Research Article

Slightly Regular Measures and Measureable Sets

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Outer and inner measures of a measure μ are defined and used to prove results involving them on a lattice ℓ and its complement ℓ' . The results concern slightly regular measures and sets such as S_μ which is the collection of μ -measureable sets.

1. Introduction

Let X be a nonempty set and ℓ a lattice of subsets of X such that $\emptyset$ and X belong to ℓ . Outer and inner measures of a measure are used to generate results on a lattice as in [1], giving more results for slightly regular measures which were studied in [2, 3]. In Section 2, these outer and inner measures along with slightly regular measures are introduced with relevant notations, definitions, and properties. Some results are presented in Section 3 for a measure on a lattice. Here, the results concern mostly slightly regular measures and measureable sets.

2. Background and Notations

We follow standard notation and definitions, according to [1–9]. Let X be a nonempty set and ℓ a lattice of subsets of X such that $\emptyset$ and X belong to ℓ . $A(\ell)$ denotes the algebra generated by ℓ , and $M(\ell)$ denotes those nontrivial, finitely additive measures on $A(\ell)$. $M_R(\ell)$ denotes those elements of $M(\ell)$ that are ℓ -regular. Note μ is ℓ -regular if we have $\mu(E) = \sup\{\mu(L) \mid L \subset E, E \in A(\ell), L \in \ell\}$. Let $M_\sigma(\ell)$ be the set of those elements of $M(\ell)$ that are σ -smooth on ℓ . Note μ is σ -smooth on ℓ if $L_n, L \in \ell$ and $L_n \downarrow \emptyset$ implies $\mu(L_n) \rightarrow 0$. $M\sigma(\ell)$ is the class of the strongly σ -smooth measures on ℓ . μ is strongly σ -smooth on ℓ if $L_n \in \ell$ and $L_n \downarrow L$ implies $\mu(L) = \inf \mu(L_n) = \lim \mu(L_n)$. $M^\sigma(\ell)$ are those elements of $M(\ell)$ that are σ -smooth on $A(\ell)$ and are, equivalently, countably additive. Note that μ is σ -smooth on $A(\ell)$ if $L_n, L \in A(\ell)$ and $L_n \downarrow \emptyset$ imply $\mu(L_n) \rightarrow 0$. We say that ℓ is a δ -lattice if $\bigcap_i L_i \in \ell$ whenever $L_i \in \ell$.

For $E \subset X$, we define two outer measures: for $\mu \in M(\ell)$, we have $\mu'(E) = \inf\{\mu(L') \mid E \subset L', L \in \ell\}$, and for $\mu \in M_\sigma(\ell)$ we have

$$\mu''(E) = \inf \left\{ \sum_i \mu(L'_i) \mid E \subset \bigcup_i L'_i, L_i \in \ell \right\}. \quad (1)$$

We have

$$\begin{aligned} \mu &\leq \mu'(\ell), \\ \mu &= \mu'(\ell') \mu = \mu'(\ell'), \\ \mu'' &\leq \mu(\ell'), \\ \mu'' &\leq \mu'. \end{aligned} \quad (2)$$

Note if $\mu \in M_\sigma(\ell)$, we have $\mu \leq \mu'(\ell)$, for $\mu \in M\sigma(\ell)$; then, $\mu' = \mu(\ell')$, and if $\mu \in M_R^\sigma(\ell)$, then $\mu' = \mu(\ell)$. We also have for our two outer measures $\mu \in M_w(\ell) \subseteq M(\ell)$ if $\mu(L') = \sup\{\mu'(B) \mid B \subseteq L', B \in \ell\}$ for $L \in \ell$ and $\mu \in M_v(\ell) \subseteq M_\sigma(\ell)$ if $\mu(L') = \sup\{\mu''(B) \mid B \subseteq L', B \in \ell\}$ for $L \in \ell$. $M_w(\ell)$ is the set of all weakly regular measures on $A(\ell)$, and $M_v(\ell)$ is the set of all vaguely regular measures on $A(\ell)$, and also $\mu \in M_S(\ell) \subseteq M(\ell)$ if $\bigcap_i L_i = L$, where $L, L_i \in \ell$; then, $\mu(L) = \inf \mu(L_i)$.

