Research Article

On a Theorem of Ziv Ran concerning Abelian Varieties Which Are Product of Jacobians

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Received 21 July 2015; Accepted 21 September 2015

Academic Editor: Kaleem R. Kazmi

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We give a new proof for a theorem of Ziv Ran which generalizes some results of Matsusaka and Hoyt. These results provide criteria for an Abelian variety to be a Jacobian or a product of Jacobians. The advantage of our method is that it works in arbitrary characteristic.

1. Introduction

In the classical papers [1] (resp. [2]), Matsusaka and Hoyt gave a necessary and sufficient criterion for an Abelian variety A for being a Jacobian, respectively, a product of Jacobians. In [3], Ran reconsiders the subject and gives a more general and probably more natural criterion for this. However, his method seems unsatisfactory in positive characteristic.

The aim of this paper is to reprove Ran's criterion, using results from [4] on the ring of the numerical algebraic cycles on A . For the particular case of the Ran-Matsusaka criterion, another proof appeared in [5]. Both proofs are independent of the characteristic of the base field.

In the sequel, for an Abelian variety A , we denote by $Z(A)$ the $\mathbb{Q}$ -vector space of the algebraic cycles with $\mathbb{Q}$ -coefficients on A and by $N(A)$ the quotient by numerical equivalence. It is well known (cf. [6] or [4]) that on $N(A)$ there are two $\mathbb{Q}$ -algebra structures given by the usual product and by the Pontrjagin product.

Throughout this paper, the latter will be very useful thanks to its geometric definition and to the fact that it gives a ring structure not only on $N(A)$ but also on $Z(A)$. Below, for $x, y \in N(A)$, we will denote the usual product by $x \cdot y$ and the Pontrjagin product by $x \star y$.

Also, for the two subvarieties X_1 and X_2 in A , we denote by $X_1 + X_2$ and $X_1 \div X_2$ their sum and difference in the group law of A , to avoid confusion with the corresponding

operations on the cycles. In the current paper, the algebraic cycles will often be divisors and 1-cycles (the latter ones being formal sums of curves) and they will always have integer coefficients.

The term curve, is reserved for integral ones, and all 1-cycles will be considered effective. Finally, a prime cycle is an irreducible subvariety of A of a corresponding codimension.

2. Generating Curves on an Abelian Variety

Let A be an Abelian variety of dimension n and E be a curve on it containing the origin 0_A of A . We consider a sequence of closed subsets in A , defined as follows:

$$\begin{aligned} E_0 &= \{0_A\} \\ E_i &= E + E + \cdots + E \quad (i \text{ terms, with } 1 \leq i \leq n), \\ E_{n+1} &= A. \end{aligned} \quad (1)$$

It is clear that this sequence is increasing, $\dim E_i \leq i$ and $\dim E_{i+1} \leq \dim E_i + 1$ for every i . As long as E is a curve, E_i is irreducible and there is a first index j such that $E_j = E_{j+1}$. Also, we have $E_i = E_j$ for all $i \geq j$ and $\dim E_j = j$. It follows that E_j is stable for the group law on A and the induced operation has 0_A as unity. Using a result of Ramanujan in ([7, chapter II, Section 4]), E_j is an Abelian subvariety of A and the points of E generate the group E_j . We denote by $\langle E \rangle$

the subvariety E_j . If $j = n$, then $\langle E \rangle = A$ and we say that E is a generating curve on A .

Remark 1. In [8] Matsusaka proves that every Abelian variety has a generating curve. Moreover, from his proof, for a projective embedding of A , every linear section with a convenient linear subspace of appropriate dimension which contains 0_A is a generating curve for A .

Using the Pontrjagin product (for cycles, not for numerical classes) it is easy to deduce the following useful fact.

