Research Article

Norm Estimates for Solutions of Polynomial Operator Equations

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We consider the equations $\sum_{k=0}^m c_{m-k} A^k X B^k = C$ and $\sum_{k=0}^m c_{m-k} A^k X B^{m-k} = C$, where $c_k \in \mathbb{C}$ ($k = 1, \dots, m$), $c_0 = 1$, A, B, C are given linear bounded operators in a Banach space and X is to be found. Representations of solutions are derived. In the cases of Euclidean and Hilbert spaces, norm estimates for the solutions are suggested.

1. Introduction and Statement of the Main Result

The objects of this paper are the equations

$$\sum_{k=0}^m c_{m-k} A^k X B^k = C, \quad (1)$$

$$\sum_{k=0}^m c_{m-k} A^k X B^{m-k} = C, \quad (2)$$

where $c_0 = 1$, $c_k \in \mathbb{C}$ ($k = 1, \dots, m$), A, B, C are given linear bounded operators in a Banach space $\mathcal{E}$ with a norm $\|\cdot\|$ and X is to be found.

There is a long tradition of finding different expressions for the solution of an operator equation in the form of operator integrals and series, some prominent examples of which occur in the works of E. Heinz, M. G. Krein, M. Rosenblum and R. Bhatia, C. Davis, and A. McIntosh. A comprehensive summary of these is contained in [1] and [2]. This tradition was continued in [3], which deals with the equation

$$\sum_{k=0}^m A^{m-k} X B^k = C. \quad (3)$$

Clearly, (3) is a particular case of (2).

The special case

$$AX - XB = C \quad (4)$$

of (2) is the much studied Sylvester equation, of great interest in operator theory, numerical analysis, and engineering; compare [4–6] and references therein. Let A^* be the operator adjoint to A . The equation

$$AX + XA^* = C \quad (5)$$

is the famous Lyapunov equation playing an essential role in the stability theory of differential equations. The equation

$$X - AXB = C \quad (6)$$

is an example of (1); in particular, the discrete Lyapunov equation

$$X - AXA^* = C \quad (7)$$

is an important tool in the theory of difference equations.

About other operator equations see [7–9] and references therein. In particular, [9] deals with necessary and sufficient conditions for the existence of solution to the systems of the general solution to a system of adjointable operator equations over Hilbert C^* modules. In [8] the nonlinear operator equations of the form $ABA = A^2$ and $BAB = B^2$ are considered. It should be noted that in the finite dimensional case the numerical methods are well developed. Besides, the traditional methods convert matrix equations into their equivalent forms by using the Kronecker product, which involve the inversion of the associated large matrix and result in increasing computation and excessive computer memory.

The recently suggested gradient based iterative methods are more powerful; compare [10–13] and references therein.

The aim of the present paper is to derive representations of solutions to (1) and (2) and to estimate the norms of these solutions. Such estimates are important, in particular, for the investigations of linear and perturbations of nonlinear differential and difference equations. Norm estimates for solutions of the Sylvester equation whose coefficients are normal operators can be found in [2]. In the finite dimensional case solution estimates for (4) and (6) have been established in [14].

In this paper, in particular, we considerably generalize the main results from [14].

Denote by $\sigma(A)$ and $r_s(A)$ the spectrum and spectral radius of A , respectively.

Theorem 1. *Let x_k ($k = 1, \dots, m$) be the roots taken with the multiplicities of the polynomial*

$$p(x) := \sum_{k=0}^m c_k x^{m-k} \quad (8)$$

and let

$$r_s(A) r_s(B) < \min_k |x_k|. \quad (9)$$

Then (1) has a unique solution X which can be represented by

$$X = (-1)^m \cdot \sum_{j_1, j_2, \dots, j_m=0}^{\infty} \frac{1}{x_1^{j_1+1} x_2^{j_2+1} \dots x_m^{j_m+1}} A^{j_1+j_2+\dots+j_m} C B^{j_1+j_2+\dots+j_m}, \quad (10)$$

and the series strongly converges.

The proof of this theorem is presented in the next section.