Note. It is not difficult to show that $M_R^\sigma(\ell) \subset M_S(\ell) \subset M^\sigma(\ell) \subset M\sigma(\ell) \subset M_\sigma(\ell)$. The proof of $M_S(\ell) \subset M_\sigma(\ell)$ is as follows.

If $L_i \downarrow \phi$, then, $\bigcap_i L_i = \phi$, where $L_i \in \ell$; then, $\mu(\phi) = \inf \mu(L_i)$ as $\mu \in M_S(\ell)$. Therefore, $\mu(\phi) + \epsilon > \mu(L_i)$, where $\epsilon > 0$. Since $\mu(\phi) = 0$, then $\mu(L_i) < \epsilon$ so $\mu \in M_\sigma(\ell)$, and the proof of $M_S(\ell) \subset M^\sigma(\ell)$ is given below.

Given that μ is a finitely subadditive outer measure, we define the notion that μ is regular if for every $A \subset X$ there exists $E \in S_\mu$ such that $A \subset E$ and $\mu(A) = \mu(E)$. In addition, if μ is a regular outer measure, $E_i \uparrow$, and $E_i \subset X$, then $\mu(\lim E_i) = \lim \mu(E_i)$, and if μ is a regular finitely additive outer measure, we have $E \in S_\mu$ if and only if $\mu(L) + \mu(L') = \mu(X)$.

We have if $\mu \in M_\sigma(\ell)$ that $\mu \in M_S(\ell)$ which implies $\mu'' = \mu'$ on ℓ' and if $\mu'' = \mu'$ on ℓ' and μ'' is regular then $\mu \in M_S(\ell)$ (see [3]).

We now define two more set functions.

Let $\mu \in M(\ell)$. For any $E \subseteq X$, we have $\mu_i(E) = \sup\{\mu(L) \mid L \subset E, L \in \ell\}$ which is an inner measure.

Let $\mu \in M_\sigma(\ell)$. For any $E \subseteq X$, we have $\mu_j(E) = \mu(X) - \mu''(E')$.

If μ'' is regular, then μ_j is an inner measure.

For $\mu \in M_\sigma(\ell)$ and $E \subseteq X$, we get

$$\begin{aligned} \mu_i(E) &\leq \mu_j(E) \\ &\leq \sup \left\{ \mu'' \left(\bigcap_1^\infty L_i \right), \bigcap L_i \subset E, L_i \in \ell \right\} \quad (3) \\ &\leq \mu''(E) \leq \mu'(E)(1). \end{aligned}$$

For the proof, see [4].

3. Results

The following theorem is proved in [2] using $I_S(\ell)$ which is a subset of $M_S(\ell)$ consisting of zero-one valued measures.

Theorem 1. If $\mu, \nu \in M_S(\ell)$, $\mu \leq \nu(\ell)$, and $\mu(X) = \nu(X)$, then $\mu = \nu$.

Proof. For $L \in \ell \subseteq X$, we have that $\mu \leq \nu(\ell)$ implies $\nu \leq \mu(\ell')$ so $\mu(L') \geq \nu(L')$. Now, $\mu(L') + \epsilon \geq \nu(L') + \epsilon = \nu''(L') + \epsilon$ as $\mu \in M_S(\ell)$, which implies $\mu'' = \mu'$ on ℓ' and $\mu = \mu'$ on ℓ' always. Now $\mu(L') + \epsilon \geq \nu(L') + \epsilon = \nu''(L') + \epsilon > \sum \nu(L'_n) > \nu(L'_n)$ by the definition of ν'' .

Therefore, $\mu(L') + \epsilon \geq \nu(L'_n)$. It follows that $-\mu(L') - \epsilon \leq -\nu(L'_n)$ and $\mu(X) - \mu(L') - \epsilon \leq \nu(X) - \nu(L'_n)$ as $\mu(X) = \nu(X)$.