Lemma 2. *Let E be a curve in A with $0_A \in E$, $\langle E \rangle$ the subvariety of A generated by E and $j = \dim \langle E \rangle$. Then, j is the maximal number i such that $E^{*i} (= E * E * \dots * E$ with i terms) is nonzero and $\langle E \rangle$ is the support of the cycle E^{*j} .*

Let us now consider a curve $E \subset A$ which does not necessarily contain the origin. It easily follows that for $x \in E$ the Abelian variety generated by $E \pm \{x\}$ does not depend on x ; it is in fact the subgroup of A generated by $E \pm E$. This Abelian variety will also be denoted by $\langle E \rangle$. If $Z = m_1 E_1 + \dots + m_k E_k$ is an effective 1-cycle, we denote by $\langle Z \rangle$ the Abelian subvariety given by $\langle E_1 \rangle + \langle E_2 \rangle + \dots + \langle E_k \rangle$ and we will say that Z is a generating 1-cycle for $\langle Z \rangle$.

Remark 3. From the definition above, we see that the construction of $\langle Z \rangle$ is independent of the numbers m_i . In particular, Z and Z_{red} generate the same subvariety and also for Z and mZ .

The next lemma will be useful in the sequel.

Lemma 4. (a) *For $Y, E \subset A$ with Y subvariety and E , a curve, both containing the origin, if $Y * E = 0$ then $E \subset Y$.*

(b) *For $Y_1, Y_2 \subset A$ Abelian subvarieties and m, n nonzero integers, if mY_1 and nY_2 are numerically equivalent, then $Y_1 = Y_2$.*

Proof. (a) We have $\dim Y \leq \dim(Y+E) \leq 1 + \dim Y$ and from $Y * E = 0$ we deduce that $\dim(Y+E) < 1 + \dim Y$. So $Y = Y+E$ and because $0_A \in Y$ it follows $E \subset Y$.

(b) Let E_1 be a generating curve for Y_1 (it exists cf. Remark 1). Then $mY_1 * E_1 = m(Y_1 * E_1) = 0$ and so $nY_2 * E_1 = n(Y_2 * E_1) = 0$. It follows that $Y_2 * E_1 = 0$, and from the first point $E_1 \subset Y_2$. The last inclusion implies that $Y_1 = \langle E_1 \rangle \subset Y_2$. In the same way the reversed inclusion is proved. $\square$

Remark 5. The point (b) above, in the case $m = n = 1$, is a result of Matsusaka in [8].

Proposition 6. (a) *For two curves $E_1, E_2 \subset A$ which are numerically equivalent, we have $\langle E_1 \rangle = \langle E_2 \rangle$.*

(b) *For a curve E and Z , a 1-cycle which is numerically equivalent with E , we have $\langle Z \rangle = \langle E \rangle$.*

(c) *Let D be an ample divisor and Z a 1-cycle which is numerically equivalent with D^{n-1} . Then Z is a generating 1-cycle for A .*

Proof. (a) Using convenient translations we can suppose that E_1 and E_2 contain 0_A . Let r, s be the dimensions of $Y_i = \langle E_i \rangle$ for $i = 1, 2$. Using Lemma 2 and the fact that E_1^{*a} is numerically equivalent with E_2^{*a} for all positive integers a , we find $r = s$. From Lemma 2 again, E_1^{*r} is a multiple of both Y_1 and Y_2 , and the conclusion follows from Lemma 4(b).

(b) As in (a), denoting $Z = m_1 E_1 + \dots + m_k E_k$ we can suppose that E and all E_i contain 0_A . By Lemma 2, for $r = \dim \langle E \rangle$, we have $E^{*(r+1)} = 0$. Now, Z^{*r} being numeric equivalent with E^{*r} , we find $Z^{*r} * E = 0$. But again from Lemma 2 we find a nonzero term in the development of Z^{*r} . With Lemma 4(a), this term which is in fact a subvariety contains E , because all terms in Z^{*r} are vanished by the Pontrjagin product with E . On the other hand, this term is contained in $\langle Z \rangle$ and so $\langle E \rangle \subset \langle Z \rangle$.

For the reverse inclusion, we consider the development of the left side of the equality $Z * E^{*r} = 0$. From Lemma 2 $E^{*r} = n \langle E \rangle$ with $n \geq 1$ is an integer. We find $m_1 n E_1 * \langle E \rangle + \dots + m_k n E_k * \langle E \rangle = 0$; therefore $E_i * \langle E \rangle = 0$ for all i and then from Lemma 4(a), $E_i \subset \langle E \rangle$, so $\langle Z \rangle \subset \langle E \rangle$.

