Research Article

Graphs Generated by Measures

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In this paper, a graph is assigned to any probability measure on the σ -algebra of Borel sets of a topological space. Using this construction, it is proved that given any number n (finite or infinite) there exists a nonregular graph such that its clique, chromatic, and dominating number equals n .

1. Introduction

The distance between two vertices in a graph is the number of edges in a shortest path connecting them. Diameter of a graph is the longest path between two vertices of the graph.

A clique is a subset of vertices of an undirected graph, such that its induced subgraph is complete; that is, every two distinct vertices in the clique are adjacent. Clique number of a graph Γ is the number of vertices of maximum clique in the graph and denoted by $\omega(\Gamma)$. The chromatic polynomial counts the number of ways a graph can be colored using no more than a given number of colors and denoted by $\chi(\Gamma)$.

A set $D \subseteq V$ of vertices in a graph $\Gamma = (V, E)$ is a dominating set if every vertex $v \in V$ is an element of D or adjacent to an element of D . The domination number $\gamma(\Gamma)$ of a graph Γ is the minimum cardinality of a dominating set of Γ .

An independent set or stable set is a set of vertices in a graph, with no two of which being adjacent. A maximum independent set is an independent set of largest possible size for a given graph Γ . This size is called the independence number of Γ and denoted $\alpha(\Gamma)$.

Lots of graphs are constructed from algebraic objects such as semigroups ([1]), groups ([2, 3]), and rings ([4, 5]). Recently some graphs have been constructed from Hilbert and topological spaces ([6, 7]) which have nice graph theoretical properties. This motivates us to construct a bigger

category of graphs which has more properties. In this paper, we introduce a class of graphs generated by measures. Using this construction, we conclude that, given any number n (finite or infinite), there exists a nonregular graph such that its clique, chromatic, and dominating number coincides and is equal to n .

Given a measure space (X, Ω, μ) , a set $E \in \Omega$ is called an atom for μ , if E has a positive measure and for every $F \in \Omega$, either $\mu(E \cap F)$ or $\mu(E \setminus F)$ is zero. It is easily obtained by definition that if E is an atom for μ and $\mu(E \cap F) > 0$, then $E \cap F$ is also an atom for μ . μ is an atomic measure iff every measurable set of positive measure contains an atom. We may say that μ is nonatomic if there are no atoms for μ . Therefore, in a nonatomic measure μ , every nonatomic measurable set of positive measure can split into two disjoint measurable sets, where both of them have positive measure. One can easily see that the zero measure is the only measure which is atomic as well as nonatomic measure.

Let (X, Ω) be a measurable space and μ and ν be two measures on it. We say that ν is absolutely continuous with respect to μ , denoted by $\nu \ll \mu$ iff, for every measurable set F , $\nu(F) = 0$ whenever $\mu(F) = 0$.

We say that μ is singular with respect to ν and denoted by $\mu \perp \nu$ if, given any $E \in \Omega$, there exists some $F \subseteq E$, such that $\nu(E) = \nu(F)$ and $\mu(F) = 0$. One can easily see that in this case $\nu(E) = \sup\{\nu(E \cap F) : \mu(F) = 0\}$. It is easily seen that the following lemma holds.

Lemma 1. Two atomic measures are mutually singular if and only if the corresponding sets of atoms do not have a common element.

Johnson in [8] proved the following useful results about atomic and nonatomic measures.

Result 1. If E is an atom for μ and $\nu \ll \mu$, then either $\nu(E) = 0$ or E is an atom for ν .

Result 2. For a given measure μ on a σ algebra S , there are unique measures μ_1 and μ_2 , such that $\mu = \mu_1 + \mu_2$, $\mu_1 \perp \mu_2$, μ_1 is atomic, and μ_2 is nonatomic.

2. Definitions and Examples

Let X be a topological space and $\mathfrak{B}(X)$ be the σ -algebra of Borel subsets of X . Let also μ be a probability measure on $\mathfrak{B}(X)$. Set

$$\begin{aligned} \mathcal{C}(\mu) \\ := \{ \nu : \mathfrak{B}(X) \longrightarrow [0, 1], \nu \text{ is a measure, } \nu \ll \mu \}. \end{aligned} \quad (1)$$

In other words, $\mathcal{C}(\mu)$ is the collection of all absolutely continuous Borel measures with respect to μ . Define the following relation on $\mathcal{C}(\mu)$:

$$\nu_1 \sim \nu_2 \quad \text{if and only if} \quad \nu_1 \ll \nu_2, \nu_2 \ll \nu_1; \quad (2)$$

in other words, $\nu_1 \sim \nu_2$ if and only if for any measurable set E we have

$$\begin{aligned} \nu_1(E) = 0 &\iff \\ \nu_2(E) = 0. \end{aligned} \quad (3)$$

Clearly, $\sim$ is an equivalence relation on $\mathcal{C}(\mu)$. Denoting the equivalence class of ν by $[\nu]$, let

$$V(\mu) = \frac{\mathcal{C}(\mu)}{\sim} \setminus \{[\mu]\} = \{[\nu] : \nu \in \mathcal{C}(\mu)\} \setminus \{[\mu]\}. \quad (4)$$

For simplicity, we write ν for $[\nu]$.

