Research Article

A Generalized Hermite-Hadamard Inequality for Coordinated Convex Function and Some Associated Mappings

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We have discussed the generalization of Hermite-Hadamard inequality introduced by Lupaş for convex functions on coordinates defined in a rectangle from the plane. Also we define that mappings are related to it and their properties are discussed.

1. Introduction

Convex functions have great importance in many areas of Mathematics. A real valued function $f : I \rightarrow \mathbb{R}$, where I is interval in $\mathbb{R}$, is said to be convex if for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y). \quad (1)$$

In particular, if $\alpha = \beta = 1/2$ we have that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}. \quad (2)$$

One of the most important inequalities that has attracted many mathematician in this field in the last few decades is the famous Hermite-Hadamard inequality, which establishes a refinement of (2) and it involves the notions of convexity.

Let $f : I \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (3)$$

is known as the Hermite-Hadamard inequality. This inequality was published by Hermite in 1883 and was independently proved by Hadamard in 1893 (see Mitrinović and Lacković [1] for the whole history). It gives us an estimation of the mean

value of a convex function f and it is important to note that (3) provides a refinement to the Jensen inequality. Fink [2] has worked on a best possible Hermite-Hadamard inequality and has deduced this inequality with least restrictions. Dragomir in [3, 4] worked on Hermite-Hadamard inequality and a mapping in connection to it. Dragomir and Milošević [5] gave some refinements of Hadamard's inequalities and its applications (see also [6–8] for more results). Dragomir and Pearce [9] wrote a monograph on selected topics on Hermite-Hadamard inequalities.

The aim of this paper is to discuss an analogue of the generalization of Hermite-Hadamard inequality introduced by Lupaş for convex functions defined in a rectangle from the plane. Also some related mappings are defined and their properties are discussed.

In [10] Dragomir has given the concept of convex functions on the coordinates in a rectangle from the plane and established the Hermite-Hadamard inequality for it.

Definition 1. Let $\Delta^2 = [a, b] \times [c, d] \subset \mathbb{R}$ with $a < b$ and $c < d$. A function $f : \Delta^2 \rightarrow \mathbb{R}$ is called convex on coordinates if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) := f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) := f(x, v)$ are convex, defined for all $y \in [c, d]$ and $x \in [a, b]$.

Note that every convex mapping $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on the coordinates but the converse is not generally true.

Theorem 2. Suppose that $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on the coordinates on Δ^2 . Then we have

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\ &\quad \left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \leq \frac{1}{(b-a)(d-c)} \\ &\quad \cdot \int_a^b \int_c^d f(x, y) dy dx \leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f(x, c) dx \right. \\ &\quad \left. + \frac{1}{b-a} \int_a^b f(x, d) dx + \frac{1}{d-c} \int_c^d f(a, y) dy \right. \\ &\quad \left. + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (4)$$

In [10], Dragomir for $f : \Delta^2 \rightarrow \mathbb{R}$, defined a mapping $H : [0, 1]^2 \rightarrow \mathbb{R}$ as

$$\begin{aligned} H(t, s) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx \right. \\ &\quad \left. + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dy dx \end{aligned} \quad (5)$$

and proved the properties of this mapping in following theorem.

Theorem 3. Suppose that $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on the coordinates on Δ^2 .

- (i) The mapping H is convex on the coordinates on $[0, 1]^2$.
- (ii) We have the bounds

$$\begin{aligned} \sup_{(s,t) \in [0,1]^2} H(s, t) \\ = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx = H(1, 1), \end{aligned} \quad (6)$$

$$\inf_{(s,t) \in [0,1]^2} H(s, t) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = H(0, 0).$$

- (iii) The mapping H is monotonic nondecreasing on the coordinates.

Also, in [10] Dragomir gave the following mapping, which is closely connected with Hadamard's inequality, $\tilde{H} : [0, 1]^2 \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} \tilde{H}(t, s) &:= \frac{1}{(b-a)^2(d-c)^2} \\ &\quad \cdot \int_a^b \int_a^b \int_c^d \int_c^d f(tx + (1-t)y, sz + (1-s)u) dx dy dz du. \end{aligned} \quad (7)$$

and proved the following properties of this mapping.

Theorem 4. Suppose that $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on the coordinates on Δ^2 .