Therefore, $\mu(L) - \epsilon \leq \nu(L_n)$ giving us $\mu(L) \geq \nu(L_n)$. This implies that $\mu(L)$ is an upper bound of $\nu(L_n)$. So $\sup \nu(L_n) \leq \mu(L)$ and it follows that $\inf \mu(L_n) \leq \inf \nu(L_n) \leq \sup \nu(L_n) \leq \mu(L)$ as $\mu \leq \nu(\ell)$. Using $\mu, \nu \in M_S(\ell)$, we get $\mu(L) \leq \nu(L) \leq \sup \nu(L_n) \leq \mu(L)$ as $\inf \mu(L_n) = \mu(L)$ and $\inf \nu(L_n) = \nu(L)$. Therefore, $\mu(L) \leq \nu(L) \leq \mu(L)$ giving us $\mu = \nu$. $\square$

Theorem 2. If $\mu \in M_S(\ell)$ and if there exists $\nu \in M_R^\sigma(\ell)$ such that $\mu \leq \nu(\ell)$ and $\mu(X) = \nu(X)$, then $\mu = \nu$.

Proof. Since $M_R^\sigma(\ell) \subset M_S(\ell)$, then $\nu \in M_R^\sigma(\ell)$ implies $\nu \in M_S(\ell)$. Therefore, $\mu = \nu$ by Theorem 1. $\square$

Theorem 3. If $\mu \in M_S(\ell)$, $\mu \leq \nu(\ell)$, ℓ is a δ -lattice, $\mu(X) = \mu'(X)$, and $\nu \in M_\sigma(\ell)$, then $\nu \in M_S(\ell)$ and therefore $\mu = \nu$.

Proof. If $L_i \downarrow L$, then, $\bigcap_i L_i = L$ and so $L = \bigcap_i L_i \subseteq L_i$. It follows that $\nu(L) \leq \nu(L_i)$ so $\nu(L)$ is a lower bound of $\nu(L_i)$. Therefore, $\nu(L) \leq \inf \nu(L_i)$. Now $\inf \nu(L_i) \leq \sup \nu(L_i)$ so $\nu(L) \leq \inf \nu(L_i) \leq \sup \nu(L_i)$. Now $\mu''(L') + \epsilon > \mu(L'_i)$ by definition of μ'' so $-\mu''(L') - \epsilon < -\mu(L'_i)$, and since $\mu(X) = \mu'(X)$ and $\mu' = \mu''$ on ℓ' , we have $\mu(X) - \mu'(L') - \epsilon < \mu(X) - \mu(L'_i)$ giving us $\mu'(L) - \epsilon < \mu(L_i)$.

Since $\mu \leq \mu'(\ell)$, we have $\mu(L) - \epsilon < \mu'(L) - \epsilon < \mu(L_i)$ giving $\mu(L) - \epsilon < \mu(L_i)$ and $\mu(L) - \epsilon < \mu(L_i) < \nu(L_i)$ as $\mu \leq \nu(\ell)$ which implies $\mu(L) \geq \nu(L_i)$ so $\mu(L)$ is an upper bound of $\nu(L_i)$ giving $\mu(L) \geq \sup \nu(L_i)$. So we have $\sup \nu(L_i) \leq \mu(L) \leq \nu(L)$ as $\mu \leq \nu(\ell)$. Remember we showed early in the proof that $\nu(L) \leq \inf \nu(L_i) \leq \sup \nu(L_i)$ and therefore we have $\nu(L) \leq \inf \nu(L_i) \leq \sup \nu(L_i) \leq \mu(L) \leq \nu(L)$ which gives us $\nu(L) = \inf \nu(L_i)$. So $\nu \in M_S(\ell)$. Therefore, $\mu = \nu$ by Theorem 1. $\square$

Theorem 4. If $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, then $\mu = \mu''(\ell)$.

Proof. We know since $\mu \in M_\sigma(\ell)$, we have $\mu \leq \mu'(\ell)$. Let $\epsilon > 0$, $L_i \downarrow L$, and $L_i \in \ell$ so then $\mu''(L') + \epsilon > \sum_i \mu(L'_i)$, where $L' \subset \bigcup_i L'_i$ for $L_i \in L$. We have $L = \bigcap (L_i \cup L)$ and $L_i \cup L \downarrow L$ as $L_i \downarrow L$ and as $\mu \in M_S(\ell)$; then, $\mu(L) + \epsilon > \mu(L_N \cup L)$ for a fixed N . Now since $\mu \leq \mu'(\ell)$, we have $\mu'(L) + \epsilon > \mu(L) + \epsilon > \mu(L_N \cup L)$ and by the monotone property of μ , we have $\mu'(L) + \epsilon > \mu(L) + \epsilon > \mu(L_N \cup L) > \mu(L)$. So $\mu'(\ell) \leq \mu(\ell)$ as $\mu'(L) + \epsilon > \mu(L)$ and since $\mu \leq \mu'(\ell)$ we get $\mu = \mu''(\ell)$. $\square$

Note for the next results $S_{\mu'}$ is the collection of μ' -measureable sets, and $S_{\mu''}$ is the collection of μ'' -measureable sets.

