A Generalized Hermite-Hadamard Inequality for Coordinated Convex Function and Some Associated Mappings

Atiq Ur Rehman,1 Gulam Farid,1 and Sidra Malik2

1COMSATS Institute of Information Technology, Attock Campus, Attock, Pakistan
2Government Islamia High School, Attock, Pakistan

Correspondence should be addressed to Atiq Ur Rehman; atiq@mathcity.org

Received 25 July 2016; Accepted 25 October 2016

Academic Editor: Nan-Jing Huang

Copyright © 2016 Atiq Ur Rehman et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Convex functions have great importance in many areas of Mathematics. A real valued function \( f: I \rightarrow \mathbb{R} \), where \( I \) is interval in \( \mathbb{R} \), is said to be convex if for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \)

\[ f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y). \quad (1) \]

In particular, if \( \alpha = \beta = 1/2 \) we have that

\[ f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}. \quad (2) \]

One of the most important inequalities that has attracted many mathematician in this field in the last few decades is the famous Hermite-Hadamard inequality, which establishes a refinement of (2) and it involves the notions of convexity.

Let \( f: I \rightarrow \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The double inequality

\[ f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (3) \]

is known as the Hermite-Hadamard inequality. This inequality was published by Hermite in 1883 and was independently proved by Hadamard in 1893 (see Mitrinovic and Lackovic [1] for the whole history). It gives us an estimation of the mean value of a convex function \( f \) and it is important to note that (3) provides a refinement to the Jensen inequality. Fink [2] has worked on a best possible Hermite-Hadamard inequality and has deduced this inequality with least restrictions. Dragomir in [3, 4] worked on Hermite-Hadamard inequality and a mapping in connection to it. Dragomir and Milošević [5] gave some refinements of Hadamard’s inequalities and its applications (see also [6–8] for more results). Dragomir and Pearce [9] wrote a monograph on selected topics on Hermite-Hadamard inequalities.

The aim of this paper is to discuss an analogue of the generalization of Hermite-Hadamard inequality introduced by Lupas for convex functions on coordinates defined in a rectangle from the plane. Also we define that mappings are related to it and their properties are discussed.

In [10] Dragomir has given the concept of convex functions on the coordinates in a rectangle from the plane and established the Hermite-Hadamard inequality for it.

Definition 1. Let \( \Delta^2 = [a, b] \times [c, d] \subset \mathbb{R} \) with \( a < b \) and \( c < d \). A function \( f: \Delta^2 \rightarrow \mathbb{R} \) is called convex on coordinates if the partial mappings \( f_y: [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y) \) and \( f_x: [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v) \) are convex, defined for all \( y \in [c, d] \) and \( x \in [a, b] \).

Note that every convex mapping \( f: \Delta^2 \rightarrow \mathbb{R} \) is convex on the coordinates but the converse is not generally true.
Theorem 2. Suppose that \( f: \Delta^2 \to \mathbb{R} \) is convex on the coordinates on \( \Delta^2 \). Then we have

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{2} \left( \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx + \frac{1}{d-c} \int_c^d f \left( a+b \frac{y}{2} \right) \, dy \right)
\]

\[
+ \frac{1}{d-c} \int_c^d f (x, y) \, dy \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f (x, y) \, dx \, dy
\]

\[
\cdot \frac{1}{b-a} \int_a^b f (x, y) \, dy \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f (x, c) \, dx + \frac{1}{d-c} \int_c^d f (b, y) \, dy \right] \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

In [10], Dragomir for \( f: \Delta^2 \to \mathbb{R} \), defined a mapping \( H: [0,1]^2 \to \mathbb{R} \) as

\[
H(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( tx + \frac{1}{2} (1-t) y \right) \, dx \, dy
\]

and proved the following theorem.

Theorem 3. Suppose that \( f: \Delta^2 \to \mathbb{R} \) is convex on the coordinates on \( \Delta^2 \).

(i) The mapping \( H \) is convex on the coordinates on \([0,1]^2\).