- (i) We have the equalities

$$\begin{aligned} \tilde{H}\left(t + \frac{1}{2}, s\right) &= \tilde{H}\left(\frac{1}{2} - t, s\right) \\ \forall t \in \left[0, \frac{1}{2}\right], s \in [0, 1], \\ \tilde{H}\left(t, s + \frac{1}{2}\right) &= \tilde{H}\left(t, \frac{1}{2} - s\right) \\ \forall t \in [0, 1], s \in \left[0, \frac{1}{2}\right], \\ \tilde{H}(1-t, s) &= \tilde{H}(t, s), \\ \tilde{H}(t, 1-s) &= \tilde{H}(t, s) \\ \forall (t, s) \in \Delta^2. \end{aligned} \quad (8)$$

- (ii) $\tilde{H}$ is convex on the coordinates.

- (iii) We have the bounds

$$\begin{aligned} \inf_{(t,s) \in [0,1]^2} \tilde{H}(t, s) &= \tilde{H}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{(b-a)^2(d-c)^2} \\ &\quad \cdot \int_a^b \int_a^b \int_c^d \int_c^d f\left(\frac{x+y}{2}, \frac{z+u}{2}\right) dx dy dz du, \\ \sup_{(t,s) \in [0,1]^2} \tilde{H}(t, s) &= \tilde{H}(0, 0) = \tilde{H}(1, 1) = \frac{1}{(b-a)(d-c)} \\ &\quad \cdot \int_a^b \int_c^d f(x, z) dz dx. \end{aligned} \quad (9)$$

- (iv) The mapping $\tilde{H}(\cdot, s)$ is monotonic nondecreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$ for all $s \in [0, 1]$. A similar property has the mapping $\tilde{H}(t, \cdot)$ for all $t \in [0, 1]$.

- (v) We have the inequality

$$\begin{aligned} \tilde{H}(t, s) &\geq \max\{H(t, s), H(1-t, s), H(t, 1-s), \\ &\quad H(1-t, 1-s)\}. \end{aligned} \quad (10)$$

A generalized form of Hermite-Hadamard inequality is given by Lupaş in [11] (see also [12, page 143]).

Theorem 5. Let p, q be given positive numbers and $a < b$. Then the inequality,

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y} \int_{A-y}^{A+y} f(t) dt \leq \frac{pf(a) + qf(b)}{p+q}, \quad (11)$$

holds for $A = (pa+qb)/(p+q)$, $y > 0$ and all continuous convex functions $f : [a, b] \rightarrow \mathbb{R}$ iff $y \leq ((b-a)/(p+q)) \min(p, q)$.

2. Main Results

The following results comprise generalization of Hermite-Hadamard inequality introduced by Lupaş on coordinates in a rectangle from the plane.

Theorem 6. Let $\Delta^2 = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ and p_1, p_2, q_1 , and q_2 be positive real numbers and

$$A_i = \frac{p_i a_i + q_i b_i}{p_i + q_i}, \quad y_i > 0, \quad (12)$$

$$\text{where } y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i) \text{ for } i = 1, 2.$$

Also let $f : \Delta^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ^2 .

$$\begin{aligned} f(A_1, A_2) &\leq \frac{1}{2} \left[\frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 \right. \\ &\quad + \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_2 \left. \right] \leq \frac{1}{4y_1 y_2} \\ &\quad \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 \\ &\leq \frac{1}{(p_1 + q_1)} \left[\frac{p_1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, a_2) ds_1 \right. \\ &\quad + \frac{q_1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, b_2) ds_1 \left. \right] \\ &\quad + \frac{1}{(p_2 + q_2)} \left[\frac{p_2}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(a_1, s_2) ds_2 \right. \\ &\quad + \frac{q_2}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(b_1, s_2) ds_2 \left. \right] \leq [p_1 p_2 f(a_1, a_2) \\ &\quad + p_1 q_2 f(a_1, b_2) + q_1 p_2 f(b_1, a_2) + q_1 q_2 f(b_1, b_2)] \\ &\quad \times \left[\frac{1}{(p_1 + q_1)} + \frac{1}{(p_2 + q_2)} \right]. \end{aligned} \quad (13)$$