We define a graph $\Gamma(\mu)$ with the set of vertices $V(\mu)$, and two vertices ν_1 and ν_2 are adjacent if and only if ν_1 and ν_2 are mutually singular. Clearly, the adjacency defined above is well-defined. The collection of all edges of $\Gamma(\mu)$ is denoted by $E(\mu)$. Therefore, we may assign a graph $\Gamma(\mu) = (V(\mu), E(\mu))$ to any probability Borel measure μ on X .

Example 2. Let $a \in X$. If $\mu = \delta_a$ is the dirac measure concentrated on a , then $V(\mu) = \{\delta_a\}$ and $E(\mu) = \emptyset$; so $\Gamma(\mu)$ consists of a vertex with no edges.

Example 3. Let $a, b \in X$, $a \neq b$, and $\lambda \in (0, 1)$. Let $\mu = \lambda\delta_a + (1 - \lambda)\delta_b$. Then $V(\mu) = \{\delta_a, \delta_b\}$ where δ_a and δ_b are adjacent.

$$\delta_b \text{ --- } \delta_a \quad (5)$$

Example 4. Let $\lambda_i \in (0, 1)$ ($i = 1, 2, 3$) be such that $\sum_{i=1}^3 \lambda_i = 1$ and a_i ($i = 1, 2, 3$) be three distinct elements of X . If $\mu = \lambda_1\delta_{a_1} + \lambda_2\delta_{a_2} + \lambda_3\delta_{a_3}$ then

$$V(\mu) = \{\delta_{a_1}, \delta_{a_2}, \delta_{a_3}, \delta_{a_1 a_2}, \delta_{a_1 a_3}, \delta_{a_2 a_3}\}, \quad (6)$$

where

$$\delta_{a_i a_j} = \lambda_i \delta_{a_i} + \lambda_j \delta_{a_j} \quad (i, j = 1, 2, 3, i \neq j). \quad (7)$$

Also, each δ_{a_i} is adjacent to δ_{a_j} for $i \neq j$. $\delta_{a_i a_j}$ is adjacent to δ_{a_k} if $i, j \neq k$. No other vertices are adjacent.

The following example is indeed the generalization of the previous examples and presents the graph corresponding to a finitely atomic measure.

Example 5. Let $\mu = \sum_{i=1}^n \lambda_i \delta_{a_i}$ be a finitely atomic measure with finite number of atoms where a_i 's are distinct points in X and $\lambda_i \in (0, 1)$ is such that $\sum_{i=1}^n \lambda_i = 1$. Then

$$\begin{aligned} V(\mu) \\ = \{ \delta_{a_{i_1} a_{i_2} \dots a_{i_k}} : 1 \leq k \leq n-1, i_s \neq i_l \text{ for } 1 \leq s, l \leq k \}, \end{aligned} \quad (9)$$

where

$$\delta_{a_{i_1} a_{i_2} \dots a_{i_k}} = \sum_{j=1}^k \lambda_{i_j} \delta_{a_{i_j}}. \quad (10)$$

By Lemma 1, two vertices $\nu_1 = \delta_{a_{i_1} a_{i_2} \dots a_{i_k}}$ and $\nu_2 = \delta_{a_{j_1} a_{j_2} \dots a_{j_l}}$ are adjacent if and only if $a_{i_s} \neq a_{j_t}$ for all $1 \leq s \leq k$ and $1 \leq t \leq l$.

3. The Graphs of Finitely Atomic Measures

In the rest of the paper, by an atomic measure, we mean a finitely atomic measure with finite number of atoms. In this section, we study the graph corresponding to atomic measures and state some of its properties. The case of atomic measures is of special interest since the corresponding graph is finite and may be visualized and enumerated.

The following theorem satisfies for atomic measures.

Theorem 6. For an atomic measure $\mu = \sum_{i=1}^n \lambda_i \delta_{a_i}$ one has the following.

