Research Article

Coefficient Bounds for Certain Subclasses of $m$-Fold Symmetric Biunivalent Functions

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We consider two new subclasses $S_{Σ_m}(α, λ)$ and $S_{Σ_m}(β, λ)$ of $Σ_m$ consisting of analytic and $m$-fold symmetric biunivalent functions in the open unit disk $U$. Furthermore, we establish bounds for the coefficients for these subclasses and several related classes are also considered and connections to earlier known results are made.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{∞} a_n z^n,$$  

(1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let $S$ be the subclass of $A$ consisting of form (1) which is also univalent in $U$.

The Koebe one-quarter theorem [1] states that the image of $U$ under every function $f$ from $S$ contains a disk of radius $1/4$. Thus, every such univalent function has inverse $f^{-1}$ which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U),$$

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}),$$

(2)

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^2 - 5a_2 a_4 + a_4) w^4 + \cdots.$$  

(3)

Function $f \in A$ is said to be biunivalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $Σ$ denote the class of biunivalent functions defined in unit disk $U$.

For a brief history and interesting examples in class $Σ$, see [2]. Examples of functions in class $Σ$ are

$$\frac{z}{1-z},$$

$$-\log(1-z),$$

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

(4)

and so on. However, the familiar Koebe function is not a member of $Σ$. Other common examples of functions in $S$ such as

$$\frac{z - z^2}{2},$$

$$\frac{z}{1 - z^2}$$

(5)

are also not members of $Σ$ (see [2]).

For each function $f \in S$, function

$$h(z) = \sqrt[1-m]{f(z^m)} \quad (z \in U, \ m \in \mathbb{N})$$

(6)

is univalent and maps unit disk $U$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see [3, 4]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{∞} a_{mk+1} z^{mk+1} \quad (z \in U, \ m \in \mathbb{N}).$$

(7)
We denote by $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (7). In fact, the functions in class $S$ are one-fold symmetric.

Analogous to the concept of $m$-fold symmetric univalent functions, we here introduced the concept of $m$-fold symmetric biunivalent functions. Each function $f \in \Sigma$ generates an $m$-fold symmetric biunivalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (7) and the series expansion for $f^{-1}$, which has been recently proven by Srivastava et al. [5], is given as follows:

\[
g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} - (3m+2)a_{m+1}a_{2m+1} + \cdots,
\]

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric biunivalent functions in $U$. For $m = 1$, formula (8) coincides with formula (3) of class $\Sigma$. Some examples of $m$-fold symmetric biunivalent functions are given as follows:

\[
\left\lfloor \frac{z^m}{1-z^m}\right\rfloor^{1/m},
\]

\[
\left[-\log(1-z^m)^{1/m}\right],
\]

\[
\left\lfloor \frac{1}{2} \log\left(\frac{1+z^m}{1-z^m}\right)^{1/m}\right\rfloor.
\]

Lewin [6] studied the class of biunivalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [8] showed that $\max|a_2| = 4/3$ if $f(z) \in \Sigma$. Brannan and Taha [9] introduced certain subclasses of biunivalent function class $S$ similar to the familiar subclasses. $S^\alpha(\beta)$ and $K(\beta)$ are of starlike and convex function of order $\beta$ ($0 \leq \beta < 1$), respectively (see [8]). Classes $S^\alpha_2(\alpha)$ and $K_2(\alpha)$ of bistarlike functions of order $\alpha$ and biconvex functions of order $\alpha$, corresponding to function classes $S^\alpha(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of function classes $S^\alpha_2(\alpha)$ and $K_2(\alpha)$, they found nonsharp estimates on the initial coefficients. In fact, the aforesaid work of Srivastava et al. [2] essentially revived the investigation of various subclasses of biunivalent function class $\Sigma$ in recent years. Recently, many authors investigated bounds for various subclasses of biunivalent functions (see [2, 10–15]). Not much is known about the bounds on general coefficient $|a_n|$ for $n \geq 4$. In the literature, only few works determine general coefficient bounds $|a_n|$ for the analytic biunivalent functions (see [16–18]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \ldots\}$) is still an open problem.

The aim of the this paper is to introduce two new subclasses of function class $\Sigma_m$ and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. We have to remember the following lemma here so as to derive our basic results.

Lemma 1 (see [4]). If $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ is an analytic function in $U$ with positive real part, then

\[
|p_n| \leq 2 \quad (n \in \mathbb{N} = \{1, 2, \ldots\}),
\]

\[
|p_{n} - p_{n+1}^2|/2 \leq 2 - |p_{1}^2/2|.
\]

2. Coefficient Bounds for Function Class $\Sigma_m(\alpha, \lambda)$

Definition 2. A function $f \in \Sigma_m$ is said to be in class $\Sigma_m(\alpha, \lambda)$ if the following conditions are satisfied:

\[
\left|\arg\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)}\right)\right| < \frac{\alpha \pi}{2},
\]

\[
f \in \Sigma, (0 < \alpha \leq 1, 0 \leq \lambda < 1, z \in U),
\]

where function $g = f^{-1}$.