Proof. Since $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on coordinates, it follows that the mapping $f_{s_1} : [a_2, b_2] \rightarrow \mathbb{R}$, $f_{s_1}(s_2) := f(s_1, s_2)$, is convex on $[a_2, b_2]$ for all $s_1 \in [a_1, b_1]$, so by inequality (11) one has

$$\begin{aligned} f_{s_1}(A_2) &\leq \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f_{s_1}(s_2) ds_2 \\ &\leq \frac{p_2 f_{s_1}(a_2) + q_2 f_{s_1}(b_2)}{p_2 + q_2}, \end{aligned} \quad (14)$$

that is

$$\begin{aligned} f(s_1, A_2) &\leq \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_2 \\ &\leq \frac{p_2 f(s_1, a_2) + q_2 f(s_1, b_2)}{p_2 + q_2}. \end{aligned} \quad (15)$$

Observe that for $0 < y_1 \leq ((b_1 - a_1)/(p_1 + q_1)) \min(p_1, q_1)$, by considering two cases ($0 < p_1 \leq q_1$ and $0 < q_1 < p_1$) we can easily verify that $a_1 \leq A_1 - y_1 < A_1 + y_1 \leq b_1$, so that f is defined on $[A_1 - y_1, A_1 + y_1]$. Integrating the above inequality on $[A_1 - y_1, A_1 + y_1]$, we have

$$\begin{aligned} \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 &\leq \frac{1}{4y_1 y_2} \\ &\quad \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 \\ &\leq \frac{1}{p_2 + q_2} \left[\frac{p_2}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, a_2) ds_1 \right. \\ &\quad + \left. \frac{q_2}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, b_2) ds_1 \right]. \end{aligned} \quad (16)$$

By a similar argument applied for the mapping $f_{s_2} : [a_1, b_1] \rightarrow \mathbb{R}$, $f_{s_2}(s_1) := f(s_1, s_2)$, we get

$$\begin{aligned} \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_2 &\leq \frac{1}{4y_1 y_2} \\ &\quad \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 \\ &\leq \frac{1}{p_1 + q_1} \left[\frac{p_1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(a_1, s_2) ds_2 \right. \\ &\quad + \left. \frac{q_1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(b_1, s_2) ds_2 \right]. \end{aligned} \quad (17)$$

Summing the inequalities (16) and (17), we get the second and third inequality in (13). Since f is convex on coordinates, using convexity of f on first coordinate and inequality (11), we have

$$f(A_1, A_2) \leq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 \quad (18)$$

and, using convexity of f on second coordinate, we have

$$f(A_1, A_2) \leq \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_2. \quad (19)$$

Adding the inequalities (18) and (19) we get the first inequality in (13). Finally, by the same Hermite-Hadamard inequality introduced by Lupaş we can also state

$$\begin{aligned} \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, a_2) ds_1 \\ \leq \frac{p_1 f(a_1, a_2) + q_1 f(b_1, a_2)}{p_1 + q_1}, \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, b_2) ds_1 \\ \leq \frac{p_1 f(a_1, b_2) + q_1 f(b_1, b_2)}{p_1 + q_1}, \end{aligned} \quad (21)$$

$$\begin{aligned} & \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(a_1, s_2) ds_2 \\ & \leq \frac{p_2 f(a_1, a_2) + q_2 f(a_1, b_2)}{p_2 + q_2}, \end{aligned} \quad (22)$$

$$\begin{aligned} & \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(b_1, s_2) ds_2 \\ & \leq \frac{p_2 f(b_1, a_2) + q_2 f(b_1, b_2)}{p_2 + q_2}. \end{aligned} \quad (23)$$

Multiply (20) and (21) by p_2 and multiply (22) and (23) by q_2 then on addition we get the last inequality in (13). $\square$

3. Some Associated Mappings

In this section we will discuss some mappings associated with generalized Hermite-Hadamard inequality introduced by Lupaş for convex mappings on coordinates.

Let p_1, p_2, q_1 , and q_2 be positive real numbers and $A_i = (p_i a_i + q_i b_i)/(p_i + q_i)$ and $y_i > 0$ where $y_i \leq ((b_i - a_i)/(p_i + q_i)) \min(p_i, q_i)$ for $i = 1, 2$. For mapping $f : \Delta^2 = [a_1, b_1] \times [a_2, b_2] \rightarrow \mathbb{R}$, we can define a mapping $H : [0, 1]^2 \rightarrow \mathbb{R}$,

$$\begin{aligned} H(u, v) := & \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(us_1 \\ & + (1-u)A_1, vs_2 + (1-v)A_2) ds_1 ds_2, \end{aligned} \quad (24)$$

where $a_1 \leq A_1 - y_1 < A_1 + y_1 \leq b_1$ and $a_2 \leq A_2 - y_2 < A_2 + y_2 \leq b_2$, so that f is defined on $[A_1 - y_1, A_1 + y_1]$ and $[A_2 - y_2, A_2 + y_2]$. The properties of this mapping are embodied in the following theorem.