(ii) We have the bounds

\[
\sup_{(s,t) \in [0,1]^2} H(s,t) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \, dy \, dx = H(1,1),
\]

\[
\inf_{(s,t) \in [0,1]^2} H(s,t) = f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) = H(0,0).
\]

(iii) The mapping \( H \) is monotonic nondecreasing on the coordinates.

Also, in [10] Dragomir gave the following mapping, which is closely connected with Hadamard’s inequality, \( \breve{H}: [0,1]^2 \to \mathbb{R} \) defined as

\[
\breve{H}(t, s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( \frac{x+y}{2}, \frac{z+u}{2} \right) \, dx \, dy \, dz \, du
\]

and proved the following properties of this mapping.

Theorem 4. Suppose that \( f: \Delta^2 \to \mathbb{R} \) is convex on the coordinates on \( \Delta^2 \).

(i) We have the equalities

\[
\breve{H} \left( \frac{1}{2}, \frac{1}{2}, s \right) = \breve{H} \left( \frac{1}{2} - t, s \right)
\]

\[
\forall t \in \left[ 0, \frac{1}{2} \right], \ s \in [0,1],
\]

\[
\breve{H} \left( t, \frac{1}{2}, \frac{1}{2} s \right) = \breve{H} \left( t, \frac{1}{2} - s \right)
\]

\[
\forall t \in [0,1], \ s \in \left[ 0, \frac{1}{2} \right],
\]

\[
\breve{H}(1-t, s) = \breve{H}(t, s),
\]

\[
\breve{H}(t, 1-s) = \breve{H}(t, s)
\]

\[
\forall (t, s) \in \Delta^2.
\]

(ii) \( \breve{H} \) is convex on the coordinates.

(iii) We have the bounds

\[
\inf_{(s,t) \in [0,1]^2} \breve{H}(s,t) = \breve{H}(0,0) = \breve{H}(1,1) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f (x, y) \, dx \, dy \, dz \, du
\]

and proved the following properties of this mapping.

A generalized form of Hermite-Hadamard inequality is given by Lupas in [11] (see also [12, page 143]).

Theorem 5. Let \( p, q \) be given positive numbers and \( a < b \). Then the inequality,

\[
f \left( \frac{pa+q}{p+q}, \frac{b}{p+q} \right) \leq \frac{1}{2y} \int_{A-y}^{A+y} f (t) \, dt \leq \frac{pf(a) + qf(b)}{p+q},
\]

holds for \( A = (pa+qb)/(p+q), \ y > 0 \) and all continuous convex functions \( f: [a, b] \to \mathbb{R} \) iff \( y \leq ((b-a)/(p+q)) \min (p, q) \).
2. Main Results

The following results comprise generalization of Hermite-Hadamard inequality introduced by Lupas on coordinates in a rectangle from the plane.

Theorem 6. Let \( \Delta^2 = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2 \) and \( p_1, p_2, q_1, \) and \( q_2 \) be positive real numbers and
\[
A_i = \frac{p_i a_i + q_i b_i}{p_i + q_i},
\]
where \( y_i > 0, \)
\[
y_i \leq \frac{b_i - a_i}{p_i + q_i} \min (p_i, q_i) \quad \text{for } i = 1, 2.
\]
Also let \( f: \Delta^2 \rightarrow \mathbb{R} \) be convex on the coordinates on \( \Delta^2. \)

\[
f (A_1, A_2) \leq \frac{1}{2} \left[ \frac{1}{y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (s_1, A_2) \, ds_1 + \frac{1}{y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (A_1, s_2) \, ds_2 \right] \leq \frac{1}{4 y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f (s_1, s_2) \, ds_1 \, ds_2
\]
\[
\leq \frac{1}{(p_1 + q_1)} \left[ \frac{p_1}{y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (s_1, a_2) \, ds_1 \right]
\]
\[
\quad + \frac{q_1}{y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (a_1, s_2) \, ds_1
\]
\[
+ \frac{1}{(p_2 + q_2)} \left[ \frac{p_2}{y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (a_1, s_2) \, ds_2 \right]
\]
\[
\quad + \frac{q_2}{y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (b_1, s_2) \, ds_2
\]
\[
\leq \left[ \frac{p_1 p_2}{(p_1 + q_1)(p_2 + q_2)} f (a_1, a_2) + q_1 q_2 f (b_1, b_2) \right]
\]
\[
\times \left[ \frac{1}{(p_1 + q_1)} + \frac{1}{(p_2 + q_2)} \right]
\]
(13)