2. Proof of Theorem 1

Following [1, Section I.3], introduce the operators A_l and B_r by $A_l X := AX$ and $B_r X := XB$, respectively. Besides, A_l and B_r commute.

Let $f(z, w)$ be a scalar function regular on $\sigma(A) \times \sigma(B)$. Define the operator valued function

$$\begin{aligned} \Phi(f, A, B) \\ := -\frac{1}{4\pi^2} \int_{L_B} \int_{L_A} f(z, w) R_z(A_l) R_w(B_r) dw dz, \end{aligned} \quad (11)$$

where L_A, L_B are closed Jordan contours surrounding $\sigma(A)$ and $\sigma(B)$, respectively. So for any bounded linear operator C

$$\begin{aligned} \Phi(f, A, B) C \\ = -\frac{1}{4\pi^2} \int_{L_B} \int_{L_A} f(z, w) R_z(A) C R_w(B) dw dz. \end{aligned} \quad (12)$$

Besides,

$$\Phi(f_1, A, B) \Phi(f_2, A, B) = \Phi(f_1 f_2, A, B), \quad (13)$$

for functions f_1, f_2 regular on $\sigma(A) \times \sigma(B)$. Take $f(z, w) = p(zw)$. If

$$p(zw) \neq 0 \quad (z \in \sigma(A), w \in \sigma(B)), \quad (14)$$

then due to formula (3.5) from [1, Section I.3] (1) can be written as

$$\Phi(p(zw), A, B) X = C. \quad (15)$$

Since

$$p(x) = \prod_{k=1}^m (x - x_k), \quad (16)$$

making use of (13), we can write

$$\begin{aligned} \Phi(zw - x_1, A, B) \\ \cdot \Phi(zw - x_2, A, B) \dots \Phi(zw - x_m, A, B) X = C. \end{aligned} \quad (17)$$

But $\Phi(zw - x_k, A, B) = A_l B_r - x_k I$. Thus,

$$\prod_{k=1}^m (A_l B_r - x_k I) X = C, \quad (18)$$

and therefore we get the following result.

Lemma 2. *Under condition (14) equation (1) has a unique solution defined by*

$$X = \prod_{k=1}^m (A_l B_r - x_k I)^{-1} C. \quad (19)$$

Furthermore, for $a \in \mathbb{C}$, satisfying

$$|a| > r_s(A) r_s(B), \quad (20)$$

consider the operator

$$Z = -\sum_{k=0}^{\infty} \frac{1}{a^{k+1}} A_l^k B_r^k C = -\sum_{k=0}^{\infty} \frac{1}{a^{k+1}} A^k C B^k. \quad (21)$$

The series converges and

$$\begin{aligned} A_l B_r Z &= AZB = -\sum_{k=0}^{\infty} \frac{1}{a^{k+1}} A^{k+1} C B^{k+1} \\ &= -\sum_{k=0}^{\infty} \frac{1}{a^k} A^k C B^k + C = aZ + C. \end{aligned} \quad (22)$$

We thus get the following result.

Lemma 3. *Let condition (20) hold. Then operator Z defined by (21) is the unique solution to the equation $AZB - aZ = C$ and $Z = (A_l B_r - aI)^{-1} C$.*

Proof of Theorem 1. Put

$$\begin{aligned} Y_1 &= (A_l B_r - x_1 I)^{-1} C, \\ Y_2 &= (A_l B_r - x_2 I)^{-1} Y_1, \\ &\vdots \\ Y_k &= (A_l B_r - x_k I)^{-1} Y_{k-1}. \end{aligned} \quad (23)$$

