (1) was not exploding, when the local characterized a and b satisfy the following conditions:

$$\begin{aligned} a(x, x, t) &\leq C(|x|^2 \rho(|x|^2) + 1), \\ |b(x, t)| &\leq C(|x| \rho(|x|^2) + 1), \end{aligned} \quad (3)$$

where $C > 0$ is a constant and $\rho \in C^1([1, +\infty))$ is a strictly positive function which satisfies the following conditions:

- (i) $\lim_{s \rightarrow +\infty} \rho(s) = +\infty$.
- (ii) $\int_1^\infty (1/(s\rho(s) + 1))ds = +\infty$.
- (iii) $\lim_{s \rightarrow +\infty} (s\rho'(s)/\rho(s)) = 0$.

And the pathwise uniqueness of the stochastic differential equations (1) is also proved, under the following conditions:

$$\begin{aligned} a(x, x, t) - 2a(x, y, t) + a(y, y, t) \\ \leq C|x - y|^2 \lambda(|x - y|^2), \\ |b(x, t) - b(y, t)| \leq C|x - y| \lambda(|x - y|^2), \end{aligned} \quad (4)$$

where $\lambda : [0, \delta_0] \rightarrow \mathbb{R}^+$ is a strictly positive C^1 function satisfying the conditions

- (i) $\lim_{s \rightarrow 0} \lambda(s) = +\infty$;
- (ii) $\int_0^{\delta_0} (1/s\lambda(s))ds = +\infty$;
- (iii) $\lim_{s \rightarrow 0} (s\lambda'(s)/\lambda(s)) = 0$.

For details, see [15]. Utteriorly, Cao proved the existence and pathwise uniqueness of solutions of SDE driven by class of special semimartingale when the coefficient of martingale term is non-Lipschitz and satisfies Taniguchi's condition (see [16]) and Yang established the pathwise uniqueness of SPDE with non-Lipschitz coefficients (see [17]).

In this paper, we derive the new sufficient conditions on the strong uniqueness and the nonexplosion for d -dimensional stochastic differential equations (1) driven by semimartingale with non-Lipschitz coefficients on $\mathbb{R}^d$ ($d > 2$), and the new conditions are sharp in a sense. The theory of related martingale would be found in [4, 5, 18].

This paper is organized as follows. In Section 2, we derive the new sufficient conditions and prove the nonexplosion

for stochastic differential equations (1) on $\mathbb{R}^d$ ($d > 2$). In Section 3, we prove the pathwise uniqueness under some new sufficient conditions and stochastic continuity.

2. No Explosion of Solutions

In this section, we prove the following nonexplosion result of stochastic differential equations (1). Let $X_t(\omega)$ be the solution of the stochastic differential equations (1) and

$$\begin{aligned} a(x, y, t) &\in C(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R}^+), \\ b(x, t) &\in C(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}). \end{aligned} \quad (5)$$

ζ is the lifetime of solution process X_t .

Theorem 1. *Let $\gamma : [0, \infty) \rightarrow \mathbb{R}^+$ be a C^1 function which satisfies the following conditions:*

- (i) $\lim_{s \rightarrow \infty} \gamma(s) = \infty$.
- (ii) $\int_0^\infty (1/(\gamma(s) + 1))ds = \infty$.
- (iii) *for any $s \in [M, \infty)$, there exists $M > 1$ such that $\gamma'(s) \geq 0$, $2\gamma(s)/s - \gamma'(s) \geq 0$.*

$\Lambda : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a positive and progressive measurable function which satisfies

$$\mathbb{E} \int_0^t \Lambda(s, \omega) ds < \infty, \quad t \geq 0. \quad (6)$$

If the condition

$$a(x, x, t) + 2\langle x, b(t, x) \rangle \leq \Lambda(t, \omega) (\gamma(|x|^2) + 1) \quad (7)$$

holds, then the stochastic differential equations (1) have no explosion $P(\zeta = +\infty) = 1$.