Theorem 7. Suppose that p_1, p_2, q_1 , and q_2 are positive real numbers and

$$\begin{aligned} A_i &= \frac{p_i a_i + q_i b_i}{p_i + q_i}, \\ y_i &> 0, \end{aligned} \quad (25)$$

$$\text{where } y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i) \text{ for } i = 1, 2.$$

Also let $f : \Delta^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ^2 .

(i) The mapping H is convex on the coordinates on $[0, 1]^2$.

(ii) We have the bounds

$$\begin{aligned} & \sup_{(u,v) \in [0,1]^2} H(u, v) \\ &= \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 \\ &= H(1, 1), \end{aligned} \quad (26)$$

$$\inf_{(u,v) \in [0,1]^2} H(u, v) = f(A_1, A_2) = H(0, 0).$$

(iii) The mapping H is monotonic nondecreasing on the coordinates.

Proof. (i) Fix $v \in [0, 1]$, for all $u_1, u_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$; we have

$$\begin{aligned} H(\alpha u_1 + \beta u_2, v) &= \frac{1}{4y_1 y_2} \\ & \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(\alpha(u_1 s_1 + (1-u_1)A_1) \\ & \cdot + \beta(u_2 s_1 + (1-u_2)A_1), vs_2 \\ & + (1-v)A_2) ds_1 ds_2 \leq \frac{\alpha}{4y_1 y_2} \\ & \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(u_1 s_1 + (1-u_1)A_1, vs_2 \\ & + (1-v)A_2) ds_1 ds_2 + \frac{\beta}{4y_1 y_2} \\ & \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(u_2 s_1 + (1-u_2)A_1, vs_2 \\ & + (1-v)A_2) ds_1 ds_2 = \alpha H(u_1, v) \\ & + \beta H(u_2, v). \end{aligned} \quad (27)$$

If $u \in [0, 1]$ is fixed then, for all $v_1, v_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we also have $H(u, \alpha v_1 + \beta v_2) \leq \alpha H(u, v_1) + \beta H(u, v_2)$ and the statement is proved.

(ii) Since f is convex on coordinates, we have by Jensen's inequality for integrals that

$$\begin{aligned} H(u, v) &\geq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f\left(us_1 + (1-u)A_1, \frac{1}{2y_2} \right. \\ & \cdot \left. \int_{A_2-y_2}^{A_2+y_2} (vs_2 + (1-v)A_2) ds_2\right) ds_1 = \frac{1}{2y_1} \\ & \cdot \int_{A_1-y_1}^{A_1+y_1} f(us_1 + (1-u)A_1, A_2) ds_1 \\ &\geq f\left(\frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} (us + (1-u)A_1) ds_1, A_2\right) \\ &= f(A_1, A_2). \end{aligned} \quad (28)$$

By the convexity of H on the coordinates, we have

$$\begin{aligned} H(u, v) &\leq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} \left[v \frac{1}{2y_2} \right. \\ & \cdot \left. \int_{A_2-y_2}^{A_2+y_2} f(us_1 + (1-u)A_1, s_2) ds_2 + (1-v) \frac{1}{2y_2} \right. \\ & \cdot \left. \int_{A_2-y_2}^{A_2+y_2} f(us_1 + (1-u)A_1, A_2) ds_2 \right] ds_1 \\ &\leq v \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} \left[u \frac{1}{2y_1} \right. \\ & \cdot \left. \int_{A_1-y_1}^{A_1+y_1} f(s_1, s_2) ds_1 + (1-u) \frac{1}{2y_1} \right. \end{aligned}$$

$$\begin{aligned}
& \cdot \int_{A_1-y_1}^{A_1+y_1} f(A_1, s_2) ds_1 \Big] ds_2 + (1-v) \frac{1}{2y_2} \\
& \cdot \int_{A_2-y_2}^{A_2+y_2} \left[u \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 + (1-u) \right. \\
& \cdot \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(A_1, A_2) ds_1 \Big] ds_2 = uv \frac{1}{4y_1 y_2} \\
& \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 + (1-u) v \\
& \cdot \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_1 ds_2 \\
& + u(1-v) \frac{1}{4y_1 y_2} \\
& \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, A_2) ds_1 ds_2 + (1-u) \\
& \cdot (1-v) \frac{1}{4y_1 y_2} \\
& \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(A_1, A_2) ds_1 ds_2 = uv \frac{1}{4y_1 y_2} \\
& \cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 + (1-u) v \frac{1}{2y_2} \\
& \cdot \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_2 + u(1-v) \frac{1}{2y_1} \\
& \cdot \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 + (1-u)(1-v) f(A_1, A_2).
\end{aligned} \tag{29}$$