Lemma 2 implies $X = Y_m$. Due to the previous lemma Y_k are solutions to the equations

$$\begin{aligned} A Y_k B - x_k Y_k &= Y_{k-1} \quad (k = 2, \dots, m), \\ A Y_1 B - x_1 Y_1 &= C, \\ Y_j &= - \sum_{k=0}^{\infty} \frac{1}{x_1^{k+1}} A^k Y_{j-1} B^k, \end{aligned} \quad (24)$$

provided (9) holds. So

$$\begin{aligned} Y_1 &= - \sum_{k=0}^{\infty} \frac{1}{x_1^{k+1}} A^k C B^k, \\ Y_2 &= - \sum_{j=0}^{\infty} \frac{1}{x_2^{j+1}} A^j Y_1 B^j = \sum_{j=0}^{\infty} \frac{1}{x_2^{j+1}} A^j \sum_{k=0}^{\infty} \frac{1}{x_1^{k+1}} A^k C B^k B^j \\ &= \sum_{j,k=0}^{\infty} \frac{1}{x_1^{k+1} x_2^{j+1}} A^{k+j} C B^{k+j}. \end{aligned} \quad (25)$$

Continuing this process for $j = 3, \dots, m$, according to Lemma 2, we prove the theorem. $\square$

3. Representations of Solutions to (2)

Assume that B is invertible; then from (2) we have

$$\sum_{k=0}^m c_{m-k} A^k X B^{-k} = \widehat{C} := C B^{-m}. \quad (26)$$

Let $r_l(B)$ denote the lower spectral radius of B : $r_l(B) = \min_{\lambda \in \sigma(B)} |\lambda|$. Taking into account the fact that $r_l(B) = 1/r_s(B^{-1})$ and applying now Theorem 1 to (26) with B^{-1} instead of B , we get the following result.

Corollary 4. *Let*

$$r_s(A) < r_l(B) \min_k |x_k|. \quad (27)$$

Then (2) has a unique solution X which can be represented by

$$X = (-1)^m \sum_{j_1, j_2, \dots, j_m=0}^{\infty} \frac{1}{x_1^{j_1+1} x_2^{j_2+1} \dots x_m^{j_m+1}} A^{j_1+j_2+\dots+j_m} C B^{-m-j_1-j_2-\dots-j_m}, \quad (28)$$

and the series strongly converges.

Condition (27) does not allow us to consider the Lyapunov equation, since $r_s(A) = r_s(A^*)$. Because of this we are going to derive the representation of solutions to (2) under other conditions. To this end put

$$\begin{aligned} \alpha(A) &= \sup \operatorname{Re} \sigma(A), \\ \beta(A) &= \inf \operatorname{Re} \sigma(A). \end{aligned} \quad (29)$$

According to (19) for a solution to (26) we have

$$X = \prod_{k=1}^m (A_l B_r^{-1} - x_k I)^{-1} \widehat{C}, \quad (30)$$

provided

$$p\left(\frac{z}{w}\right) \neq 0 \quad (z \in \sigma(A), w \in \sigma(B)). \quad (31)$$

Here B_r^{-1} is defined as $B_r^{-1} C = C B^{-1}$. Thus

$$\begin{aligned} X &= \prod_{k=1}^m (A_l B_r^{-1} - x_k I)^{-1} B_r^{-m} B_r^m \widehat{C} \\ &= \prod_{k=1}^m (A_l - x_k B_r)^{-1} C. \end{aligned} \quad (32)$$

Now the invertibility condition for B can be removed. We thus have proved the following result.

Lemma 5. *Let the condition*

$$z - x_k w \neq 0 \quad (k = 1, \dots, m; z \in \sigma(A), w \in \sigma(B)) \quad (33)$$

hold. Then (2) has a unique solution X which can be represented by (32).

Lemma 6. *Let the condition*

$$\beta(A) > \alpha(B) \quad (34)$$

hold. Then (4) has a unique solution X_S , which can be represented as

$$X_S = \int_0^{\infty} e^{-At} C e^{Bt} dt. \quad (35)$$

Proof. For the brevity put $b = \alpha(B)$ and $c = (\beta(A) - \alpha(B))/2$. Then $\beta(A) = b + 2c$ and $\alpha(A - I(b + c)) = -c$. Moreover, $\alpha(-(A - I(b + c))) = \alpha(-A) + b + c = -\beta(A) + b + c = -c$. Equation (4) is equivalent to the following one:

