Research Article
Mathematical Analysis of a Reactive Viscous Flow through a Channel Filled with a Porous Medium

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An investigation has been carried out to study entropy generation in a viscous, incompressible, and reactive fluid flowing steadily through a channel with porous materials. Approximate solutions for both velocity and temperature fields are obtained by using a rapidly convergent Adomi and decomposition method (ADM). These solutions are then used to determine the heat irreversibility and Bejan number of the problem. Variations of other important fluid parameters are conducted, presented graphically, and discussed.

1. Introduction


From applications’ point of view, studies on transport reactive fluids in porous media are very important since they occur in many important areas like water treatment using fixed beds, agriculture, oil recovery, groundwater flows, geothermal engineering, exhaust systems in combustion, material processing, and reservoir engineering. Recently, Rundora and his associates [20–22] documented several investigations on unsteady reactive fluid flow in porous medium and how the flow evolved to the steady state. Bég et al. [23] examined the flow of viscoelastic fluid through a non-Darcian porous medium. Makinde [24] studied the inherent heat irreversibility in reactive fluid through a channel filled with porous material.

In all the studies above, the entropy productions in the flow of viscous incompressible fluid flow through porous medium have not been investigated. Therefore, the work done
in [24] can be further extended to give more interesting results on the thermodynamics and heat transfer properties of the fluid flow. This is because huge amount of money and effort could be wasted if the inherent irreversibility in the fluid flow is not well addressed. Therefore, the specific objective of this article is to examine the rate at which entropy is produced in a viscous fluid flow system through a porous medium. The problem under consideration is nonlinear due to the exponential nature of the rate law in Arrhenius kinetics for combustible fluids. In view of this, exact solution for the temperature field may not be possible to get. To solve the problem, we seek Adomian series solution to avoid linearization of the exponential term. The Adomian decomposition method is a straightforward way of solving all kinds of differential equations arising from many physical scenarios. It has been used extensively in the last few decades as reported in the bibliography by Rach [25], and, more recently, the method has been used in [26–31]. The plan of the article is as follows: the problem is formulated and the mathematical analysis is presented in Section 2. Section 3 of the work gives the Adomian method of solution. Graphical results are presented and interpreted in Section 4 while, in Section 5, concluding remarks are given.

2. Mathematical Analysis

The steady flow of viscous incompressible reactive fluid through parallel plate immersed in a porous medium is studied. The flow is assumed to be full-developed and driven by an applied pressure gradient. The channel wall temperatures are kept constant. Then, the balanced governing equations are [24]

\[ 0 = -\frac{dP}{dx} + \mu \frac{d^2u'}{dy^2} - \frac{\mu u'}{K}, \]

\[ 0 = \frac{d^2T}{dy'^2} + \frac{Q}{k} \frac{C_0A}{k} e^{-E/RT} + \frac{\mu}{k} \left( \frac{du'}{dy'} \right)^2 + \frac{\mu u'^2}{kK}. \]

with the following boundary conditions:

\[ T(0) = T(h) = T_0, \]

\[ u'(0) = 0 = u'(h). \]  

(2)

Under these assumptions, entropy generation equation becomes

\[ E_G = \frac{1}{T_0^2} \left( \frac{dT}{dy'} \right)^2 + \frac{\mu}{T_0} \left( \frac{du'}{dy'} \right)^2 + \frac{\mu u'^2}{T_0 K}. \]

To nondimensionalize (1)–(3), we need the following parameters and variables:

\[ \theta = \frac{E(T - T_0)}{RT_0^2}, \]

\[ Da = \frac{K}{h^2}, \]

\[ N_s = \frac{h^2 E^2 C_0}{k R^2 T_0}, \]

\[ \epsilon = \frac{RT_0}{E} \]

\[ \lambda = \frac{Q E AC_0 h^2 e^{-E/RT_0}}{RT_0 ^2 k}, \]

\[ M = -\frac{h^2 \frac{dP}{dx}}{\mu U d^2 x}, \]

\[ \delta = \frac{U^2 \mu M^2 e^{E/RT_0}}{Q AC_0 h^2}, \]

\[ \beta^2 = \frac{1}{Da}. \]

(4)

to get the dimensionless problems:

\[ \frac{d^2 u}{dy'^2} - \beta^2 u = -1; \quad u(0) = u(1) = 0, \]  

\[ \frac{d^2 \theta}{dy'^2} + \lambda \left\{ e^{\theta/(1+\epsilon \theta)} + \delta \left( \frac{du'}{dy'} \right)^2 + \beta \theta u'^2 \right\} = 0; \]

\[ \theta(0) = \theta(1) = 0 \]  