Proof. Since \( f: \Delta^2 \rightarrow \mathbb{R} \) is convex on coordinates, it follows that the mapping \( f_{s_i}: [a_2, b_2] \rightarrow \mathbb{R}, f_{s_1} (s_2) = f (s_1, s_2), \) is convex on \( [a_2, b_2] \) for all \( s_1 \in [a_1, b_1], \) so by inequality (11) one has
\[
f_{s_1} (A_2) \leq \frac{1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f_{s_1} (s_2) \, ds_2
\]
\[
\leq \frac{p_2 f_{s_1} (a_2) + q_2 f_{s_1} (b_2)}{p_2 + q_2},
\]
that is
\[
f (s_1, A_2) \leq \frac{1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (s_1, s_2) \, ds_2
\]
\[
\leq \frac{p_2 f (s_1, a_2) + q_2 f (s_1, b_2)}{p_2 + q_2}.
\]
(14)

Observe that for \( 0 < y_1 \leq ((b_i - a_i)/(p_i + q_i)) \min (p_i, q_i), \) by considering two cases \( 0 < p_i \leq q_i \) and \( 0 < q_i \leq p_i \) we can easily verify that \( a_i \leq A_i - y_i < A_i + y_i \leq b_i, \) so that \( f \) is defined on \([A_1 - y_1, A_1 + y_1].\) Integrating the above inequality on \([A_1 - y_1, A_1 + y_1],\) we have
\[
\frac{1}{2 y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (s_1, A_2) \, ds_1 \leq \frac{1}{4 y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f (s_1, s_2) \, ds_1 \, ds_2
\]
\[
\leq \frac{1}{y_1} \int_{A_1 - y_1}^{A_1 + y_1} \left[ \frac{p_2}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (s_1, a_2) \, ds_2 \right]
\]
\[
+ \frac{q_2}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (b_1, s_2) \, ds_2
\]
\[
\leq \frac{1}{y_1} \int_{A_1 - y_1}^{A_1 + y_1} \left[ \frac{p_1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (a_1, s_2) \, ds_2 \right]
\]
\[
+ \frac{q_1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (b_1, s_2) \, ds_2
\]
(16)

By a similar argument applied for the mapping \( f_{s_2}: [a_1, b_1] \rightarrow \mathbb{R}, f_{s_2} (s_1) = f (s_1, s_2), \) we get
\[
\frac{1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (A_1, s_2) \, ds_2 \leq \frac{1}{4 y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f (s_1, s_2) \, ds_1 \, ds_2
\]
\[
\leq \frac{1}{y_2} \int_{A_2 - y_2}^{A_2 + y_2} \left[ \frac{p_1}{2 y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (a_1, s_2) \, ds_1 \right]
\]
\[
+ \frac{q_1}{2 y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (b_1, s_2) \, ds_1
\]
(17)

Summing the inequalities (16) and (17), we get the second and third inequality in (13). Since \( f \) is convex on coordinates, using convexity of \( f \) on first coordinate and inequality (11), we have
\[
f (A_1, A_2) \leq \frac{1}{2 y_1} \int_{A_1 - y_1}^{A_1 + y_1} f (s_1, A_2) \, ds_1
\]
(18)

and, using convexity of \( f \) on second coordinate, we have
\[
f (A_1, A_2) \leq \frac{1}{2 y_2} \int_{A_2 - y_2}^{A_2 + y_2} f (A_1, s_2) \, ds_2.
\]
(19)
\[
\frac{1}{2y_2} \int_{A_1-y_2}^{A_1+y_2} f(a_1, s_2) \, ds_2 \\
\leq p_2 f(a_1, a_2) + q_2 f(a_1, b_2) \quad \frac{p_2}{p_2+q_2},
\]
(22)

\[
\frac{1}{2y_2} \int_{A_1-y_2}^{A_1+y_2} f(b_1, s_2) \, ds_2 \\
\leq p_2 f(b_1, a_2) + q_2 f(b_1, b_2) \quad \frac{p_2}{p_2+q_2}.
\]
(23)