By Hadamard's inequality introduced by Lupas (11), we also have

$$\begin{aligned}
f(A_1, s_2) &\leq \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, s_2) ds_1, \\
f(s_1, A_2) &\leq \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_2.
\end{aligned} \tag{30}$$

Thus by integration, we get that

$$\begin{aligned}
& \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(A_1, s_2) ds_2 \\
& \leq \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2, \\
& \frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} f(s_1, A_2) ds_1 \\
& \leq \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2.
\end{aligned} \tag{31}$$

Also using the result

$$f(A_1, A_2) \leq \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 \tag{32}$$

we deduce the inequality

$$\begin{aligned}
H(u, v) &\leq [uv + (1-u)v + u(1-v) + (1-u)(1-v)] \\
&\cdot \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2 = \frac{1}{4y_1 y_2} \\
&\cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_1 ds_2,
\end{aligned} \tag{33}$$

for all $(u, v) \in [0, 1]^2$ and the second bound in (ii) is proved.

(iii) Firstly, we show that $H(u, v) \geq H(0, v)$ for all $[u, v] \in [0, 1]^2$. By Jensen's inequality for integrals, we have

$$\begin{aligned}
H(u, v) &\geq \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{1}{2y_1} \right. \\
&\cdot \int_{A_1-y_1}^{A_1+y_1} (us_1 + (1-u)A_1) ds_1, vs_2 + (1-v) \\
&\cdot A_2 \Big) ds_2 = \frac{1}{2y_2} \int_{A_2-y_2}^{A_2+y_2} f(A_1, vs_2 \\
&\cdot + (1-v)A_2) ds_2 = H(0, v)
\end{aligned} \tag{34}$$

for all $[u, v] \in [0, 1]^2$.

Now let $0 \leq u_1 \leq u_2 \leq 1$. By the convexity of mapping $H(\cdot, v)$ for all $[u, v] \in [0, 1]^2$, we have

$$\frac{H(u_2, v) - H(u_1, v)}{u_2 - u_1} \geq \frac{H(u_1, v) - H(0, v)}{u_1} \geq 0. \tag{35}$$

□

The following theorem also holds.

Theorem 8. Suppose that p_1, p_2, q_1 , and q_2 are positive real numbers and

$$\begin{aligned}
A_i &= \frac{p_i a_i + q_i b_i}{p_i + q_i}, \\
y_i &> 0,
\end{aligned} \tag{36}$$

$$\text{where } y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i) \text{ for } i = 1, 2.$$

Also let $f : \Delta^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ^2 .

(i) The mapping H is convex on $[0, 1]^2$.

(ii) Define the mapping $h : [0, 1] \rightarrow \mathbb{R}$, $h(u) := H(u, u)$. Then h is convex, monotonic nondecreasing on $[0, 1]$ and one has the bounds

$$\begin{aligned}
\sup_{u \in [0,1]} h(u) &= h(1) \\
&= \frac{1}{4y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f(s_1, s_2) ds_1 ds_2, \quad (37) \\
\inf_{u \in [0,1]} h(u) &= h(0) = f(A_1, A_2).
\end{aligned}$$

Proof. (i) Let $(u_1, u_2), (v_1, v_2) \in [0, 1]^2$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since $f : \Delta^2 \rightarrow \mathbb{R}$ is convex on Δ^2 , we have

$$\begin{aligned}
H(\alpha(u_1, u_2) + \beta(v_1, v_2)) &= H(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) = \frac{1}{4y_1 y_2} \\
&\cdot \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f(\alpha(u_1 s_1 + (1 - u_1)A_1, u_2 s_2 + (1 - u_2)A_2) \\
&\cdot + \beta(v_1 s_1 + (1 - v_1)A_1, v_2 s_2 + (1 - v_2)A_2)) ds_1 ds_2 \\
&\leq \alpha \cdot \frac{1}{4y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f(u_1 s_1 + (1 - u_1)A_1, u_2 s_2 \\
&\cdot + (1 - u_2)A_2) ds_1 ds_2 + \beta \cdot \frac{1}{4y_1 y_2} \\
&\cdot \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f(v_1 s_1 + (1 - v_1)A_1, v_2 s_2 + (1 - v_2) \\
&\cdot A_2) ds_1 ds_2 = \alpha H(u_1, u_2) + \beta H(v_1, v_2), \quad (38)
\end{aligned}$$

which shows H is convex on $[0, 1]^2$.