Theorem 3. Let $f$ given by (7) be in class $\Sigma_m(\alpha, \lambda), 0 < \alpha \leq 1$. Then,

\[
|a_{m+1}| \leq \frac{2\alpha}{m(1-\lambda)\sqrt{\alpha + 1}},
\]

\[
|a_{2m+1}| \leq \frac{\alpha}{m(1-\lambda)} + \frac{2(m+1)\alpha^2}{m^2(1-\lambda)^2}.
\]

Proof. Let $f \in \Sigma_m(\alpha, \lambda)$. Then,

\[
z f'(z) / (1-\lambda)f(z) + \lambda zf'(z) = [p(z)]^\alpha,
\]

\[
\lambda g'(w) / (1-\lambda)g(w) + \lambda wg'(w) = [q(w)]^\alpha,
\]

where $g = f^{-1}$ and $p, q \in P$ have the following forms:

\[
p(z) = 1 + p_mz^m + p_{2m}z^{2m} + \cdots,
\]

\[
q(w) = 1 + q_mw^m + q_{2m}w^{2m} + \cdots.
\]

Now, equating the coefficients in (13), we get

\[
m(1-\lambda)a_{m+1} = \alpha p_m,
\]

\[
m(1-\lambda)\left[2a_{2m+1} - (\lambda m + 1)a_{m+1}\right] = \alpha p_{2m} + \frac{\alpha(\alpha - 1)}{2}p_m^2,
\]

\[
-m(1-\lambda)a_{m+1} = \alpha q_m,
\]

\[
m(1-\lambda)\left[(1 + m(2-\lambda))a_{m+1}^2 - 2a_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha - 1)}{2}q_m^2.
\]
From (15) and (17), we obtain
\[
P_m = -q_m, \quad (19)
\]
\[
2m^2 (1 - \lambda)^2 a_{m+1}^2 = \alpha^2 \left( p_m^2 + q_m^2 \right). \quad (20)
\]
Also from (16), (18), and (20), we have
\[
2m^2 (1 - \lambda)^2 a_{m+1}^2 = \alpha (p_{2m} + q_{2m})
+ \frac{\alpha (\alpha - 1)}{2} (p_m^2 + q_m^2)
= \alpha (p_{2m} + q_{2m})
+ \frac{\alpha (\alpha - 1)}{2} 2m^2 (1 - \lambda)^2 \alpha^2 a_{m+1}^2.
\]
Therefore, we have
\[
a_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{m^2 (1 - \lambda)^2 (\alpha + 1)}. \quad (22)
\]
Applying Lemma 1 for coefficients \(p_{2m}\) and \(q_{2m}\), we obtain
\[
|a_{m+1}| \leq \frac{2\alpha}{m (1 - \lambda) \sqrt{\alpha + 1}}. \quad (23)
\]
Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (18) from (16), we obtain
\[
4m (1 - \lambda) a_{2m+1} - 2m (m + 1) (1 - \lambda) a_{m+1}^2
= \alpha (p_{2m} - q_{2m})
+ \frac{\alpha (\alpha - 1)}{2} (p_m^2 - q_m^2). \quad (24)
\]
Then, in view of (19) and (20) and applying Lemma 1 for coefficients \(p_{2m}, p_m\) and \(q_{2m}, q_m\), we have
\[
|a_{2m+1}| \leq \frac{\alpha}{m (1 - \lambda)} + \frac{2 (m + 1) \alpha^2}{m^2 (1 - \lambda)^2}. \quad (25)
\]
This completes the proof of Theorem 3. \(\square\)

3. Coefficient Bounds for Function Class \(S_{2m} (\beta, \lambda)\)

**Definition 4.** Function \(f \in \Sigma_m\) given by \((7)\) is said to be in class \(S_{2m} (\beta, \lambda)\) if the following conditions are satisfied:

\[
\text{Re} \left( \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} \right) > \beta
\]

\(f \in \Sigma, \quad 0 \leq \beta < 1, \quad 0 \leq \lambda < 1, \quad z \in U), \quad (26)
\]

\[
\text{Re} \left( \frac{\lambda g'(w)}{(1 - \lambda)g(w) + \lambda wg'(w)} \right) > \beta
\]

\(0 \leq \beta < 1, \quad 0 \leq \lambda < 1, \quad w \in U), \quad (26)
\]

where function \(g = f^{-1}\).

**Theorem 5.** Let \(f\) given by \((7)\) be in class \(S_{2m} (\beta, \lambda), 0 \leq \beta < 1\). Then,

\[
|a_{m+1}| \leq \frac{\sqrt{2(1 - \beta)}}{m (1 - \lambda)}, \quad (27)
\]

\[
|a_{2m+1}| \leq \frac{2 (m + 1) (1 - \beta)^2}{m^2 (1 - \lambda)^2} + \frac{1 - \beta}{m (1 - \lambda)}. \quad (28)
\]

**Proof.** Let \(f \in S_{2m} (\beta, \lambda)\). Then,

\[
\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} = \beta + (1 - \beta) p(z),
\]

\[
\frac{\lambda g'(w)}{(1 - \lambda)g(w) + \lambda wg'(w)} = \beta + (1 - \beta) q(w),
\]

where \(p, q \in P\) and \(g = f^{-1}\).