Proof. Define

$$\tilde{\gamma}(x) = \begin{cases} x^2 \gamma\left(\frac{1}{x}\right), & 0 < x < 1, \\ \gamma(x), & x \geq 1. \end{cases} \quad (8)$$

According to the continuity of a and b , there exists $\Lambda \geq 0$ such that condition (7) holds with respect to $\tilde{\gamma}(x)$ for any x . Let $\Gamma(x) = \int_0^x (1/(\tilde{\gamma}(s) + 1))ds$; then $\Gamma'(x) = 1/(\tilde{\gamma}(x) + 1)$,

$$\Gamma''(x) = \begin{cases} -\frac{(2x\gamma(1/x) - \gamma'(1/x))}{(x^2\gamma(1/x) + 1)^2}, & 0 < x < 1, \\ -\frac{\gamma'(x)}{(\gamma(x) + 1)^2}, & x \geq 1. \end{cases} \quad (9)$$

Due to (iii) of condition (7), we know that there exists $M > 1$ such that $\Gamma''(x) \leq 0$ for $x < 0$ or $x > M$. It is obvious that the function $\Gamma''(x)$ is not continuous at the point $x = 1$; hence the Itô formula is not applied for Γ . Now take $\tilde{\Gamma} \in C^2(\mathbb{R})$ such that $\tilde{\Gamma}(x) \geq \Gamma(x)$, and

$$\tilde{\Gamma}(x) = \Gamma(x), \quad \text{for } x < \frac{1}{M} \text{ or } x > M. \quad (10)$$

In fact, $\Gamma''(x)$ exists for any $x \neq 1$; we can take

$$\begin{aligned} \tilde{\Gamma}(x) &= \begin{cases} \in C^2\left(\left[\frac{1}{M}, M\right]\right), \tilde{\Gamma} \geq \Gamma, & x \in \left[\frac{1}{M}, M\right], \\ \Gamma(x), & x < \frac{1}{M} \text{ or } x > M \end{cases} \end{aligned} \quad (11)$$

and $\tilde{\Gamma}''(1/M) = \Gamma''(1/M)$, $\tilde{\Gamma}''(M) = \Gamma''(M)$.

Let $\xi_t(\omega) = |X_t(\omega)|^2$; define

$$K = \sup_{x \in [1/M, M]} (|\Gamma'(x)| + |\Gamma''(x)|), \quad (12)$$

$$\tau_R = \inf \{t > 0, \xi_t \geq R\}, \quad R > 0;$$

then $\tau_R \uparrow \zeta$ as $R \uparrow \infty$. Let

$$I_\omega = \left\{t > 0, \xi_t(\omega) \in \left[\frac{1}{M}, M\right]\right\}. \quad (13)$$

If $s \in I_\omega$, then we have

$$a(X_s, X_s, s) + 2\langle X_s, b(X_s, s) \rangle \leq C_M \Lambda(s, \omega), \quad (14)$$

where $C_M = \sup_{x \in [1/M, M]} (\gamma(x) + 1)$. Note that

$$\begin{aligned} d\xi_t &= 2X_t M(X_t, dt) + 2X_t b(X_t, t) dt \\ &\quad + a(X_t, X_t, t) dt, \end{aligned} \quad (15)$$

and the stochastic covariation $d\xi_t \cdot d\xi_t$ is given by

$$d\xi_t \cdot d\xi_t = 4a(X_t, X_t, t) X_t^2 dt. \quad (16)$$

By the Itô formula and (15) and (16), we have

$$\begin{aligned} \tilde{\Gamma}(\xi_{t \wedge \tau_R}) &= \tilde{\Gamma}(\xi_0) + 2 \int_0^{t \wedge \tau_R} \tilde{\Gamma}'(\xi_s) \langle X_s, M(X_s, ds) \rangle \\ &\quad + 2 \int_0^{t \wedge \tau_R} \tilde{\Gamma}'(\xi_s) \langle X_s, b(X_s, s) \rangle ds \\ &\quad + \int_0^{t \wedge \tau_R} \tilde{\Gamma}'(\xi_s) a(X_s, X_s, s) ds \\ &\quad + 2 \int_0^{t \wedge \tau_R} \tilde{\Gamma}''(\xi_s) a(X_s, X_s, s) X_s^2 ds. \end{aligned} \quad (17)$$

Taking expectation for (17) and according to (7) and (14), we have

$$\begin{aligned} \mathbb{E} \tilde{\Gamma}(\xi_{t \wedge \tau_R}) &= \tilde{\Gamma}(\xi_0) + \mathbb{E} \left(\int_0^{t \wedge \tau_R} \tilde{\Gamma}'(\xi_s) \right. \\ &\quad \cdot (2\langle X_s, b(X_s, s) \rangle + a(X_s, X_s, s)) ds \\ &\quad + 2\mathbb{E} \left(\int_0^{t \wedge \tau_R} \tilde{\Gamma}''(\xi_s) a(X_s, X_s, s) X_s^2 ds \right) := \tilde{\Gamma}(\xi_0) \\ &\quad + J_1 + J_2. \end{aligned} \quad (18)$$