Multiply (20) and (21) by \(p_2\) and multiply (22) and (23) by \(q_2\) then on addition we get the last inequality in (13).

\[
\textbf{3. Some Associated Mappings}
\]

In this section we will discuss some mappings associated with generalized Hermite-Hadamard inequality introduced by Lupas for convex mappings on coordinates.

Let \(p_1, p_2, q_1,\) and \(q_2\) be positive real numbers and \(A_i = (p_1 a_i + q_i b_i)/(p_1 + q_i)\) and \(y_i > 0\) where \(y_i \leq ((b_i - a_i)/(p_1 + q_i))\min(p_i, q_i)\) for \(i = 1, 2.\) For mapping \(f : \Delta^2 \to [a_1, b_1] \times [a_2, b_2] \to \mathbb{R},\) we can define a mapping \(H : [0, 1]^2 \to \mathbb{R},\)

\[
H(u, v) = \frac{1}{4y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(us_1) \\
+ (1 - u) A_1, vs_2 + (1 - v) A_2) \, ds_1 \, ds_2,
\]
(24)

where \(a_1 \leq A_1 - y_1 < A_1 + y_1 \leq b_1\) and \(a_2 \leq A_2 - y_2 < A_2 + y_2 \leq b_2,\) so that \(f\) is defined on \([A_1-y_1, A_1+y_1]\) and \([A_2-y_2, A_2+y_2].\) The properties of this mapping are embodied in the following theorem.

**Theorem 7.** Suppose that \(p_1, p_2, q_1,\) and \(q_2\) are positive real numbers and

\[
A_i = \frac{p_1 a_i + q_i b_i}{p_1 + q_i},
\]

\[y_i > 0,
\]

where \(y_i \leq \frac{b_i - a_i}{p_1 + q_i} \min(p_i, q_i)\) for \(i = 1, 2.\)

Also let \(f : \Delta^2 \to \mathbb{R}\) be convex on the coordinates on \(\Delta^2:\)

(i) The mapping \(H\) is convex on the coordinates on \([0, 1]^2.\)

(ii) We have the bounds

\[
\sup_{(u,v)\in[0,1]^2} H(u,v) = \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) \, ds_1 \, ds_2 \\
= \frac{1}{4y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) \, ds_1 \, ds_2 \\
= H(1, 1),
\]

\[
\inf_{(u,v)\in[0,1]^2} H(u,v) = f(A_1, A_2) = H(0, 0).
\]

(iii) The mapping \(H\) is monotonic nondecreasing on the coordinates.

**Proof.** (i) Fix \(v \in [0, 1],\) for all \(u_1, u_2 \in [0, 1]\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1;\) we have

\[
H(\alpha u_1 + \beta u_2, v) = \frac{1}{4y_1y_2} \\
\frac{1}{y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(\alpha (u_1 s_1 + (1 - u_1) A_1) \\
+ \beta (u_2 s_1 + (1 - u_2) A_1), vs_2 \\
+ (1 - v) A_2) \, ds_1 \, ds_2 \\
\leq \frac{\alpha}{4y_1y_2} \\
\int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(u_1 s_1 + (1 - u_1) A_1, vs_2 \\
+ (1 - v) A_2) \, ds_1 \, ds_2 = \alpha H(u_1, v) \\
+ \beta H(u_2, v).
\]

If \(u \in [0, 1]\) is fixed then, for all \(v_1, v_2 \in [0, 1]\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1;\) we also have \(H(u, \alpha v_1 + \beta v_2) \leq \alpha H(u, v_1) + \beta H(u, v_2)\) and the statement is proved.