(c) Let m be a positive integer with $|mD|$ very ample. The cycles $m^{n-1} D^{n-1}$ and $m^{n-1} Z$ are therefore numerical equivalents and there will exist an integer curve E in the same numeric class with $m^{n-1} D^{n-1}$ and so with $m^{n-1} Z$. From (b) we have $\langle E \rangle = \langle m^{n-1} Z \rangle = \langle Z \rangle$. But from Remark 1 $\langle E \rangle = A$, so Z is a generating 1-cycle. $\square$

The point (c) above is a slight generalization of the result from Remark 1 and will be used to deduce the Matsusaka-Hoyt criterion from that of Ran.

3. Algebraic Cycles Constructed from Generating Curves

We recall a result from [4] which will be the main tool in the proof of Ran's theorem. Let E be a generating curve of the n -dimensional Abelian variety A . We consider on A the following cycles: $W_n(E) = \{0_A\}$ and $W_i(E) = (1/(n-i)!) E^{*(n-i)}$ for $0 \leq i \leq n-1$. From the definition of the Pontrjagin product, $W_i(E)$ is a cycle with irreducible support of codimension i on A . In particular $W_1(E)$ is a divisor and there exists $\alpha_E \in \mathbb{Q}$ such that $W_0(E) = \alpha_E \cdot 1_A$, where 1_A is the fundamental cycle on A .

The result we need from [4] is the following.

Proposition 7. *All cycles $W_i(E)$ have integer coefficients and in particular $\alpha_E \in \mathbb{Z}$, being evidently positive. Also, $W_1(E)^i = i! \alpha_E^{i-1} W_i(E)$ for $1 \leq i \leq n$. In particular, $W_1(E)^n > 0$ and so $W_1(E)$ is ample.*

Remark 8. For E a smooth curve and A its Jacobian, these divisors are well-known.

A first application of the proposition above is point (b) in the following.

Proposition 9. (a) *Let D be an effective divisor and Z a generating 1-cycle. Then $D \cdot Z > 0$.*

(b) *If moreover D is ample, then $D \cdot Z \geq n = \dim A$.*

Proof. (a) We can suppose that D is a prime divisor. Let $E_1, \dots, E_k$ be the components of Z . We have $D \cdot E_i \geq 0$ for all i , because the general translation of E_i cuts properly D . It is therefore sufficient to find an i such that $D \cdot E_i > 0$. Suppose there is no such i . Then using a result from [7, chapter2, Section 6] translations with elements of the form $\{x\} \div \{y\}$ with $x, y \in E_i$ leave D invariant. But Z is a generating 1-cycle, and therefore every element in A is a sum of elements of this form. So D is invariant with respect to any translation and then numerically equivalent with 0, in contradiction with its effectiveness.

(b) Consider a first case where $Z = E$ is a prime cycle (i.e., E is a curve) and without loss of generality $0_A \in E$. Let t be a variable and consider the polynomial $P(t) = (t \cdot W_1(E) + D)^n = n! \alpha_E^{n-1} t^n + n! \alpha_E^{n-2} (D \cdot E) t^{n-1} + \dots + D^n$. Because $W_1(D)$ is ample and D is nondegenerate, the index theorem for Abelian varieties compare [7] asserts that all roots of P are real and negative. So the means inequality gives $D \cdot E \geq n(\chi(\mathcal{O}_A(D)) \cdot \alpha_E)^{1/n} \geq n$.

For the general case, let $Z = m_1 E_1 + \dots + m_k E_k$ with all $m_i > 0$, $X_i = \langle E_i \rangle$, and D_i be the restriction of D to X_i . The projection formula gives $D \cdot E_i = D_i \cdot E_i \geq \dim X_i$ from the particular case above. So, $D \cdot Z = \sum_i m_i D \cdot E_i \geq \sum_i D \cdot E_i \geq \sum_i \dim X_i \geq \dim A = n$, because Z is a generating 1-cycle. $\square$

The following consequence of the above proposition will be useful in the last part of the paper.