(ii) Let $u_1, u_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

$$\begin{aligned}
h(\alpha u_1 + \beta u_2) &= H(\alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2) \\
&= H(\alpha(u_1, u_1) + \beta(u_2, u_2))
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha H(u_1, u_1) + \beta H(u_2, u_2) \\
&= \alpha h(u_1) + \beta h(u_2), \quad (39)
\end{aligned}$$

which shows the convexity of h on $[0, 1]$.

We have, by the above theorem, that

$$\begin{aligned}
h(u) &= H(u, u) \geq H(0, 0) = f(A_1, A_2), \\
&u \in [0, 1], \\
h(u) &= H(u, u) \leq H(1, 1) \quad (40)
\end{aligned}$$

$$= \frac{1}{4y_1 y_2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} f(s_1, s_2) ds_1 ds_2,$$

$$u \in [0, 1],$$

which proves the required bounds.

Now let $0 \leq u_1 \leq u_2 \leq 1$. By the convexity of mapping h , we have that

$$\frac{h(u_2) - h(u_1)}{u_2 - u_1} \geq \frac{h(u_1) - h(0)}{u_1} \geq 0, \quad (41)$$

and the theorem is proved. $\square$

Next, for positive real numbers p_1, p_2, q_1, q_2 and $A_i = (p_i a_i + q_i b_i)/(p_i + q_i)$ and $y_i > 0$ where $y_i \leq ((b_i - a_i)/(p_i + q_i)) \min(p_i, q_i)$ for $i = 1, 2$, then for mapping $f : \Delta^2 = [a_1, b_1] \times [a_2, b_2] \rightarrow \mathbb{R}$ we shall consider the mapping $\tilde{H} : [0, 1]^2 \rightarrow [0, \infty)$, which is given as

$$\tilde{H}(u, v) := \frac{1}{16y_1^2 y_2^2} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_1 - y_1}^{A_1 + y_1} \int_{A_2 - y_2}^{A_2 + y_2} \int_{A_2 - y_2}^{A_2 + y_2} f(us_1 + (1 - u)t_1, vs_2 + (1 - v)t_2) dt_2 ds_2 dt_1 ds_1, \quad (42)$$

where $a_1 \leq A_1 - y_1 < A_1 + y_1 \leq b_1$ and $a_2 \leq A_2 - y_2 < A_2 + y_2 \leq b_2$, so that f is defined on $[A_1 - y_1, A_1 + y_1]$ and $[A_2 - y_2, A_2 + y_2]$.

The next theorem contains the main properties of this mapping.

Theorem 9. Suppose that p_1, p_2, q_1 , and q_2 are positive real numbers and

$$\begin{aligned}
A_i &= \frac{p_i a_i + q_i b_i}{p_i + q_i}, \\
y_i &> 0, \quad (43)
\end{aligned}$$

$$\text{where } y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i) \text{ for } i = 1, 2.$$

Also let $f : \Delta^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ^2 .

(i) We have the equalities

$$\begin{aligned}
\tilde{H}\left(u + \frac{1}{2}, v\right) &= \tilde{H}\left(\frac{1}{2} - u, v\right), \\
\forall u \in \left[0, \frac{1}{2}\right], v \in [0, 1], \\
\tilde{H}\left(u, v + \frac{1}{2}\right) &= \tilde{H}\left(u, \frac{1}{2} - v\right), \\
\forall u \in \left[0, \frac{1}{2}\right], v \in [0, 1], \quad (44)
\end{aligned}$$

$$\tilde{H}(1 - u, v) = \tilde{H}(u, v),$$

$$\tilde{H}(u, 1 - v) = \tilde{H}(u, v),$$

$$\forall (u, v) \in \Delta^2.$$

(ii) $\tilde{H}$ is convex on the coordinates.