(ii) Since \(f\) is convex on coordinates, we have by Jensen’s inequality for integrals

\[
H(u, v) \geq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(us_1 + (1 - u) A_1, \frac{1}{2y_2} \\
\int_{A_2-y_2}^{A_2+y_2} (us_2 + (1 - v) A_2) \, ds_2) \, ds_1 = \frac{1}{2y_1} \\
\int_{A_1-y_1}^{A_1+y_1} f(us_1 + (1 - u) A_1, A_2) \, ds_1 \\
\geq f(\frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} (us + (1 - u) A_1) \, ds_1, A_2) \\
= f(A_1, A_2).
\]

By the convexity of \(H\) on the coordinates, we have

\[
H(u, v) \leq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} \left[ \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) \, ds_2 + (1 - v) \frac{1}{2y_2} \right] \\
\int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) \, ds_2 \\
\leq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} \left[ \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) \, ds_2 + (1 - u) \frac{1}{2y_1} \right] \\
\int_{A_1-y_1}^{A_1+y_1} f(s_1, s_2) \, ds_1 \\
= f(A_1, A_2).
\]
\[
\int_{A_1}^{A_2+y_1} \int_{s_1}^{s_2} f(A_1, s_1) ds_1 \, ds_2 + (1 - v) \frac{1}{2y_2}
\]
\[
\int_{A_1}^{A_2+y_1} \int_{s_1}^{s_2} f(s_1, A_2) ds_1 \, ds_2 + (1 - u)
\]
\[
\int_{A_1}^{A_2+y_1} f(A_1, s_1) ds_1 \, ds_2 = uv \frac{1}{4y_1y_2}
\]
\[
\int_{A_1}^{A_2+y_1} \int_{s_1}^{s_2} f(s_1, s_2) ds_1 \, ds_2 + (1 - u) v \frac{1}{2y_2}
\]
\[
\int_{A_1}^{A_2+y_1} f(s_1, A_2) ds_1 \, ds_2 + u (1 - v) \frac{1}{2y_1}
\]
\[
\int_{A_1}^{A_2+y_1} f(s_1, A_2) ds_1 \, ds_2 + (1 - u) (1 - v) f(A_1, A_2).
\]

By Hadamard's inequality introduced by Lupas (11), we also have
\[
f(A_1, s_2) \leq \frac{1}{2y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1, \tag{30}
\]
\[
f(s_1, A_2) \leq \frac{1}{2y_2} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_2.
\]
Thus by integration, we get that
\[
\frac{1}{2y_2} \int_{A_1}^{A_2+y_1} f(A_1, s_2) ds_2 \leq \frac{1}{4y_1y_2} \int_{A_1}^{A_2+y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1 \, ds_2,
\]
\[
\frac{1}{2y_1} \int_{A_1}^{A_2+y_1} f(s_1, A_2) ds_1 \leq \frac{1}{4y_1y_2} \int_{A_1}^{A_2+y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1 \, ds_2.
\]

Also using the result
\[
f(A_1, A_2) \leq \frac{1}{4y_1y_2} \int_{A_1}^{A_2+y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1 \, ds_2 \tag{32}
\]
we deduce the inequality
\[
H(u, v) \leq [uv + (1 - u) v + u (1 - v) + (1 - u) (1 - v)] \frac{1}{4y_1y_2}
\]
\[
\int_{A_1}^{A_2+y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1 \, ds_2 = \frac{1}{4y_1y_2}
\]
\[
\int_{A_1}^{A_2+y_1} \int_{A_1}^{A_2+y_1} f(s_1, s_2) ds_1 \, ds_2,
\]
for all \((u, v) \in [0, 1]^2\) and the second bound in (ii) is proved.