Corollary 10. *Let A be an Abelian variety, $D = \sum_{i=1}^r m_i D_i$ an ample effective divisor, and $Z = \sum_{j=1}^s n_j E_j$ a generating 1-cycle of A (the coefficients are supposed to be nonzero). If $D \cdot Z = n = \dim A$ then $m_i = n_j = 1$ for all i, j .*

Proof. We have $n = D \cdot Z = \sum_{i=1}^r m_i D_i \cdot Z \geq \sum_{i=1}^r D_i \cdot Z \geq n$ because $\sum_{i=1}^r D_i$ is ample and one can apply Proposition 9(b). So $m_i D_i \cdot Z = D_i \cdot Z$ and because the last term is nonzero by Proposition 9(b), we find $m_i = 1$ for all i .

In the same way, $n = D \cdot Z = \sum_{j=1}^s n_j D \cdot E_j \geq \sum_{j=1}^s D \cdot E_j = D \cdot \sum_{j=1}^s E_j \geq n$ because $\sum_{j=1}^s E_j$ remains a generator 1-cycle by Remark 3. So $n_j D \cdot E_j = D \cdot E_j$ and D being ample, the last term is positive. It results that $n_j = 1$ for all j . $\square$

We can now prove the following result, which is nothing else but Ran's version of the Matsusaka theorem.

Theorem 11. *Let D be an ample divisor on the Abelian variety A and let E be a generating curve such that $D \cdot E = n = \dim A$. Then E is smooth, A is its Jacobian, and D is a translation of $W_1(E)$.*

Proof. In the proof of point (b) from Proposition 9 we have the inequality $D \cdot E \geq n(\chi(\mathcal{O}_A(D)) \cdot \alpha_E)^{1/n} \geq n$. If $D \cdot E = n$ we will have $\chi(\mathcal{O}_A(D)) = \alpha_E = 1$ and so $D^n = n!$. In this case the polynomial $P(t)$ from the same proposition becomes $P(t) = n! t^n + n! \cdot n \cdot t^{n-1} + \dots + n!$. It follows that the arithmetic and geometric means of the roots coincide and so all the roots have the form $-\lambda$ for a positive value of λ . So $P(t) = n!(t + \lambda)^n$ and by identification, $\lambda^n = 1$. It follows that $\lambda = 1$ and then

$W_1(E)^{n-1} \cdot D = W_1(E)^{n-2} \cdot D^2 = n!$. These relations imply that $(D - W_1(E)) \cdot W_1(E)^{n-1} = (D - W_1(E))^2 \cdot W_1(E)^{n-2} = 0$. The Hodge index theorem asserts that D is numerically equivalent with $W_1(E)$, and because $W_1(E)$ is a principal polarization (from Proposition 7 and equality $\alpha_E = 1$), one may deduce that D is a translation of $W_1(E)$.

Consider the normalization $f_0 : T \rightarrow E$ for E , and let $f : J \rightarrow A$ be a prolongation of f_0 , where J is a Jacobian of T . If we choose a base point in the construction of J , one on T which sits above $0_A \in E$, f will be a morphism of Abelian varieties, sending origin to origin. Also, f is surjective because E is generating for A and for $g = \text{genus of } T$ we have $g \geq n$.

Let us denote by $W_i = W_i(T)$ the canonical cycles on the Jacobian J . Therefore $f_*(W_{g-i}) = W_{n-i}(E)$ for $1 \leq i \leq n$: for $i = 1$ this is clear because $W_{g-1} = T$ and for $i > 1$ it is a consequence of the definitions for W_{g-i} and $W_{g-i}(E)$ and also from the fact that f_* commute with the Pontrjagin product. In particular $f_*(W_{g-n}) = W_0(E) = \alpha_E \cdot 1_A$ and so $\alpha_E = 1$ is the degree of the restriction of f to W_{g-n} . Therefore this restriction is a birational morphism and has an inverse: $A \dashrightarrow W_{g-n}$. This inverse, considered as a rational map from A to J can be extended over all the A giving a morphism $A \rightarrow J$ compare [7]. As a consequence, the restriction g of f to W_{g-n} will be an isomorphism and W_{g-n} will be an Abelian subvariety of J . But W_{g-n} contains $W_{g-1} = T$ which generates J and so $W_{g-n} = J$. In this case we have $g = n$ and f is birational from J to A hence an isomorphism. $\square$