(iii) We have the bounds

$$\begin{aligned} \inf_{(u,v) \in [0,1]^2} \tilde{H}(u,v) &= \tilde{H}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{16y_1^2 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, \frac{s_2+t_2}{2}\right) dt_2 ds_2 dt_1 ds_1, \\ \sup_{(u,v) \in [0,1]^2} \tilde{H}(u,v) &= \tilde{H}(0,0) = \tilde{H}(1,1) = \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_2 ds_1. \end{aligned} \quad (45)$$

(iv) The mapping $\tilde{H}(\cdot, v)$ is monotonic nonincreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$ for all $v \in [0, 1]$. A similar property has the mapping $\tilde{H}(u, \cdot)$ for all $u \in [0, 1]$.

(v) We have the inequality

$$\begin{aligned} \tilde{H}(u,v) &\geq \max \{H(s_1, s_2), H(1-s_1, s_2), \\ &H(s_1, 1-s_2), H(1-s_1, 1-s_2)\}. \end{aligned} \quad (46)$$

Proof. (i) and (ii) are obvious.

(iii) By the convexity of f in the first variable, we get that

$$\begin{aligned} &\frac{1}{2} [f(us_1 + (1-u)t_1, vs_2 + (1-v)t_2) \\ &+ f((1-u)s_1 + ut_1, vs_2 + (1-v)t_2)] \\ &\geq f\left(\frac{s_1+t_1}{2}, vs_2 + (1-v)t_2\right) \end{aligned} \quad (47)$$

for all $(s_1, t_1) \in [a_1, b_1]^2$, $(s_2, t_2) \in [a_2, b_2]^2$, and $(u, v) \in [0, 1]^2$. Integrating on $[A_1 - y_1, A_1 + y_1]^2$, we get

$$\frac{1}{4y_1^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} f(us_1 + (1-u)t_1, vs_2)$$

Similarly,

$$\begin{aligned} &\frac{1}{4y_2^2} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, vs_2 + (1-v) \right. \\ &\cdot t_2) ds_2 dt_2 \geq \frac{1}{4y_2^2} \\ &\cdot \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, \frac{s_2+t_2}{2}\right) ds_2 dt_2. \end{aligned} \quad (49)$$

Now integrating this inequality on $[A_1 - y_1, A_1 + y_1]^2$ and taking into account the above inequality, we deduce

$$\tilde{H}(u,v) \geq \frac{1}{16y_1^2 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, \frac{s_2+t_2}{2}\right) dt_2 ds_2 dt_1 ds_1 \quad (50)$$

for $(u, v) \in [0, 1]^2$. The first bound in (iii) is proved and the second bound goes on likewise.

(iv) The monotonicity of $\tilde{H}(\cdot, s)$ follows by a similar argument from (iii).

(v) By Jensen's integral inequality, we have successively for all $(u, v) \in [0, 1]^2$ that

$$\begin{aligned} \tilde{H}(u,v) &\geq \frac{1}{8y_1 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{1}{2y_1} \int_{A_1-y_1}^{A_1+y_1} (us_1 + (1-u)t_1) dt_1, vs_2 + (1-v)t_2\right) dt_2 ds_2 ds_1 \\ &= \frac{1}{8y_1 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f(us_1 + (1-u)A_1, vs_2 + (1-v)t_2) dt_2 ds_2 ds_1 \\ &\geq \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(us_1 + (1-u)A_1, vs_2 + (1-v)A_2) ds_2 ds_1 = H(s_1, s_2). \end{aligned} \quad (51)$$

Similarly, we can easily show $\tilde{H}(1-u, v) \geq H(1-s_1, s_2)$, $\tilde{H}(u, 1-v) \geq H(s_1, 1-s_2)$, and $\tilde{H}(1-u, 1-v) \geq H(1-s_1, 1-s_2)$.

In addition, as $\tilde{H}(u, v) = \tilde{H}(1-u, v) = \tilde{H}(u, 1-v) = \tilde{H}(1-u, 1-v)$ for all $(u, v) \in [0, 1]^2$, so we deduce $\tilde{H}(u, v) \geq \max\{H(s_1, s_2), H(1-s_1, s_2), H(s_1, 1-s_2), H(1-s_1, 1-s_2)\}$.