$$(A - (\alpha(B) + c) I) X - X (B - (\alpha(B) + c) I) = C. \quad (36)$$

Due to Theorem 9.2 from [2] a solution of (36) is defined by the equality

$$\begin{aligned} X_S &= \int_0^{\infty} e^{-(A - I(\alpha(B) + c))t} C e^{(B - (\alpha(B) + c)I)t} dt \\ &= \int_0^{\infty} e^{-At} C e^{Bt} dt, \end{aligned} \quad (37)$$

as claimed. $\square$

Corollary 7. Let the condition

$$\beta(A) > \alpha(aB) \quad (a \in \mathbb{C}) \quad (38)$$

hold. Then the equation

$$AX - aXB = C \quad (39)$$

has a unique solution X_a , which can be represented as

$$X_a = \int_0^\infty e^{-At} C e^{aBt} dt. \quad (40)$$

A solution of (39) is also given by $X_a = (A_I - aB_r)^{-1} C$. So under condition (38) we have

$$(A_I - aB_r)^{-1} C = \int_0^\infty e^{-At} C e^{aBt} dt. \quad (41)$$

Now assume that

$$\beta(A) > \max_k \alpha(x_k B). \quad (42)$$

Then by (41)

$$(A_I - x_k B_r)^{-1} = \int_0^\infty e^{-At} C e^{x_k Bt} dt. \quad (43)$$

Let $x_k = a_k + ib_k$. Then $\alpha(x_k B) = \sup_{s \in \sigma(B)} (a_k \operatorname{Re} s - b_k \operatorname{Im} s)$. So condition (42) means that $\beta(A) > \sup_{s \in \sigma(B)} (a_k \operatorname{Re} s - b_k \operatorname{Im} s)$.

Put

$$\begin{aligned} W_1 &= (A_I - x_1 B_r)^{-1} C, \\ W_2 &= (A_I - x_2 B_r)^{-1} Y_1, \\ &\vdots \\ W_k &= (A_I - x_k B_r)^{-1} W_{k-1}. \end{aligned} \quad (44)$$

Then $X = W_m$. Due to the previous lemma Y_k are solutions to the equations

$$\begin{aligned} AW_k - x_k B W_{k-1} &= W_{k-1} \quad (k = 2, \dots, m), \\ AY_1 B - x_1 Y_1 &= C, \\ W_j &= - \int_0^\infty e^{-At} W_{j-1} e^{x_j Bt} dt. \end{aligned} \quad (45)$$

provided (42) holds. So

$$\begin{aligned} W_1 &= - \int_0^\infty e^{-At_1} C e^{x_1 Bt_1} dt_1, \\ W_2 &= - \int_0^\infty e^{-At_2} W_1 e^{x_2 Bt_2} dt_2 \\ &= \int_0^\infty \int_0^\infty e^{-(t_1+t_2)A} C e^{(x_2 t_2 + x_1 t_1)B} dt_1 dt_2. \end{aligned} \quad (46)$$

Continuing this process for $j = 3, \dots, m$, according to Lemma 5, we obtain

$$\begin{aligned} X &= \int_0^\infty \dots \int_0^\infty e^{-(t_1+\dots+t_m)A} C e^{(x_1 t_1 + \dots + x_m t_m)B} dt_1 \dots dt_m. \end{aligned} \quad (47)$$

We arrive at the following result.

Theorem 8. Let x_k ($k = 1, \dots, m$) be the roots of $p(x)$ and let condition (42) hold. Then (2) has a unique solution X , which can be represented by (47).

4. Solution Estimates in the Finite Dimensional Case

4.1. Equation (1). In this section $\mathcal{E} = \mathbb{C}^n$ is a Euclidean space. Let $\lambda_k(A)$ ($k = 1, \dots, n$) be the eigenvalues of a matrix A counted with their multiplicities. The following quantity (the departure from normality of A) plays a key role hereafter:

$$g(A) = \left[N_2(A)^2 - \sum_{k=1}^n |\lambda_k(A)|^2 \right]^{1/2}, \quad (48)$$

where $N_2(A) = (\operatorname{Trace} AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt norm) of A .