For J_1 , due to (14), we have

$$\begin{aligned} J_1 &= \mathbb{E} \left(\int_0^{t \wedge \tau_R} \mathbb{I}_{I_\omega}(s) \tilde{\Gamma}'(\xi_s) \right. \\ &\quad \cdot (2\langle X_s, b(X_s, s) \rangle + a(X_s, X_s, s)) ds \\ &\quad + \mathbb{E} \left(\int_0^{t \wedge \tau_R} \mathbb{I}_{I_\omega^c}(s) \tilde{\Gamma}'(\xi_s) \right. \\ &\quad \cdot (2\langle X_s, b(X_s, s) \rangle + a(X_s, X_s, s)) ds \\ &\leq KC_M \mathbb{E} \left(\int_0^{t \wedge \tau_R} \Lambda(s, \omega) ds \right) \\ &\quad + \mathbb{E} \left(\int_0^{t \wedge \tau_R} \frac{1}{\tilde{\gamma}(\xi_s) + 1} \Lambda(s, \omega) \tilde{\gamma}(\xi_s) ds \right). \end{aligned} \quad (19)$$

For J_2 , noting that $\Gamma''(x) \leq 0$ for $x < 0$ or $x > M$, we have

$$\begin{aligned} J_2 &= 2\mathbb{E} \left(\int_0^{t \wedge \tau_R} \mathbb{I}_{I_\omega}(s) \tilde{\Gamma}''(\xi_s) a(X_s, X_s, s) X_s^2 ds \right) \\ &\quad + 2\mathbb{E} \left(\int_0^{t \wedge \tau_R} \mathbb{I}_{I_\omega^c}(s) \tilde{\Gamma}''(\xi_s) a(X_s, X_s, s) X_s^2 ds \right) \\ &\leq 2\mathbb{E} \left(\int_0^{t \wedge \tau_R} \mathbb{I}_{I_\omega}(s) \tilde{\Gamma}''(\xi_s) a(X_s, X_s, s) X_s^2 ds \right) \\ &\leq 2KMC_M \mathbb{E} \left(\int_0^{t \wedge \tau_R} \Lambda(s, \omega) ds \right). \end{aligned} \quad (20)$$

Combining (19) and (20), it holds that

$$\begin{aligned} \mathbb{E} \tilde{\Gamma}(\xi_{t \wedge \tau_R}) &\leq \tilde{\Gamma}(\xi_0) \\ &\quad + (1 + KC_M + 2KMC_M) \mathbb{E} \left(\int_0^{t \wedge \tau_R} \Lambda(s, \omega) ds \right) \\ &:= C_t < \infty. \end{aligned} \quad (21)$$

Hence,

$$\mathbb{E} \Gamma(\xi_{t \wedge \tau_R}) \leq \mathbb{E} \tilde{\Gamma}(\xi_{t \wedge \tau_R}) \leq C_t < \infty. \quad (22)$$

Letting $R \uparrow \infty$, by the Fatou lemma, we have

$$\mathbb{E} \int_0^{\xi_{t \wedge \zeta}} \frac{1}{\tilde{\gamma}(x) + 1} ds \leq C_t < \infty. \quad (23)$$

If $P(\zeta < \infty) > 0$, then there exists $T > 0$ such that $P(\zeta < T) > 0$; take $t = T$ in (23); we have $\mathbb{E}(\mathbb{I}_{\zeta \leq T} \Gamma(\xi_\zeta)) \leq C_t$, which is impossible, because of $\Gamma(\xi_\zeta) = \infty$. $\square$

3. Pathwise Uniqueness of Solutions

In this section, we will prove the following pathwise uniqueness result; when the driving process is Brownian motion, such kind of properties was studied for non-Lipschitzian coefficients in [8].