(5)

(6)

(7)

Setting

\[ N_1 = \left( \frac{d\theta}{dy'} \right)^2, \]

\[ N_2 = \frac{\delta \lambda}{\epsilon} \left( \left( \frac{du'}{dy'} \right)^2 + \beta \theta u'^2 \right), \]  

then, the irreversibility ratio becomes

\[ Be = \frac{N_1}{N_1 + N_2} = \frac{1}{1 + \Phi}, \]

\[ \Phi = \frac{N_2}{N_1}, \]

(9)

From (9), it is evident that

\[ \Phi \begin{cases} 0, & N_2 \gg N_1 \\ 0.5 N_1 = N_2 \quad & N_2 \ll N_1. \end{cases} \]

(10)
3. Adomian Method of Solution

A direct integration of (5)-(6) leads to the integral equations

\[
u(y) = \int_0^y \frac{du(0)}{dY} dY + \int_0^y \int_0^y (\beta^2 u) dY dY
\]

(11)

with

\[
\theta(y) = \int_0^y \frac{d\theta(0)}{dY} dY
\]

(12)

\[
= \int_0^y \int_0^y \left\{ \exp \left( \frac{\theta(y)}{1 + \epsilon \sum_{n=0}^\infty \theta_n(y)} \right) \right\} \left( \frac{du}{dY} \right)^2 + \delta \beta^2 u^2 \right\} dY dY.
\]

Due to the exponential nonlinearity in (12), we now define a series of functions defined by

\[
u(y) = \sum_{n=0}^\infty u_n(y),
\]

(13)

\[
th(y) = \sum_{n=0}^\infty \theta_n(y).
\]

(14)

Substituting (13) into the integral equations (11)-(12), we obtain

\[
\sum_{n=0}^\infty u_n(y) = \int_0^y \frac{du(0)}{dY} dY + \int_0^y \int_0^y (\beta^2 \sum_{n=0}^\infty \theta_n(y)) dY dY,
\]

(14)

\[
\sum_{n=0}^\infty \theta_n(y) = \int_0^y \frac{d\theta(0)}{dY} dY
\]

(12)

\[
= -\int_0^y \int_0^y \lambda \left\{ \exp \left( \frac{\theta(y)}{1 + \epsilon \sum_{n=0}^\infty \theta_n(y)} \right) \right\} \left( \frac{du}{dY} \right)^2 + \delta \beta^2 u^2 \right\} dY dY.
\]

The nonlinear term in (15) represented by

\[
B_n = \exp \left( \frac{\sum_{n=0}^\infty \theta_n(y)}{1 + \epsilon \sum_{n=0}^\infty \theta_n(y)} \right)
\]

(16)

is expanded by Taylor's series to get the following Adomian polynomials:

\[
B_0 = e^{\theta(1+\epsilon)},
\]

\[
B_1 = \frac{\theta_1}{(1+\epsilon e^\theta)} e^{\theta(1+\epsilon)},
\]

\[
B_2 = \frac{(1-2e-2e^2 \theta_0) \theta_1^2 + 2(1+e(\theta_0)^2 \theta_1)}{2(1+e\theta_1)^2} \]

(17)

The zeroth-order components of the series solutions (14) and (15) are

\[
u_0(y) = a_0 y - \frac{y^2}{2},
\]

(18)

\[
\theta_0(y) = 0.
\]

Since the integral of a continuous function is continuous, then each term of the series can be uniquely determined by

\[
u_{n+1}(y) = \beta^2 \int_0^y \int_0^y u_n dY dY \quad n \geq 1
\]

\[
\theta_1(y) = \int_0^y a_1 dY
\]

(19)

\[
-\lambda \int_0^y \int_0^y \left( B_0 + \delta \left( \frac{du}{dY} \right)^2 + \delta \beta^2 u^2 \right) dY dY
\]

\[
\theta_{n+1}(y) = -\lambda \int_0^y \int_0^y B_n dY dY \quad n \geq 1,
\]

where \( du(0)/dY = a_0 \) and \( d\theta(0)/dY = a_1 \) are the parameters to be determined.