The following relations are checked in [15, Section 2.1]:

$$\begin{aligned} g^2(A) &\leq N_2(A)^2(A) - |\operatorname{Trace} A^2|, \\ g^2(A) &\leq \frac{N_2^2(A_I)}{2}, \end{aligned} \quad (49)$$

where $A_I = (A - A^*)/2i$. If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \leq g(A_1) + g(A_2)$. By the inequality between geometric and arithmetic mean values we have

$$\left(\frac{1}{n} \sum_{k=1}^n |\lambda_k(A)|^2 \right)^n \geq \left(\prod_{k=1}^n |\lambda_k(A)| \right)^2. \quad (50)$$

So $g^2(A) \leq N_2(A)^2 - n(\det A)^{2/n}$. Obviously $g(aA) = |a|g(A)$ ($a \in \mathbb{C}$).

By Corollary 2.7.2 from [15], one has

$$\|A^w\| \leq \sum_{k=0}^{n-1} \frac{w! g^k(A) r_s^{w-k}(A)}{(w-k)! (k!)^{3/2}} \quad (w = 1, 2, \dots). \quad (51)$$

Note that $1/(w-k)! = 0$ ($s < k$). Furthermore, due to Example 1.10.3 from [15] we have

$$\|e^{At}\| \leq e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}} \quad (t \geq 0). \quad (52)$$

From (21) and Lemma 3 we have

$$\begin{aligned} (A_I B_r - aI)^{-1} C &= - \sum_{w=0}^\infty \frac{1}{a^{w+1}} A^w C B^w \\ &\quad (s = 1, \dots, m), \end{aligned} \quad (53)$$

provided condition (20) holds. Hence, and from (51), it directly follows

$$\|(A_I B_r - x_s I)^{-1}\| \leq J(A, B, x_s), \quad (54)$$

where

$$J(A, B, a) = \sum_{j,k=0}^{n-1} \frac{g^k(A) g^j(B)}{(k!j!)^{3/2}} \cdot \sum_{w=0}^{\infty} \frac{1}{|a|^{w+1}} \frac{(w!)^2 r_s^{w-k}(A) r_s^{w-j}(B)}{(w-k)!(w-j)!}. \quad (55)$$

Now Lemma 2 and (54) imply the following result.

Corollary 9. *Let condition (9) hold. Then a unique solution X to (1) in $\mathbb{C}^n$ satisfies the inequality*

$$\|X\| \leq \|C\| J(A, B, x_1) J(A, B, x_2) \cdots J(A, B, x_m). \quad (56)$$

If A is normal, then $g(A) = 0$ and

$$J(A, B, a) = \sum_{j=0}^{n-1} \frac{g^j(B)}{(j!)^{3/2}} \sum_{w=0}^{\infty} \frac{1}{|a|^{w+1}} \frac{w! r_s^w(A) r_s^{w-j}(B)}{(w-j)!}. \quad (57)$$

But

$$\begin{aligned} & \sum_{w=0}^{\infty} \frac{w! r_s^w(A) r_s^{w-j}(B)}{|a|^{w+1} (w-j)!} \\ &= \frac{r_s^j(A)}{|a|^{j+1}} \sum_{w=0}^{\infty} \frac{w! (r_s(A) r_s(B))^{w-j}}{|a|^{w-j} (w-j)!} \\ &= \frac{r_s^j(A)}{|a|^{j+1}} \frac{d^j}{dx^j} \sum_{w=0}^{\infty} x^w = j! \frac{r_s^j(A)}{|a|^{j+1}} (1-x)^{-j-1} \\ & \quad \left(x = \frac{1}{|a|} r_s(A) r_s(B) \right). \end{aligned} \quad (58)$$

Thus, we have

$$J(A, B, a) \leq \sum_{j=0}^{n-1} \frac{g^j(B) r_s^j(A)}{(j!)^{1/2} (|a| - r_s(A) r_s(B))^{j+1}}, \quad (59)$$

provided A is normal. If both A and B are normal, then

$$J(A, B, a) \leq \frac{1}{|a| - r_s(A) r_s(B)}. \quad (60)$$

Furthermore, obviously,

$$\begin{aligned} \sum_{w=0}^{\infty} \frac{(w!)^2 b^{2w}}{(w-k)!(w-j)!} &\leq \sum_{t=0}^{\infty} \frac{t! b^t}{(t-k)!} \sum_{w=0}^{\infty} \frac{w! b^w}{(w-j)!} \\ &= b^{k+j} \sum_{t=0}^{\infty} \frac{t! b^{t-k}}{(t-k)!} \sum_{w=0}^{\infty} \frac{w! b^{w-j}}{(w-j)!} \\ &= \frac{j! k! b^{k+j}}{(1-b)^{k+1} (1-b)^{j+1}} \\ & \quad (0 < b < 1). \end{aligned} \quad (61)$$