4. Proof of Ran Theorem

The purpose of this section is to give a proof for Ran's full theorem. Some points are as in [3] and are included only for the sake of completeness. The modifications appear from the replacement of Lemma II.8 from [3] with the result below whose proof is very simple.

Lemma 12. *Let D be a prime divisor on an Abelian variety A . Then, there exists an Abelian variety B , a surjective morphism of Abelian varieties $f : A \rightarrow B$ and an ample divisor F on B such that $f^{-1}(F) = D$ as schemes.*

Proof. We consider the closed subgroup K of A defined by $K := \{x \in A \mid \{x\} + D = D\}$ and the Abelian subvariety K_0 of A which is the connected component of 0_A in K . We denote by B the quotient A/K_0 and by $f : A \rightarrow B$ the quotient morphism. Finally we denote by F the closed irreducible subset $f(D)$ with the reduced structure. We easily find $\dim F = \dim B - 1$, so F is a divisor on B and set theoretically $f^{-1}(F) = D + K_0 = D$ because $K_0 \subset K$. Let $x \in A$ such that $\{f(x)\} + F = F$. Applying f^{-1} we find $\{x\} + D + K_0 = D + K_0$, and because $D + K_0 = D$ we find $\{x\} + D = D$ and so $x \in K$. Therefore, the elements in B which leave F invariant by translations are from $f(K)$. They are then in a finite number, because the index $[K : K_0]$ is finite. So F is an ample divisor on B . Finally the equality $f^{-1}(F) = D$ also holds at the schemes level, because f is smooth from its construction. $\square$

The result we are interested in is the following theorem of Ran.

Theorem 13. *Let A be an Abelian variety of dimension n , $D = \sum_{i=1}^r m_i D_i$ an ample effective divisor, and $Z = \sum_{j=1}^s n_j E_j$ a generating 1-cycle such that $D \cdot Z = n$. Then $m_i = n_j = 1$ for all i, j , $r = s$ and there are r smooth curves $T_1, \dots, T_r$ with Jacobians $B_1, \dots, B_r$ and an isomorphism of Abelian varieties $h : B_1 \times \dots \times B_r \rightarrow A$ such that, for every i , E_i is a translation of $h(\{0\} \times \dots \times \{0\} \times T_i \times \{0\} \times \dots \times \{0\})$ (T_i on the i th place) and D_i is a translation of $h(B_1 \times \dots \times B_{i-1} \times W_i \times B_{i+1} \times \dots \times B_r)$, where W_i is the canonical divisor $W_1(T_i)$ on B_i .*

Proof. The fact that $m_i = n_j = 1$ for all i, j is Corollary 10. For the other points, the proof follows closely the one from [3] with some modifications of the arguments. We began with three preliminary steps.

Step 1. We prove that for every j there is a unique i such that $D_i \cdot E_j \neq 0$. We translate the curves E_j such that they contain the origin and denote the result with the same letter. Let $A_j = \langle E_j \rangle$ and $d_j = \dim A_j$, so that E_j is a generating curve for A_j . Denote by e_j the inclusion $A_j \hookrightarrow A$ and by the same letter D a translation of the divisor D which has a proper intersection with every A_j . Therefore, $e_j^*(D) := D'_j$ is defined as a cycle and is an ample divisor on A_j . The projection formula gives

$$D \cdot E_j = D'_j \cdot E_j \quad (2)$$

and so

$$n = D \cdot E = \sum_{j=1}^s D \cdot E_j = \sum_{j=1}^s D'_j \cdot E_j \geq \sum_{j=1}^s d_j \geq n. \quad (3)$$

The first inequality comes from the fact that on A_j one has $D'_j \cdot E_j \geq d_j$ according to Proposition 9(b), and the last one is due to the fact that Z is a generating 1-cycle. So $D'_j \cdot E_j = d_j$, and E_j being a generating curve for A_j , from Theorem 11 one finds that E_j is smooth, A_j is its Jacobian, and D'_j is a translation of the canonical divisor on A_j ; so D'_j is prime as any divisor numeric equivalent with it (it is a principal polarization).