Then, (17)-(19) are evaluated using MATHEMATICA and the solutions are obtained as finite series:

\[
u(y) = \sum_{n=0}^m u_n(y),
\]

(20)

\[
\theta(y) = \sum_{n=0}^m \theta_n(y).
\]

The series solutions are shown to be convergent and twice differentiable (see Tables 1 and 2). Next, we establish the
uniqueness solution of (20). It is well known that the Lipschitz condition is sufficient for the uniqueness of solution. Therefore, we first seek for a Lipschitz constant $\phi$ such that

$$
\| F(\theta) - F(\bar{\theta}) \| \leq \phi \| \theta(y) - \bar{\theta}(\bar{y}) \| \tag{21}
$$

is satisfied. To do this, the boundary-valued problems (6) are converted to system first-order differential equations by introducing the following transformations:

$$
y_1 = y, \quad y_2 = u(y), \quad y_3 = \theta(y), \quad y_4 = u'(y), \quad y_5 = \theta'(y). \tag{22}
$$

With (22), (6) can now be written as

$$
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}' =
\begin{pmatrix}
1 \\
y_4 \\
y_5 \\
\beta^2 y_2 - 1 \\
-\lambda (e^{y_3/(1+y_3)} + \delta (y_2^2 + \beta^2 y_2^2))
\end{pmatrix};
\tag{23}
$$

where $c_i (i = 1, 2)$, the guess values that will ensure the boundary conditions, are satisfied. Then,

$$
\begin{pmatrix}
y_1(0) \\
y_2(0) \\
y_3(0) \\
y_4(0) \\
y_5(0)
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
c_1 \\
c_2
\end{pmatrix},
$$

since $df_i/dy_j, i, j = 1, 2, \ldots, 5$, exist and are continuous in the domain $[0, 1]$. Hence, the Lipschitz constant $\phi$ with the property

$$
\left| \frac{df_i}{dy_j} \right| \leq \phi \tag{24}
$$

exists.

**Uniqueness Analysis.** The Adomian series solutions (20) of the nonlinear problem (6) converges if $0 < \alpha < 1$ and $|\theta_0(y)| < \infty$, where $\alpha = \lambda KY^2/2$.

**Proof.** Let $(C(J, \| \cdot \|)$ be a Banach space for all continuous functions on $J$ with the norm

$$
\| F(Y) \| = \max_{y \in J} |F(Y)|. \tag{26}
$$

Let $\theta(y)$ and $\bar{\theta} (\bar{y})$ be any two solutions of the integral equation (12); then,

$$
|\theta(y) - \bar{\theta}(\bar{y})| = \left| - \int_0^y \int_0^y \lambda \{ F(\theta) - F(\bar{\theta}) \} dY d\bar{Y} \right|. \tag{27}
$$

This implies that

$$
|\theta(y) - \bar{\theta}(\bar{y})| \leq \int_0^y \int_0^y \lambda \{ F(\theta) - F(\bar{\theta}) \} dY d\bar{Y}. \tag{28}
$$

In view of (21), we then have

$$
|\theta(y) - \bar{\theta}(\bar{y})| \leq \int_0^y \int_0^y |\theta(y) - \bar{\theta}(\bar{y})| dY d\bar{Y} \leq |\theta(y) - \theta(\bar{y})| \int_0^y \int_0^y \lambda dY d\bar{Y} \tag{29}
$$

$$
\leq \frac{\lambda \Phi Y^2}{2} |\theta(y) - \theta(\bar{y})|. \tag{29}
$$
Table 1: Uniqueness results $m = 3, \beta = 1$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$y_u$-exact</th>
<th>$u$-Adomian</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0412846</td>
<td>0.0412847</td>
<td>$8.58689 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.072741</td>
<td>0.072742</td>
<td>$1.72597 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0953855</td>
<td>0.0953858</td>
<td>$2.61030 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.108743</td>
<td>0.108744</td>
<td>$3.51816 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.113181</td>
<td>0.113182</td>
<td>$4.44491 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.108743</td>
<td>0.108744</td>
<td>$5.34572 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0953855</td>
<td>0.0953861</td>
<td>$6.06498 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0729741</td>
<td>0.0729747</td>
<td>$6.1915 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0412846</td>
<td>0.0412851</td>
<td>$4.79648 \times 10^{-7}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Convergence results $\beta = 1 = \lambda = \delta, \epsilon = 0.1$.