Theorem 2. Let $\eta : [0, 1] \rightarrow \mathbb{R}^+$, $\eta(0) = 0$, be a positive and continuous function on some intersection $[0, \epsilon)$ ($\epsilon > 0$), and $\int_0^{+\infty} (1/\eta(s))ds = \infty$ and $\Theta : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a positive and progressive measurable function satisfying

$$\mathbb{E} \int_0^t \Theta(s, \omega) ds < \infty, \quad t > 0. \quad (24)$$

If a and b satisfy the condition

$$\begin{aligned} & a(x, x, t) - 2a(x, y, t) + a(y, y, t) \\ & + 2\langle x - y, b(t, x) - b(t, y) \rangle \\ & \leq \Theta(t, \omega) \eta(|x - y|^2), \end{aligned} \quad (25)$$

then the stochastic differential equations (1) have the pathwise uniqueness.

Proof. Let $\eta_t = X_t - Y_t$ and $\xi_t = \eta_t^2$, where (X, B) and (Y, B) are the solutions of stochastic differential equations (1) with the same initial value. Let $\theta > 0$; define the function $\Gamma_\theta : [0, 1] \rightarrow \mathbb{R}^+$ as follows:

$$\Gamma_\theta(x) = \int_0^x \frac{1}{\eta(s) + \theta} ds. \quad (26)$$

It is obvious that, for any $0 < s < 1$, we have

$$\Gamma_\theta(x) \uparrow \Gamma_0(x) = \int_0^x \frac{1}{\eta(s)} ds = \infty, \quad \text{as } \theta \downarrow 0. \quad (27)$$

Because of

$$\begin{aligned} \frac{\Gamma_\theta(x_2) - \Gamma_\theta(x_1)}{x_2 - x_1} &\geq \frac{\Gamma_\theta(x_3) - \Gamma_\theta(x_2)}{x_3 - x_2}, \\ 0 < x_1 < x_2 < x_3 < \epsilon, \end{aligned} \quad (28)$$

we get the conclusion that Γ_θ is a concave function on $[0, \epsilon)$. Letting

$$\tilde{\Gamma}_\theta(x) = \begin{cases} \Gamma_\theta(x), & x \in [0, \epsilon), \\ \text{a concave function}, & x \in \mathbb{R} \setminus [0, \epsilon), \end{cases} \quad (29)$$

then Γ_θ'' is a nonpositive random measure in the distributive sense.

Let $\tau = \inf\{t > 0, \xi_t \geq \epsilon\}$; by Itô formula, we have

$$\begin{aligned} \tilde{\Gamma}_\theta(\xi_{t \wedge \tau}) &= \Gamma_\theta(\xi_{t \wedge \tau}) = 1 + 2 \int_0^{t \wedge \tau} \Gamma_\theta'(\xi_s) \\ &\cdot \langle \eta_s, M(X_s, ds) - M(Y_s, ds) \rangle + 2 \int_0^{t \wedge \tau} \Gamma_\theta'(\xi_s) \\ &\cdot \langle \eta_s, b(X_s, s) - b(Y_s, s) \rangle ds + \int_0^{t \wedge \tau} \Gamma_\theta'(\xi_s) \\ &\cdot [a(X_s, X_s, s) - 2a(X_s, Y_s, s) + a(Y_s, Y_s, s)] ds \\ &+ 2 \int_0^{t \wedge \tau} \Gamma_\theta''(\xi_s) \eta_s^2 \\ &\cdot [a(X_s, X_s, s) - 2a(X_s, Y_s, s) + a(Y_s, Y_s, s)] ds. \end{aligned} \quad (30)$$

Notice that $\Gamma_\theta''(\xi) \leq 0$, $\xi \in [0, \epsilon)$; applying hypothesis (25) and taking expectation for (30), we get

$$\begin{aligned} \mathbb{E} \Gamma_\theta(\xi_{t \wedge \tau}) &\leq 2 \mathbb{E} \int_0^{t \wedge \tau} \Gamma_\theta'(\xi_s) \\ &\cdot \langle \eta_s, b(X_s, s) - b(Y_s, s) \rangle ds + \mathbb{E} \int_0^{t \wedge \tau} \Gamma_\theta'(\xi_s) \\ &\cdot [a(X_s, X_s, s) - 2a(X_s, Y_s, s) + a(Y_s, Y_s, s)] ds \\ &\leq \mathbb{E} \int_0^{t \wedge \tau} \frac{1}{\eta(\xi_s) + \theta} \Theta(s, \omega) \eta(\xi_s) ds \\ &\leq \mathbb{E} \int_0^t \Theta(s, \omega) ds := C_t < \infty. \end{aligned} \quad (31)$$