Taking

$$b = b(A, B, a) = \sqrt{\frac{r_s(A) r_s(B)}{|a|}}, \quad (62)$$

we get

$$J(A, B, a) \leq \hat{J}(A, B, a), \quad (63)$$

where

$$\hat{J}(A, B, a) = \frac{1}{|a|} \cdot \sum_{j,k=0}^{n-1} \frac{g^k(A) g^j(B) b^{k+j}(A, B, a)}{(k!j!)^{1/2} (1-b(A, B, a))^{k+1} (1-b(A, B, a))^{j+1}}. \quad (64)$$

Now Corollary 9 implies the following result.

Corollary 10. *Let condition (9) hold. Then a unique solution X of (1) in $\mathbb{C}^n$ satisfies the inequality*

$$\|X\| \leq \|C\| \hat{J}(A, B, x_1) \hat{J}(A, B, x_2) \cdots \hat{J}(A, B, x_m). \quad (65)$$

Theorem 1 and simple calculations imply the following result.

Corollary 11. *Let $r_s(A) < 1$. Then (7) has a unique solution X_L , which satisfies the inequalities*

$$\|X_L\| \leq \|C\| \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{g^k(A) m! r_s^{m-k}(A)}{(k!)^{3/2} (m-k)!} \right)^2. \quad (66)$$

From Corollary 11 and (63) it follows that

$$\|X_L\| \leq \|C\| \left(\sum_{k=0}^{n-1} \frac{g^k(A)}{(k!)^{1/2} (1-r_s(A))^{k+1}} \right)^2. \quad (67)$$

4.2. Equation (2). Due to (41) and (52) under (38) we have

$$\begin{aligned} \|(A_l - aB_r)^{-1}\| &\leq \int_0^{\infty} \|e^{-As}\| \|e^{aBs}\| ds \\ &\leq \int_0^{\infty} e^{-sc} \sum_{j,k=0}^{n-1} \frac{s^{k+j} g^k(A) g^j(aB)}{(k!j!)^{3/2}} ds, \end{aligned} \quad (68)$$

where $c = \beta(A) - \alpha(aB)$. But

$$\int_0^{\infty} s^{k+j} e^{-cs} ds = \frac{(k+j)!}{c^{k+j+1}}. \quad (69)$$

Thus

$$\|(A_l - aB_r)^{-1}\| \leq \gamma(a, A, B), \quad (70)$$

where

$$\gamma(a, A, B) := \sum_{j,k=0}^{n-1} \frac{(k+j)! g^k(A) g^j(aB)}{(\beta(A) - \alpha(aB))^{k+j+1} (k!j!)^{3/2}}, \quad (71)$$

provided $\beta(A) > \alpha(aB)$. Due to Lemma 5 we arrive at the following result.

Corollary 12. *Let condition (42) hold. Then a unique solution X to (2) in $\mathbb{C}^n$ satisfies the inequality*

$$\|X\| \leq \|C\| \gamma(A, B, x_1) \gamma(A, B, x_2) \cdots \gamma(A, B, x_m). \quad (72)$$

If A is normal, then $g(A) = 0$ and therefore

$$\gamma(a, A, B) := \sum_{j=0}^{n-1} \frac{|a|^j g^j(B)}{(\beta(A) - \alpha(aB))^{j+1} (j!)^{1/2}}. \quad (73)$$

If both A and B are normal, then

$$\gamma(a, A, B) \leq \frac{1}{\beta(A) - \alpha(aB)}. \quad (74)$$

For the Sylvester equation we have the following result.