<table>
<thead>
<tr>
<th>$N_b$</th>
<th>$b_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.537883</td>
</tr>
<tr>
<td>2</td>
<td>0.590362</td>
</tr>
<tr>
<td>3</td>
<td>0.589506</td>
</tr>
<tr>
<td>4</td>
<td>0.589078</td>
</tr>
<tr>
<td>5</td>
<td>0.589096</td>
</tr>
<tr>
<td>6</td>
<td>0.589100</td>
</tr>
<tr>
<td>7</td>
<td>0.589100</td>
</tr>
<tr>
<td>8</td>
<td>0.589100</td>
</tr>
</tbody>
</table>

Let $\alpha = \lambda \phi Y^2 / 2$; then,

$$|\theta (y) - \theta (\bar{y})| \leq \alpha |\theta (y) - \theta (\bar{y})|$$  \hfill (30)

or

$$(1 - \alpha) |\theta (y) - \theta (\bar{y})| \leq 0.$$  \hfill (31)

Hence, the problem will have a unique solution whenever $0 < \alpha < 1$ and as such

$$\theta (y) = \theta (\bar{y}).$$  \hfill (32)

Kindly see Tables 1 and 2 for the numerical results.

4. Results and Discussion

In this section, the effects of pertinent fluid parameters on the velocity and temperature profiles are shown graphically. Figure 1 shows the effect of medium porosity on the entropy generation rate. Since an increase in the porous permeability parameter implies a decrease in the medium porosity, this caused a decreased flow and heat trapping strategy. The net effect is seen here; that is, entropy production increases only in the centerline of the channel while it decreases at the walls. Figure 2 represents the effect of activation energy parameter on the entropy generation rate. Since activation energy decreases the fluid temperature, it is therefore expected to decrease the entropy generated in the flow region. This is true since an increase in the Frank-Kamenetskii parameter is known to enhance the fluid temperature. Therefore, by increasing values of this parameter, entropy generated is expected to be on the increase as shown in Figure 3. In Figure 4, the effect of the viscous heating parameter is shown. The result shows that entropy generation rate increases with increasing values of the viscous heating parameter due to frictional interaction in the fluid layers. Figure 5 represents the Bejan number’s variations with the activation energy. From the plot, as the activation energy parameter increases, heat irreversibility due to heat transfer decreases. As a result, fluid friction irreversibility dominates over heat transfer irreversibility within the porous medium. Moreover, as the porous permeability parameter increases in Figure 6, there are reduced flow and fluid temperature rises. The net effect shows that heat transfer irreversibility dominates over irreversibility due to fluid friction. Similar behaviour is observed in Figure 7 as the exothermic Frank-Kamenetskii parameter increases. Finally, as viscous heating parameter increases, the viscous heat dissipation increases in the flow.
channel, and fluid friction irreversibility dominates over heat transfer irreversibility as observed in Figure 8.

5. Conclusion
The entropy generation rate in the flow of reactive fluid through a porous medium has been investigated. Adomian decomposition method is used to obtain approximate solution to the strongly nonlinear boundary-valued problem for the dimensionless energy equation. The main contributions to knowledge from the present analysis are as follows: a reactive fluid flowing through a porous medium, entropy generation is minimum at the centerline of the channel;
as a result, fluid friction irreversibility dominates over heat transfer irreversibility in the centerline. Secondly, porous medium with low permeability is one major factor that depletes the useful available energy in a thermofluid.

Nomenclature

\begin{align*}
T &: \text{Fluid temperature} \\
P &: \text{Pressure} \\
T_0 &: \text{Wall temperature} \\
k &: \text{Thermal conductivity of the material} \\
K &: \text{Porous permeability respectively} \\
\mu &: \text{Dynamic viscosity} \\
Q &: \text{Heat of reaction} \\
A &: \text{Rate constant} \\
E &: \text{Activation energy} \\
R &: \text{Universal gas constant} \\
C_0 &: \text{Initial concentration of the reactant species} \\
h &: \text{Channel half width} \\
(x, y) &: \text{Cartesian coordinates system} \\
U &: \text{Fluid characteristic velocity} \\
M &: \text{Dimensionless axial pressure gradient} \\
\lambda &: \text{Frank-Kamenetskii parameter} \\
\varepsilon &: \text{Activation energy parameter} \\
\delta &: \text{Viscous heating parameter} \\
\beta &: \text{Porous medium permeability parameter} \\
Da &: \text{Darcy number, respectively} \\
E_G N_s &: \text{Dimensional and dimensionless entropy generation.}
\end{align*}

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

fluid through a porous saturated medium with asymmetric convective boundary conditions,” *Journal of Hydrodynamics*, vol. 27, no. 6, pp. 934–944, 2015.


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