(iii) Firstly, we show that \(H(u, v) \geq H(0, v)\) for all \([u, v] \in [0, 1]^2\). By Jensen's inequality for integrals, we have
\[
H(u, v) \geq \frac{1}{2y_2} \int_{A_1}^{A_2+y_1} f \left( \frac{1}{2y_1} \right) \int_{A_1}^{A_2+y_1} \left( u s_1 + (1 - u) A_1 \right) ds_1 \, ds_2 + (1 - v) \frac{1}{2y_2}
\]
\[
\int_{A_1}^{A_2+y_1} \left( u s_1 + (1 - u) A_1 \right) ds_1 \, ds_2 = \frac{1}{2y_2} \int_{A_1}^{A_2+y_1} f \left( A_1, s_2 \right) ds_2 + (1 - v) \frac{1}{2y_2}
\]
\[
\int_{A_1}^{A_2+y_1} f \left( A_1, s_2 \right) ds_2 + (1 - v) \frac{1}{2y_2}
\]
\[
\int_{A_1}^{A_2+y_1} f \left( A_1, s_2 \right) ds_2 = H(0, v)
\]
for all \([u, v] \in [0, 1]^2\).

Now let \(0 \leq u_1 \leq u_2 \leq 1\). By the convexity of mapping \(H(\cdot, v)\) for all \([u, v] \in [0, 1]^2\), we have
\[
\frac{H(u_2, v) - H(u_1, v)}{u_2 - u_1} \geq H(u_1, v) - H(0, v) \geq 0. \tag{35}
\]

The following theorem also holds.

**Theorem 8.** Suppose that \(p_1, p_2, q_1,\) and \(q_2\) are positive real numbers and
\[
A_i = \frac{p_i a_i + q_i b_i}{p_i + q_i},
\]
\[
y_i > 0, \tag{36}
\]
where \(y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i)\) for \(i = 1, 2\).

Also let \(f : \Delta^2 \rightarrow \mathbb{R}\) be convex on the coordinates on \(\Delta^2\).

(i) The mapping \(H\) is convex on \([0, 1]^2\).

(ii) Define the mapping \(h : [0, 1] \rightarrow \mathbb{R}, h(u) := H(u, u)\). Then \(h\) is convex, monotonic nondecreasing on \([0, 1]\) and one has the bounds
\[
\sup_{u \in [0,1]} h(u) = h(1) = \frac{1}{4y_1y_2} \int_{A_1, y_1}^{A_1 + y_1} \int_{A_2, y_2}^{A_2 + y_2} f(s_1, s_2) \, ds_1 \, ds_2,
\]
(37)

\[
\inf_{u \in [0,1]} h(u) = h(0) = f(A_1, A_2).
\]

Proof. (i) Let \((u_1, u_2), (v_1, v_2) \in [0, 1]^2\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\). Since \(f : \Delta^2 \rightarrow \mathbb{R}\) is convex on \(\Delta^2\), we have

\[
H(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) \\
= \frac{1}{4y_1y_2} \int_{A_1, y_1}^{A_1 + y_1} \int_{A_2, y_2}^{A_2 + y_2} f(s_1, s_2) \, ds_1 \, ds_2,
\]
(38)

which shows \(H\) is convex on \([0, 1]^2\).

(ii) Let \(u_1, u_2 \in [0, 1]\) and \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\).

\[
h(\alpha a_1 + \beta a_2) = \inf_{u \in [0,1]} h(a_1, a_2) \\
= H(\alpha u_1, \alpha u_2) + \beta h(u_2, \alpha u_2),
\]

which proves the required bounds.

Next, for positive real numbers \(p_1, p_2, q_1, q_2\) and \(A_1 = (p_1a_1 + q_1b_1)/(p_1 + q_1)\) and \(y_i > 0\) where \(y_i \leq ((b_i - a_i)/(p_i + q_i)) \min(p_i, q_i)\) for \(i = 1, 2\), then for mapping \(f : \Delta^2 = [a_1, b_1] \times [a_2, b_2] \rightarrow \mathbb{R}\) we shall consider the mapping \(\tilde{H} : [0, 1]^2 \rightarrow [0, \infty)\), which is given as

\[
\tilde{H}(u, v) = \frac{1}{16y_1y_2} \int_{A_1, y_1}^{A_1 + y_1} \int_{A_2, y_2}^{A_2 + y_2} f(s_1, s_2) \, ds_1 \, ds_2,
\]
(42)

where \(a_1 \leq A_1 - y_1 < A_1 + y_1 \leq b_1\) and \(a_2 \leq A_2 - y_2 < A_2 + y_2 \leq b_2\), so that \(f\) is defined on \([A_1 - y_1, A_1 + y_1]\) and \([A_2 - y_2, A_2 + y_2]\).