Let us fix j , and consider for any i a translation of D_i which cuts A_j properly. Every such translation, also denoted by D_i , restricted to A_j either is an effective divisor or has an empty intersection with A_j , in which case $D_i \cdot E_j = 0$. But the sum of these restrictions is numerically equivalent with D'_j and so there cannot be two indexes i with $D_i \cdot E_j \neq 0$, because in such a case D'_j which is prime would be the sum of two effective divisors. The existence of an i with $D_i \cdot E_j \neq 0$ comes from the fact that D is ample.

Step 2. This part consists in the proof of the following fact: for an n -dimensional Abelian variety A , a prime ample divisor D , and a generating 1-cycle $Z = T_1 + \dots + T_s$ with $D \cdot Z = n$ one has $s = 1$ (i.e., Z is irreducible and reduced).

The proof is due to Ran compare Lemma III.2 from [3]. Denote by $A_1 = \langle T_1 \rangle$. From the first step, we know that A_1

is in fact the Jacobian of the smooth curve T_1 ; in particular it is principally polarized and isomorphic with its dual. It will suffice to prove that $A_1 = A$, because in this case T_1 will be a generating curve, and the fact that D is ample together with the inequalities $n \leq D \cdot T_1 \leq D \cdot Z = n$ implies that $s = 1$ as desired.

For the time being, we replace D with a translation whose restriction $D|_{A_1} := D_1$ is well defined as divisor on A_1 . As in the proof of Step 1, D_1 is numerically equivalent with $W_1(T_1)$. Let $s : A \times A_1 \rightarrow A$ the morphism given by $s(r, y) = r + y$ and p, p_1 be the projections. Consider on $A \times A_1$ the line bundle $\mathcal{M} = s^*(\mathcal{O}_A(D)) \otimes p_1^*(\mathcal{O}_{A_1}(-D_1))$ and on $A_1 \times A_1$ the line bundle $\mathcal{P} = (s|_{A_1 \times A_1})^*(\mathcal{O}_{A_1}(D_1)) \otimes q_1^*(\mathcal{O}_{A_1}(-D_1)) \otimes q_2^*(\mathcal{O}_{A_1}(-D_1))$, where q_1, q_2 are the projections on the factors of $A_1 \times A_1$. Using the fact that A_1 is a Jacobian (and therefore it is its own Picard variety with the Poincare bundle equal to $\mathcal{P}$), we deduce the existence of a morphism $f : A \rightarrow A_1$ and of a line bundle $\mathcal{N}$ on A such that

$$\mathcal{M} \otimes p^*(\mathcal{N}) \simeq (f \times \text{Id}_{A_1})^*[\mathcal{P}]. \quad (4)$$

Restricting (4) on the fiber $\{x\} \times A_1$, for $x \in A$, one finds an isomorphism

$$e^*(t_x^*(\mathcal{O}_A(D))) \simeq t_{f(x)}^*(\mathcal{O}_{A_1}(D_1)), \quad (5)$$

where e is the embedding $A_1 \hookrightarrow A$. Because D_1 is a principal polarisation, the point $f(x)$ is uniquely defined by the above property, which can be written in divisorial terms as $(\{-x\} + D)|_{A_1} = \{-f(x)\} + D_1$, at least for x general such that the divisor $(\{-x\} + D)|_{A_1}$ is well defined. From this one deduces that points in A_1 are fixed by f and so f is surjective with $K \cap A_1 = \{0_A\}$, where K is the kernel of f .