It follows from (28) that

\[
m (1 - \lambda) a_{m+1} = (1 - \beta) p_m,
\]

\[
m (1 - \lambda) \left[ 2a_{2m+1} - (\lambda m + 1) a_{m+1}^2 \right] = (1 - \beta) p_{2m},
\]

\[
- m (1 - \lambda) a_{m+1} = (1 - \beta) q_m,
\]

\[
m (1 - \lambda) \left[ (1 + m (2 - \lambda)) a_{m+1}^2 - 2a_{2m+1} \right]
= (1 - \beta) q_{2m}.
\]

From (29) and (31), we obtain

\[
p_m = -q_m
\]

\[
2m^2 (1 - \lambda)^2 a_{m+1}^2 = (1 - \beta)^2 (p_m^2 + q_m^2).
\]

Adding (30) and (32), we have

\[
2m^2 (1 - \lambda)^2 a_{m+1}^2 = (1 - \beta) (p_{2m} + q_{2m}). \quad (34)
\]

Therefore, we obtain

\[
a_{m+1}^2 = \frac{(1 - \beta) (p_{2m} + q_{2m})}{2m^2 (1 - \lambda)^2}. \quad (35)
\]

Applying Lemma 1 for coefficients \(p_{2m}\) and \(q_{2m}\), we obtain

\[
|a_{m+1}| \leq \frac{\sqrt{2 (1 - \beta)}}{m (1 - \lambda)}. \quad (36)
\]

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (32) from (30), we obtain

\[
4m (1 - \lambda) a_{2m+1} - 2m (m + 1) (1 - \lambda) a_{m+1}^2
= (1 - \beta) (p_{2m} - q_{2m}). \quad (37)
\]

Then, in view of (33), applying Lemma 1 for coefficients \(p_{2m}, p_m\) and \(q_{2m}, q_m\), we have

\[
|a_{2m+1}| \leq \frac{2 (m + 1) (1 - \beta)^2}{m^2 (1 - \lambda)^2} + \frac{1 - \beta}{m (1 - \lambda)}. \quad (38)
\]

This completes the proof of Theorem 5. \(\square\)
If we set \( \lambda = 0 \) in Theorems 3 and 5, then classes \( S_{\Sigma_m}^{\alpha, \lambda} \) and \( S_{\Sigma_m}^{\alpha, \lambda} \) reduce to classes \( S_{\Sigma_m}^{\alpha} \) and \( S_{\Sigma_m}^{\beta} \), and thus we obtain the following corollaries.

**Corollary 6.** Let \( f \) given by (7) be in class \( S_{\Sigma_m}^{\alpha} (0 < \alpha \leq 1) \). Then,
\[
|a_{m+1}| \leq \frac{2\alpha}{m \sqrt{\alpha + 1}},
\]
\[
|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.
\]

**Corollary 7.** Let \( f \) given by (7) be in class \( S_{\Sigma_m}^{\beta} (0 \leq \beta < 1) \). Then,
\[
|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m},
\]
\[
|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}.
\]

Classes \( S_{\Sigma_m}^{\alpha} \) and \( S_{\Sigma_m}^{\beta} \) are, respectively, defined as follows.

**Definition 8.** Function \( f \in \Sigma_m \) given by (7) is said to be in class \( S_{\Sigma_m}^{\alpha} \) if the following conditions are satisfied:
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad f \in \Sigma, \quad (0 < \alpha \leq 1, \ z \in U),
\]
\[
\left| \arg \left( \frac{\lambda g'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, \ w \in U),
\]
where function \( g = f^{-1} \).

**Definition 9.** Function \( f \in \Sigma_m \) given by (7) is said to be in class \( S_{\Sigma_m}^{\beta} \) if the following conditions are satisfied:
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad f \in \Sigma, \quad (0 \leq \beta < 1, \ z \in U),
\]
\[
\Re \left( \frac{\lambda g'(w)}{g(w)} \right) > \beta, \quad (0 \leq \beta < 1, \ w \in U),
\]
where function \( g = f^{-1} \).

For one-fold symmetric bi-univalent functions and \( \lambda = 0 \), Theorems 3 and 5 reduce to Corollaries 10 and 11, respectively, which were proven earlier by Murugusundaramoorthy et al. [19].

**Corollary 10.** Let \( f \) given by (7) be in class \( S_{\Sigma_m}^{\alpha} (0 < \alpha \leq 1) \). Then,
\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}},
\]
\[
|a_3| \leq 4\alpha^2 + \alpha.
\]

**Corollary 11.** Let \( f \) given by (7) be in class \( S_{\Sigma_m}^{\beta} (0 \leq \beta < 1) \). Then,
\[
|a_2| \leq \sqrt{2(1-\beta)},
\]
\[
|a_3| \leq 4(1-\beta)^2 + (1-\beta).
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


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