The next theorem contains the main properties of this mapping.

**Theorem 9.** Suppose that \(p_1, p_2, q_1, q_2\) and \(q_2\) are positive real numbers and

\[
A_i = \frac{p_i a_i + q_i b_i}{p_i + q_i},
\]

\(y_i > 0\),

where \(y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i)\) for \(i = 1, 2\).

Also let \(f : \Delta^2 \rightarrow \mathbb{R}\) be convex on the coordinates on \(\Delta^2\).

We have, by the above theorem, that

\[
h(u) = H(u, u) \geq H(0, 0) = f(A_1, A_2),
\]
\(u \in [0, 1]\),

\[
h(u) = H(u, u) \leq H(1, 1)
\]
(40)

which shows the convexity of \(h\) on \([0, 1]\).

and the theorem is proved.

\[
\tilde{H}(u + \frac{1}{2} v) = \tilde{H}(u, \frac{1}{2} v),
\]
\(\forall u \in [0, 1], v \in [0, 1]\),

\[
\tilde{H}(u, v + \frac{1}{2} v) = \tilde{H}(u, \frac{1}{2} v) + \frac{1}{4} \tilde{H}(u, v),
\]
(44)

\(\forall u \in [0, 1], v \in [0, 1]\),

\[
\tilde{H}(1 - u, v) = \tilde{H}(u, v),
\]
\(\tilde{H}(u, 1 - v) = \tilde{H}(u, v),
\)
\(\forall (u, v) \in \Delta^2\).
(ii) $\tilde{H}$ is convex on the coordinates.

(iii) We have the bounds

$$
\inf_{(u,v)\in[0,1]^2} \tilde{H}(u,v) = \tilde{H}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{16y_1^2 y_2^2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f\left(\frac{s_1 + t_1}{2}, \frac{s_2 + t_2}{2}\right) dt_2 ds_2 dt_1 ds_1,
$$

$$
\sup_{(u,v)\in[0,1]^2} \tilde{H}(u,v) = \tilde{H}(0,0) = \tilde{H}(1,1) = \frac{1}{4y_1 y_2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f(s_1, s_2) ds_2 ds_1.
$$

(iv) The mapping $\tilde{H}(\cdot, v)$ is monotonic nonincreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$ for all $v \in [0, 1]$. A similar property has the mapping $\tilde{H}(u, \cdot)$ for all $u \in [0, 1]$.

(v) We have the inequality

$$
\tilde{H}(u, v) \geq \max \{H(s_1, s_2), H(1-s_1, s_2), H(s_1, 1-s_2), H(1-s_1, 1-s_2)\}.
$$

Proof. (i) and (ii) are obvious.

(iii) By the convexity of $f$ in the first variable, we get that

$$
\frac{1}{2} \left[ f(us_1 + (1-u)t_1, vs_2 + (1-v)t_2) + f((1-u)s_1 + ut_1, vs_2 + (1-v)t_2) \right] \geq f\left(\frac{s_1 + t_1}{2}, vs_2 + (1-v)t_2\right)
$$

for all $(s_1, t_1) \in [a_1, b_1]^2$, $(s_2, t_2) \in [a_2, b_2]^2$, and $(u, v) \in [0, 1]^2$. Integrating on $[A_{1-y_1}, A_1 + y_1]^2$, we get

$$
\frac{1}{4y_1^2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f(us_1 + (1-u)t_1, vs_2 + (1-v)t_2) ds_2 dt_2 ds_1 dt_1
$$

for $(u, v) \in [0, 1]^2$. The first bound in (iii) is proved and the second bound goes on likewise.