Theorem 5. If $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, ℓ is a δ -lattice, and $\mu(X) = \mu'(X)$, then $\ell \subset S_{\mu'}$.

Proof. We have $\mu_j = \mu' = \mu''$ on ℓ ; as for $L \in \ell \subseteq X$, we have $\mu_j(L) = \mu(X) - \mu''(L')$. Since $\mu \in M_S(\ell)$, then $\mu'' = \mu'$ on ℓ' so $\mu_j(L) = \mu(X) - \mu'(L') = \mu'(L)$ as $\mu'(L') + \mu'(L) = \mu'(X)$. Therefore, $\mu_j = \mu'$ on ℓ . $\square$

From $\mu_i(E) \leq \mu_j(E) \leq \sup\{\mu''(\bigcap_1^\infty L_i), \bigcap L_i \subset E, L_i \in \ell\} \leq \mu''(E) \leq \mu'(E)(1)$, we have with $E = L$ that $\mu_i(L) \leq \mu_j(L) \leq \sup\{\mu''(\bigcap_1^\infty L_i), \bigcap L_i \subset L, L_i \in \ell\} \leq \mu''(L) \leq \mu'(L)$. Therefore, $\mu_i(L) \leq \mu_j(L) \leq \sup\{\mu''(\bigcap_1^\infty L_i), \bigcap L_i \subset L, L_i \in \ell\} \leq \mu''(L) \leq \mu'(L) = \mu_j(L)$ as $\mu_j = \mu'$ on ℓ . Therefore, $\mu_j = \mu' = \mu''$ on ℓ , and using $\mu_i(E) \leq \mu_j(E) \leq \sup\{\mu''(\bigcap_1^\infty L_i), \bigcap L_i \subset E, L_i \in \ell\} \leq \mu''(E) \leq \mu'(E)(1)$ for $E \in \ell$, we get $\mu'(E) = \sup\{\mu''(\bigcap_1^\infty L_i), \bigcap L_i \subset E, L_i \in \ell\}$. Also since ℓ is a δ -lattice, we get $\bigcap_1^\infty L_i = L \in \ell$ so we get $\mu'(E) = \sup\{\mu''(L), L \subset E, L \in \ell\}$. Now $\mu = \mu''(\ell)$ from Theorem 4 so $\mu'(E) = \sup\{\mu(L), L \subset E, L \in \ell\}$ and it follows from [3] that $E \in S_{\mu'}$. Since $E \in \ell$, we have $\ell \subset S_{\mu'}$.

Theorem 6. If $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, ℓ is a δ -lattice, and $\mu(X) = \mu'(X)$, then $\ell \subset S_{\mu''}$.

Proof. Since $\mu \in M_S(\ell)$ and ℓ is a δ -lattice, then $S_{\mu'} = S_{\mu''}$ from Theorem 4.8 part (c) of [3]. Now since $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, ℓ is a δ -lattice, and $\mu(X) = \mu'(X)$, then $\ell \subset S_{\mu'}$ from Theorem 5. So since $S_{\mu'} = S_{\mu''}$, we get $\ell \subset S_{\mu''}$. $\square$

Theorem 7. *If $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, ℓ is a δ -lattice, and $\mu(X) = \mu'(X)$, then $\mu \in M^\sigma(\ell)$.*

Proof. From Theorem 6, we have $\ell \subset S_{\mu''}$, so μ'' is countably additive on $A(\ell)$. Now since $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, then $\mu = \mu''(\ell)$ from Theorem 4 so μ is countably additive on $A(\ell)$. Therefore, $\mu \in M^\sigma(\ell)$ by definition. $\square$