Corollary 13. *Let $\alpha(B) < \beta(A)$. Then (4) in $\mathbb{C}^n$ has a unique solution Y_S , which satisfies the inequality*

$$\|Y_S\| \leq \sum_{j,k=0}^{n-1} \frac{(k+j)! g^k(A) g^j(B)}{(\beta(A) - \alpha(B))^{k+j+1} (k!j!)^{3/2}}. \quad (75)$$

Consider the Lyapunov equation (5). Taking in Corollary 13 $-A$ instead of A and A^* instead of B , since $\alpha(A^*) = \alpha(A)$ and $\beta(-A) = -\alpha(A)$, we get the following result.

Corollary 14. *Let $\alpha(A) < 0$. Then (5) in $\mathbb{C}^n$ has a unique solution Y_L , which satisfies the inequality*

$$\|Y_L\| \leq \|C\| \sum_{j,k=0}^{n-1} \frac{(k+j)! g^{k+j}(A)}{(2|\alpha(A)|)^{k+j+1} (k!j!)^{3/2}}. \quad (76)$$

5. Solution Estimates in the Infinite-Dimensional Case

5.1. Equation (1). In this section $\mathcal{E} = H$, a separable Hilbert space. It is assumed that

$$A - A^* \text{ and } B - B^* \text{ are Hilbert-Schmidt operators.} \quad (77)$$

Put

$$g_I(A) = \left(\frac{1}{2} N_2^2(A - A^*) - 2 \sum_{k=1}^{\infty} |\operatorname{Im} \lambda_k(A)|^2 \right)^{1/2}. \quad (78)$$

Recall that $N_2(A) = (\operatorname{Trace} AA^*)^{1/2}$ and $\lambda_k(A)$ are the eigenvalues of a matrix A counted with their multiplicities. If A is a Hilbert-Schmidt operator, then $g_I(A) = g(A)$, where

$$g(A) = \left(N_2^2(A - A^*) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 \right)^{1/2}; \quad (79)$$

compare Lemma 6.5.2 [15]. Due to Example 7.10.2 from [15] we can write

$$\|A^w\| \leq \sum_{k=0}^{\infty} \frac{w! g_I^k(A) r_s^{w-k}(A)}{(w-k)! (k!)^{3/2}} \quad (w = 1, 2, \dots). \quad (80)$$

As it was shown in Section 4

$$(A_I B_r - x_s I)^{-1} C = - \sum_{w=0}^{\infty} \frac{1}{x_s^{w+1}} A^w C B^w \quad (81)$$

$(s = 1, \dots, m),$

provided (9) holds. Hence, and from (80), it directly follows that

$$\|(A_I B_r - x_s I)^{-1}\| \leq J_H(A, B, x_k), \quad (82)$$

where

$$J_H(A, B, a) = \sum_{j,k=0}^{\infty} \frac{g_I^k(A) g_I^j(B)}{(k!j!)^{3/2}} \cdot \sum_{w=0}^{\infty} \frac{1}{|a|^{w+1}} \frac{(w!)^2 r_s^{w-k}(A) r_s^{w-j}(B)}{(w-k)! (w-j)!}. \quad (83)$$

Now Lemma 2 implies the following result.

Corollary 15. *Let conditions (9) and (77) hold. Then the unique solution X to (1) in H satisfies the inequality*

$$\|X\| \leq \|C\| J_H(A, B, x_1) J_H(A, B, x_2) \cdots J_H(A, B, x_m). \quad (84)$$

If A is normal, then $g(A_I) = 0$ and according to (59)

$$J_H(A, B, a) \leq \sum_{j=0}^{\infty} \frac{g_I^j(B) r_s^j(A)}{(j!)^{1/2} (|a| - r_s(A) r_s(B))^{j+1}}. \quad (85)$$

If both A and B are normal, then

$$J_H(A, B, a) \leq \frac{1}{|a| - r_s(A) r_s(B)}. \quad (86)$$

Furthermore, according to (63), we get $J_H(A, B, a) \leq \hat{J}_H(A, B, a)$, where

$$\hat{J}_H(A, B, a) = \frac{1}{|a|} \cdot \sum_{j,k=0}^{\infty} \frac{g_I^k(A) g_I^j(B) b^{k+j}(A, B, a)}{(k!j!)^{1/2} (1 - b(A, B, a))^{k+1} (1 - b(A, B, a))^{j+1}}, \quad (87)$$

where $b(A, B, a)$ is defined by (62). Now Corollary 15 implies the following result.