(iv) The monotonicity of $\tilde{H}(\cdot, s)$ follows by a similar argument from (iii).

(v) By Jensen’s integral inequality, we have successively for all $(u, v) \in [0, 1]^2$ that

$$
\tilde{H}(u, v) \geq \frac{1}{8y_1 y_2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f\left(\frac{1}{2}, \frac{1}{2}\right) ds_2 dt_2 ds_1 dt_1 (us_1 + (1-u)t_1, vs_2 + (1-v)t_2)
$$

$$
= \frac{1}{8y_1 y_2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f(us_1 + (1-u)t_1, vs_2 + (1-v)t_2) dt_2 ds_2 ds_1
$$

$$
\geq \frac{1}{4y_1 y_2} \int_{A_{1-y_1}}^{A_{1+y_1}} \int_{A_{1-y_2}}^{A_{1+y_2}} f(us_1 + (1-u)t_1, vs_2 + (1-v)t_2) ds_2 ds_1 = H(s_1, s_2).
$$
Similarly, we can easily show \( \tilde{H}(1-u,v) \geq H(1-s_1,s_2) \), \( \tilde{H}(u,1-v) \geq H(s_1,1-s_2) \), and \( H(1-u,1-v) \geq H(1-s_1,1-s_2) \).

In addition, as \( \tilde{H}(u,v) = \tilde{H}(1-u,1-v) = \tilde{H}(1-u,1-v) \) for all \((u,v) \in [0,1]^2\), so we deduce \( \tilde{H}(u,v) \geq \max\{H(s_1,s_2), H(1-s_1,s_2), H(s_1,1-s_2), H(1-s_1,1-s_2)\} \).

The theorem is thus proved.

**Theorem 10.** Suppose that \( p_1, p_2, q_1, \) and \( q_2 \) are positive real numbers and

\[
A_i = p_i a_i + q_i b_i / p_i + q_i,
\]

\[
\inf_{u \in [0,1]} \bar{h}(u) = \bar{h}(1) = \bar{h}(0) = \frac{1}{16y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1,s_2) ds_2 ds_1,
\]

\[
\sup_{u \in [0,1]} \bar{h}(u) = \bar{h}(1) = \bar{h}(0) = \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1,s_2) ds_2 ds_1.
\]

(iii) One has the inequality

\[
\bar{h}(u) \geq \max\{h(u), h(1-u)\}. \tag{54}
\]

\[
\bar{H}(u_1, u_2) + \beta (v_1, v_2) = \bar{H}(u_1 + \beta v_1, u_2 + \beta v_2) = \frac{1}{16y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(u_1,1-u_2) du_1 du_2 \leq \frac{1}{16y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(u_1,1-u_2) du_1 du_2 \leq \bar{H}(u_1, u_2) + \beta \bar{H}(v_1, v_2),
\]

which shows \( \bar{H} \) is convex on \( \Delta^2 \).

(ii) Let \( u_1, u_2, v \in [0,1] \) and \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

\[
\bar{h}(u_1, u_2) = \bar{H}(u_1, u_2) + \beta \bar{H}(u_2, u_2) = \alpha \bar{H}(u_1, u_2) + \beta \bar{H}(u_2, u_2),
\]

which shows the convexity of \( \bar{h} \) on \([0,1] \).

As

\[
\bar{h}(u) = \bar{H}(u, u) \geq \bar{H}(1/2, 1/2) = \frac{1}{16y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1,s_2) ds_2 ds_1,
\]

\[
\bar{h}(u) = \bar{H}(u, u) \leq \bar{H}(0,0) \leq \bar{H}(0,0) = \frac{1}{4y_1y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1,s_2) ds_2 ds_1,
\]

which proves the required bounds.
It is obvious that $\tilde{h}$ is monotonic nonincreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$.

(iii) As we know

$$
\tilde{h}(u) = \tilde{H}(u, u) \geq \max \{H(u, u), H(1-u, 1-u)\}
= \max \{h(u), h(1-u)\}
$$

(58)

and the theorem is proved.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References