Letting $\theta \downarrow 0$ in inequality (31), we have $\mathbb{E} \int_0^{\xi_{t \wedge \tau}} (1/\eta(s)) ds \leq C_t < \infty$. It means that, for any t given,

$$\xi_{t \wedge \tau} = 0, \text{ a.s.} \quad (32)$$

If $P(\tau < +\infty) > 0$, then there exists some $T > 0$ big enough such that $P(\tau \leq T) > 0$. By (32), for all $t \in \mathbb{Q} \cap [0, T]$, $\xi_{t \wedge \tau} = 0$, a.s. It follows that, on $\{\tau \leq T\}$, $\xi_\tau = 0$, which is absurd with the definition of τ . Therefore $\tau = +\infty$ almost surely and, for any t given, $\xi_t = 0$ almost surely. By the continuity of samples, two solutions are indistinguishable. $\square$

Remark 3. The proved method of [15] is not suitable here. In fact, let $\Phi_\theta(\xi) = e^{\Gamma_\theta(\xi)}$; we get $\Phi_\theta''(\xi) = \Phi_\theta(\xi)((1 - \eta'(\xi))/(\eta(\xi) + \theta)^2)$; by (30) we know that

$$\begin{aligned} \int_0^{t \wedge \tau} \Phi_\theta''(\xi_s) d\xi_s \cdot d\xi_s &\leq \int_0^{t \wedge \tau} \Phi_\theta(\xi_s) \left| \frac{1 - \eta'(\xi_s)}{(\eta(\xi_s) + \theta)^2} \right| \\ &\cdot \eta(\xi_s) \cdot \xi_s ds. \end{aligned} \quad (33)$$

Notice that $|(1 - \eta'(\xi_s))/(\eta(\xi_s) + \theta)^2| \cdot \eta(\xi_s) \cdot \xi_s$ may be unbounded, such that Gronwall lemma cannot be used here.

By the above pathwise uniqueness, we can also obtain the following continuity result.

Theorem 4. Suppose that the coefficients a and b satisfy assumptions (7) and (25). Let $X_t(x_0)$ be the solution of the SDE (1) with initial value x_0 . Then, for any $\epsilon > 0$, one has

$$\lim_{y \rightarrow x} P \left(\sup_{0 \leq s \leq t} |X(s, y) - X(s, x)| > \epsilon \right) = 0. \quad (34)$$

Proof. Consider x and y satisfying $|y - x| < \epsilon$; let

$$\xi_t = |X_t(y) - X_t(x)|^2. \quad (35)$$

Define $\tau(x, y) = \inf\{t > 0, \xi_t > \epsilon^2\}$; let Γ_θ be the function defined in Theorem 2. By Itô formula, we have

$$\mathbb{E} \Gamma_\theta(\xi_{t \wedge \tau(x, y)}) \leq \Gamma_\theta(\xi_0) + C_t, \quad \forall t > 0, \theta > 0, \quad (36)$$

where C_t is the same as in (31). Let $\theta = |x - y|$ in the above inequality; it holds that

$$\mathbb{E}\Gamma_\theta(\xi_{t\wedge\tau(x,y)}) \leq \int_0^{\theta^2} \frac{1}{\eta(s) + \theta} ds + C_t \leq \theta + C_t \quad (37)$$

so

$$P(\tau(x, y) < t) \Gamma_\theta(\epsilon^2) \leq \mathbb{E}\Gamma_\theta(\xi_{t\wedge\tau(x,y)}) \leq \theta + C_t. \quad (38)$$

Then we can obtain

$$\begin{aligned} & P\left(\sup_{0 \leq s \leq t} |X(s, y) - X(s, x)| > \epsilon\right) \\ &= P(\tau(x, y) < t) \leq \Gamma_\theta^{-1}(\epsilon^2)(\theta + C_t) \rightarrow 0 \end{aligned} \quad (39)$$

as $\theta = |x - y| \rightarrow 0$. The proof is completed. $\square$

Remark 5. We should point out that the results (Theorems 1 and 2) of [15] are the special case of this paper result (Theorems 1 and 2). In fact, let $\gamma(x) = x\rho(x)$, $\eta(x) = x\lambda(x)$ (where $\rho(x), \lambda(x)$ are the corresponding function of [15]); it is clear that $\gamma(x), \eta(x)$ satisfy conditions (7) and (25), respectively. Hence, the result of this paper includes the result of [15]. What is more, η only need continuity in Theorem 2; however, λ must be differentiable function in [8, 15], and our proof is more directive.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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