Theorem 8. *If $E \in S_{\mu'}$, then $\mu' = \mu_i$.*

Proof. From Lemma 4.1 of [3], $E \in S_{\mu'}$ if and only if $\mu'(E) = \sup\{\mu(L), L \subset E, | L, E \in \ell\}$. Now $\mu_i(E) = \sup\{\mu(L) \mid L \subset E, L \in \ell\}$; therefore, $\mu' = \mu_i$. $\square$

Theorem 9. *If $E \in S_{\mu'}$, then $\mu' = \mu_i = \mu_j = \mu''$.*

Proof. If $E \in S_{\mu'}$, then $\mu' = \mu_i$ from Theorem 8. Now using $\mu_i(E) \leq \mu_j(E) \leq \sup\{\mu''(\bigcap_{i=1}^\infty L_i), \bigcap_{i=1}^\infty L_i \subset E, | L_i \in \ell\} \leq \mu'(E) \leq \mu'(E)(1)$ for $E \in S_{\mu'}$ we get $\mu' = \mu_i = \mu_j = \mu''$. $\square$

Theorem 10. *If $\mu \in M_S(\ell) \cap M_\sigma(\ell)$, $\mu' = \mu_i$, and ℓ is a δ -lattice, then $E \in S_{\mu'}$.*

Proof. Since $\mu' = \mu_i$, $\mu_i(E) \leq \mu_j(E) \leq \sup\{\mu''(\bigcap_{i=1}^\infty L_i), \bigcap_{i=1}^\infty L_i \subset E, | L_i \in \ell\} \leq \mu''(E) \leq \mu'(E)(1)$ becomes $\mu'(E) = \sup\{\mu''(\bigcap_{i=1}^\infty L_i), \bigcap_{i=1}^\infty L_i \subset E, | L_i \in \ell\}$. As ℓ is a δ -lattice, we get $\bigcap_{i=1}^\infty L_i = L \in \ell$ so we get $\mu'(E) = \sup\{\mu''(L), L \subset E, | L \in \ell\}$ and since $\mu \in M_S(\ell) \cap M_\sigma(\ell)$ we have by Theorem 4 that $\mu = \mu''(\ell)$ so $\mu'(E) = \sup\{\mu(L), L \subset E, | L \in \ell\}$, and from [3] this implies $E \in S_{\mu'}$. $\square$

Theorem 11. *If $\mu \in M_S(\ell)$ and ℓ is a δ -lattice, then $S_{\mu''} \cap \ell = \{L \in \ell \mid \mu(L) = \mu''(L)\}$.*

Proof. Let $L \in S_{\mu''} \cap \ell$; then, $L \in S_{\mu'} \cap \ell$ as $S_{\mu'} = S_{\mu''}$ from [3], and in [3] we have that if $\mu \in M(\ell)$, then $S_{\mu'} \cap \ell = \{L \in \ell \mid \mu(L) = \mu'(L)\}$. So $L \in S_{\mu'} \cap \ell$ implies $\mu(L) = \mu'(L)$. However, $\mu \in M_S(\ell)$, and ℓ is a δ -lattice implying $\mu'' = \mu'$ everywhere in [3] so $\mu(L) = \mu''(L)$.

Now let $\mu(L) = \mu''(L)$ for $L \in \ell$; then, $\mu(L) = \mu'(L)$ as $\mu'' = \mu'$ everywhere. Therefore, $L \in S_{\mu'} \cap \ell$ and as $S_{\mu'} = S_{\mu''}$ we have $L \in S_{\mu''} \cap \ell$. $\square$

We now end with a sort of converse to Theorem 4.

Theorem 12. *If $\mu \in M_\sigma(\ell)$, $\mu(X) = \mu''(X)$, $\mu = \mu''(\ell)$, $\ell \subset S_{\mu'}$, and μ'' is regular, then $\mu \in M_S(\ell)$.*

Proof. Now $\mu(L') = \mu(X) - \mu(L) = \mu''(X) - \mu''(L)$ as $\mu(X) = \mu''(X)$ and $\mu = \mu''(\ell)$. Therefore, $\mu(L') = \mu''(X) - \mu''(L)$ giving us $\mu(L') = \mu''(L')$ as μ'' is regular, so we have $\mu = \mu''(\ell')$. Then, it follows $\mu \in M_S(\ell)$ from [3]. $\square$

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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