Because K cuts A_1 only in 0_A , the sum morphism $K \times A_1 \rightarrow A$ is injective and so we will have $\dim(K + D_1) = n - \dim A_1 + \dim A_1 - 1 = n - 1$. Now, for a general $x \in A$, we have $\{-f(x)\} + D_1 = (\{-x\} + D)|_{A_1} \subset \{-x\} + D$. So, $x - f(x) \in \text{Tran}(D_1, D) := \{y \in A \mid \{y\} + D_1 \subset D\}$. But $\text{Tran}(D_1, D)$ is closed and so for any $x \in A$ we have $\{x - f(x)\} + D_1 \subset D$.

Then for $x \in K$, $\{x\} + D_1 \subset D$ and therefore $K + D_1 \subset D$. For K_0 the connected component of the origin in K , we have $K_0 + D_1 \subset D$. But $K_0 + D_1$ is a divisor and D is prime, so the previous inclusion is an equality. Now,

$$K_0 + D = K_0 + (K_0 + D_1) = K_0 + D_1 = D. \quad (6)$$

But ample D implies that K_0 is finite and prime D implies that $K_0 = \{0_A\}$ which is equivalent with $A = A_1$.

Step 3. Within this step we prove that for any i there is a unique j such that $D_i \cdot E_j \neq 0$. For this, we consider for all i , an Abelian variety B_i , an ample divisor F_i on B_i , and a surjective morphism $f_i : A \rightarrow B_i$ such that $f_i^{-1}(F_i) = D_i$. Their existence follows from Lemma 12.

We have

$$n = D \cdot Z = \sum_{i=1}^r f_i^{-1}(F_i) \cdot Z = \sum_{i=1}^r F_i \cdot (f_i)_* Z \geq \sum_{i=1}^r l_i, \quad (7)$$

where $l_i = \dim B_i$ and the last inequality is from Proposition 9(b). We examine the last sum using the effective

construction of B_i 's from Lemma 12. There, B_i is of the form A/K_i where K_i is an Abelian subvariety of A . As consequence, $l_i = \text{codim } K_i$ and so

$$\sum_{i=1}^r l_i = \sum_{i=1}^r \text{codim } K_i \geq \text{codim } (\cap K_i) = n \quad (8)$$

(by definition of K_i and the ampleness of D , the intersection $\cap K_i$ is finite).

It results in that

$$n \geq \sum_{i=1}^r F_i \cdot (f_i)_* Z \geq \sum_{i=1}^r l_i \geq n, \quad (9)$$

and so $F_i \cdot (f_i)_* Z = l_i$. But F_i is a prime divisor and from Step 2 there is a unique j_i with $(f_i)_* E_{j_i}$ a curve. All the other curves from the support of Z will be therefore contracted. We now fix i and compute $D_i \cdot E_j = f_i^{-1}(F_i) \cdot E_j = F_i \cdot f_{i*} E_j$. This last number is 0 if $j \neq j_i$ and nonzero for $j = j_i$ because F_i is ample. This conclude the third step.

From the first and third steps we find that $i \rightarrow j_i$ is a bijection and so $r = s$. Also one can reorder the curves E_j (such that E_{j_i} will be numbered by E_i) and so we can suppose that for all i, j we have $D_i \cdot E_j \neq 0 \Leftrightarrow i = j$. To conclude the proof, we consider all the requirements supposed above.

In the first place we review B_i 's. Let T_i be the cycle $(f_i)_* Z$. From the third step, T_i is in fact a curve, namely, $f_i(E_i)$. Also we have seen that $F_i \cdot T_i = l_i = \dim B_i$ and therefore Theorem 11 implies that B_i is the Jacobian of T_i . To see this, we need only to prove that T_i is a generating curve of B_i and this is implied by the fact that, as we have seen, f_i contracts all the curves E_j for $j \neq i$ and as far as these contain 0_A , the contraction will be to 0_{B_i} . So $f_i(A) = f_i(A_i) = B_i$ and because E_i generates A_i , T_i generates B_i . So, by Theorem 11, F_i is a translation of the canonical divisor on B_i .