Corollary 16. *Let conditions (9) and (77) hold. Then a unique solution X of (1) in H satisfies the inequality*

$$\|X\| \leq \|C\| \hat{J}_H(A, B, x_1) \hat{J}_H(A, B, x_2) \cdots \hat{J}_H(A, B, x_m). \quad (88)$$

By virtue of Corollary 15 we can assert that a unique solution X_L of (7) satisfies the inequality

$$\|X_L\| \leq \|C\| \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{g_I^k(A) m! r_s^{m-k}(A)}{(k!)^{\frac{3}{2}} (m-k)!} \right)^2, \quad (89)$$

provided A satisfies conditions (77) and $r_s(A) < 1$. From Corollary 16 it follows that

$$\|X_L\| \leq \|C\| \left(\sum_{k=0}^{\infty} \frac{g_I^k(A)}{(k!)^{1/2} (1 - r_s(A))^{k+1}} \right)^2. \quad (90)$$

5.2. Equation (2). Due to Example 7.10.3 from [15] we have

$$\|e^{aAt}\| \leq e^{\alpha(aA)t} \sum_{k=0}^{\infty} \frac{t^k |a|^k g^k(A_I)}{(k!)^{3/2}} \quad (a \in \mathbb{C}, t \geq 0), \quad (91)$$

provided A satisfies (77). Now let us consider (2), assuming that condition (42) holds. Due to (41) and (91),

$$\begin{aligned} \|(A_I - aB_r)^{-1}\| &\leq \int_0^{\infty} \|e^{-As}\| \|e^{aBs}\| ds \\ &\leq \int_0^{\infty} e^{-sc} \sum_{j,k=0}^{\infty} \frac{s^{k+j} g_I^k(A) g_I^j(aB)}{(k!j!)^{3/2}} ds, \end{aligned} \quad (92)$$

where $c = \beta(A) - \alpha(aB)$. Thus

$$\|(A_I - aB_r)^{-1}\| \leq \gamma_H(A, B, a), \quad (93)$$

where

$$\gamma_H(a, A, B) := \sum_{j,k=0}^{\infty} \frac{(k+j)! g_I^k(A) g_I^j(aB)}{(\beta(A) - \alpha(aB))^{k+j+1} (k!j!)^{3/2}}, \quad (94)$$

provided $\beta(A) > \alpha(aB)$. Due to Lemma 5 we arrive at the following result.

Corollary 17. *Let conditions (42) and (77) hold. Then the unique solution X to (2) in H satisfies the inequality*

$$\begin{aligned} \|X\| &\leq \|C\| \gamma_H(A, B, x_1) \gamma_H(A, B, x_2) \cdots \gamma_H(A, B, x_m). \end{aligned} \quad (95)$$

Let $\beta(A) > \alpha(aB)$. Then, if A is normal, then $g_I(A) = 0$ and therefore

$$\gamma_H(a, A, B) := \sum_{j=0}^{\infty} \frac{|a|^j g_I^j(B)}{(\beta(A) - \alpha(aB))^{j+1} (j!)^{1/2}}. \quad (96)$$

If both A and B are normal, then

$$\gamma_H(a, A, B) = \frac{1}{\beta(A) - \alpha(aB)}. \quad (97)$$

According to Corollary 17 we get the following result.

Corollary 18. *Let A satisfies conditions (77) and $\alpha(A) < 0$. Then the solution Y_L to the Lyapunov equation (5) in H satisfies the inequality*

$$\|Y_L\| \leq \|C\| \sum_{j,k=0}^{\infty} \frac{(k+j)! g_I^{k+j}(A)}{(2|\alpha(A)|)^{k+j+1} (k!j!)^{3/2}}. \quad (98)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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