The theorem is thus proved. $\square$

Theorem 10. Suppose that p_1, p_2, q_1 , and q_2 are positive real numbers and

$$A_i = \frac{p_i a_i + q_i b_i}{p_i + q_i},$$

$y_i > 0$,

$$\text{where } y_i \leq \frac{b_i - a_i}{p_i + q_i} \min(p_i, q_i) \text{ for } i = 1, 2. \quad (52)$$

Also let $f : \Delta^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ^2 .

- (i) The mapping $\tilde{H}$ is convex on Δ^2 .
- (ii) Define the mapping $\tilde{h} : [0, 1] \rightarrow \mathbb{R}$, $\tilde{h}(u) := \tilde{H}(u, u)$. Then $\tilde{h}$ is convex, monotonic nonincreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$ and one has the bounds

$$\begin{aligned} \inf_{u \in [0, 1]} \tilde{h}(u) &= \tilde{h}\left(\frac{1}{2}\right) = \frac{1}{16y_1^2 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, \frac{s_2+t_2}{2}\right) dt_2 ds_2 dt_1 ds_1, \\ \sup_{u \in [0, 1]} \tilde{h}(u) &= \tilde{h}(1) = \tilde{h}(0) = \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_2 ds_1. \end{aligned} \quad (53)$$

- (iii) One has the inequality

$$\tilde{h}(u) \geq \max\{h(u), h(1-u)\}. \quad (54)$$

Proof. (i) Let $(u_1, u_2), (v_1, v_2) \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since f is convex on Δ^2 , we have

$$\begin{aligned} \tilde{H}(\alpha(u_1, u_2) + \beta(v_1, v_2)) &= \tilde{H}(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2) = \frac{1}{16y_1^2 y_2^2} \\ &\cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f(\alpha(u_1 s_1 + (1-u_1)t_1, u_2 s_2 + (1-u_2)t_2) + \beta(v_1 s_1 + (1-v_1)t_1, v_2 s_2 + (1-v_2)t_2)) dt_2 ds_2 dt_1 ds_1 \\ &\leq \alpha \cdot \frac{1}{16y_1^2 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f(u_1 s_1 + (1-u_1)t_1, u_2 s_2 + (1-u_2)t_2) dt_2 ds_2 dt_1 ds_1 + \beta \cdot \frac{1}{16y_1^2 y_2^2} \\ &\cdot \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f(v_1 s_1 + (1-v_1)t_1, v_2 s_2 + (1-v_2)t_2) dt_2 ds_2 dt_1 ds_1 = \alpha \tilde{H}(u_1, u_2) + \beta \tilde{H}(v_1, v_2), \end{aligned} \quad (55)$$

which shows $\tilde{H}$ is convex on Δ^2 .

- (ii) Let $u_1, u_2, \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

$$\begin{aligned} \tilde{h}(\alpha u_1 + \beta u_2) &= \tilde{H}(\alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2) \\ &= \tilde{H}(\alpha(u_1, u_1) + \beta(u_2, u_2)) \end{aligned}$$

$$\leq \alpha \tilde{H}(u_1, u_1) + \beta \tilde{H}(u_2, u_2)$$

$$= \alpha \tilde{h}(u_1) + \beta \tilde{h}(u_2),$$

(56)

which shows the convexity of $\tilde{h}$ on $[0, 1]$.

As

$$\tilde{h}(u) = \tilde{H}(u, u) \leq \tilde{H}\left(\frac{1}{2}, \frac{1}{2}\right) = \tilde{h}\left(\frac{1}{2}\right) = \frac{1}{16y_1^2 y_2^2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} \int_{A_2-y_2}^{A_2+y_2} f\left(\frac{s_1+t_1}{2}, \frac{s_2+t_2}{2}\right) dt_2 ds_2 dt_1 ds_1, \quad (57)$$

$$\tilde{h}(u) = \tilde{H}(u, u) \leq \tilde{H}(0, 0) = \tilde{H}(0, 0) = \frac{1}{4y_1 y_2} \int_{A_1-y_1}^{A_1+y_1} \int_{A_2-y_2}^{A_2+y_2} f(s_1, s_2) ds_2 ds_1,$$

which proves the required bounds.

It is obvious that $\tilde{h}$ is monotonic nonincreasing on $[0, 1/2]$ and nondecreasing on $[1/2, 1]$.

(iii) As we know

$$\begin{aligned}\tilde{h}(u) &= \tilde{H}(u, u) \geq \max\{H(u, u), H(1-u, 1-u)\} \\ &= \max\{h(u), h(1-u)\}\end{aligned}\quad (58)$$

and the theorem is proved. $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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