Let us recall that in the first step we supposed (using appropriate translations) that all D_i 's cut properly the subvarieties A_j 's, which means that either $e_j^*(D_i)$ is an effective divisor on A_j or $D_i \cap A_j$ is empty, in which case $e_j^*(D_i) = 0$. The former case can happen only for $j = i$, because in this situation $e_j^*(D_i) \cdot E_j \neq 0$ (more precisely, the projection formula gives $e_j^*(D_i) \cdot E_j = D_i \cdot e_{j*} E_j = D_i \cdot E_j$). So $e_j^*(D_i) \neq 0 \Leftrightarrow j = i$ and we have

$$\begin{aligned} D'_j &= e_j^*(D) = e_j^*(D_j) = e_j^* f_j^*(F_j) \\ &= (f_j \circ e_j)^*(F_j). \end{aligned} \quad (10)$$

Let us consider the morphism $f_j \circ e_j : A_j \rightarrow B_j$. It sends the generating curve E_j of A_j on the generating curve T_j of B_j , and therefore it is surjective; so $d_j \geq l_j$. But, from the first and third steps, $n = \sum_{j=1}^r d_j = \sum_{j=1}^r l_j$; this implies that $f_j \circ e_j$ has a finite nonzero degree. On the other hand $f_j \circ e_j$ pull back the principal polarization F_j from B_j to the principal polarization D'_j on A_j . So its degree is 1 and it is an isomorphism with inverse denoted by g_j .

Let $h : B_1 \times \cdots \times B_r \rightarrow A$ be defined by $h(b_1, \dots, b_r) = \sum_{i=1}^r g_i(b_i)$ and $g : A \rightarrow B_1 \times \cdots \times B_r$ be defined by

$g(a) = (f_1(a), \dots, f_r(a))$. Then $h \circ g$ is the identity, being the identity on every A_i . Also, $g \circ h$ is the identity, being the identity on every $\{0_{B_1}\} \times \cdots \times B_i \times \cdots \times \{0_{B_r}\}$.

So h is an isomorphism, B_i is the Jacobian of T_i , and the last part of the theorem concerning the form of the divisors D_i and curves T_i is obvious due to the fact that the transformations of D_i and T_i were translations. $\square$

Finally, we formulate the following corollary which is the result of Hoyt from [2].

Corollary 14. *Let A be an Abelian variety, D an ample divisor with $D^n = n!$, and Z a 1-cycle such that D^{n-1} is numerically equivalent with $(n-1)!Z$. Then the conclusion of Theorem 13 holds true.*

Proof. We have $D^n = (n-1)!D \cdot Z$, so $D \cdot Z = n$. On the other hand, from Proposition 6(c), $(n-1)!Z$ is a generating 1-cycle and therefore Z is a generating 1-cycle. Now everything is a consequence of Theorem 13. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] I. Matsusaka, "On a characterization of a Jacobian variety," *Memoirs of the College of Science, University of Kyoto, Series A: Mathematics*, vol. 32, pp. 1–19, 1959.
- [2] W. L. Hoyt, "On products and algebraic families of Jacobian Varieties," *The Annals of Mathematics*, vol. 77, no. 3, pp. 415–423, 1963.
- [3] Z. Ran, "On subvarieties of abelian varieties," *Inventiones Mathematicae*, vol. 62, no. 3, pp. 459–479, 1980.
- [4] N. Buruiana, "Algebraic cycles on symmetric products and Abelian varieties," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 45, no. 4, pp. 593–608, 2000.
- [5] A. Collino, "A new proof of the Ran-Matsusaka criterion for Jacobians," *Proceedings of the American Mathematical Society*, vol. 92, no. 3, pp. 329–331, 1984.
- [6] S. Lang, *Abelian Varieties*, *Interscience Tracts in Pure and Applied Mathematics*, vol. 7, 1959.
- [7] D. Mumford, *Abelian Varieties*, Oxford University Press, 1970.
- [8] I. Matsusaka, "On a generating curve of an Abelian variety," *Natural Science Report of the Ochanomizu University*, vol. 3, pp. 1–4, 1952.