- (1) $V(\mu) = \cup_{k=1}^n V_k$, where $V_k = \{ \delta_{a_{i_1} a_{i_2} \dots a_{i_k}} : 1 \leq i_j \leq n \}$ and V_k 's are disjoint.
- (2) $\deg(v) = 2^{n-k} - 1$ for any $v \in V_k$.

Proof. Part (1) holds, since the support of each element of $V(\mu)$ should be contained in $\{a_1, a_2, \dots, a_n\}$.

To prove part (2), let $v \in V_k$ and then $v = \sum_{j=1}^k \lambda_{a_{i_j}} \delta_{a_{i_j}}$. As mentioned in Example 5, a vertex w is adjacent to v if and only if $\{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \cap \text{supp}(w) = \emptyset$. Hence w should be of the form $w = \sum_{i_l \neq i_1, \dots, i_k} \lambda_{i_l} \delta_{i_l}$.

The number of such vertices is

$$\binom{n-k}{1} + \binom{n-k}{2} + \dots + \binom{n-k}{n-k} = 2^{n-k} - 1 \quad (11)$$

which completes the proof. $\square$

Theorem 7. For an atomic measure $\mu = \sum_{i=1}^n \lambda_i \delta_{a_i}$, ($n > 3$) one has the following:

- (1) $\omega(\Gamma(\mu)) = \chi(\Gamma(\mu)) = \gamma(\Gamma(\mu)) = n$.
- (2) $\text{diam}(\Gamma(\mu)) = 3$.

Proof. (1) Since $\{\delta_{a_1}, \delta_{a_2}, \dots, \delta_{a_n}\}$ is a clique set for $\Gamma(\mu)$, it follows that $\omega(\Gamma(\mu)) = n$.

Since all elements of $\{\delta_{a_1}, \delta_{a_2}, \dots, \delta_{a_n}\}$ are pairwise adjacent then $\chi(\Gamma(\mu)) \geq n$. To prove the equality, we present a coloring of $\Gamma(\mu)$ with exactly n colors. For $i \in \{1, 2, 3, \dots, n\}$, let c_i be a color assigned to the vertex δ_{a_i} . Now, for $v = \delta_{a_{i_1} a_{i_2} \dots a_{i_k}} = \sum_{j=1}^k \lambda_{i_j} \delta_{a_{i_j}}$, assign the color c_{i_v} , where

$$i_v := \min \{i_j : j = 1, 2, \dots, k\}. \quad (12)$$

Clearly, there are exactly n colors used for the previous coloring. Also, no adjacent vertices have the same colors, since the two vertices $v_1 = \delta_{a_{i_1} a_{i_2} \dots a_{i_k}}$ and $v_2 = \delta_{a_{j_1} a_{j_2} \dots a_{j_l}}$ are adjacent if and only if $\{i_1, i_2, \dots, i_k\} \cap \{j_1, j_2, \dots, j_l\} \neq \emptyset$; therefore

$$\min \{i_1, i_2, \dots, i_k\} \neq \min \{j_1, j_2, \dots, j_l\}; \quad (13)$$

so $c_{i_{v_1}} \neq c_{i_{v_2}}$. It completes the proof of $\chi(\Gamma(\mu)) = n$.

Finally, since $\{\delta_{a_1}, \delta_{a_2}, \dots, \delta_{a_n}\}$ is a dominating set for $\Gamma(\mu)$ then $\gamma(\Gamma(\mu)) \leq n$.

Now suppose $D = \{\mu_0, \dots, \mu_{n-l}\}$, $1 < l < n+1$, is a dominating set for $\Gamma(\mu)$. Set D_i the set of atoms in which their coefficients in μ_i are nonzero. $D' = \{D_0, \dots, D_{n-l}\}$ is a poset with the inclusion relation. By changing the indices we can write $D' = \{D_0, \dots, D_{r-1}, D_r, \dots, D_{n-l}\}$, where any D_i is a minimal set for every $0 \leq i < r$ and is not minimal set for other indices. By the axiom of choice, for every $0 \leq i < r$, one can choose an atom $\alpha_i \in D_i \setminus D_{i+1}$, where $i \equiv \bar{i}$ (module r) and for every $r \leq i \leq n-l$ one can choose $\alpha_i \notin D_i$.

Now define $\mu^* = \sum_{i=0}^{n-l} (1/(n-l+1)) \delta_{\alpha_i}$. It is easy to verify that μ^* is neither adjacent nor equal to any element of D , which is contradiction to the assumption that D is a dominating set.

(2) Since each vertex of $\Gamma(\mu)$ is adjacent to at least an element of the set $V_1 = \{\delta_{a_1}, \delta_{a_2}, \dots, \delta_{a_n}\}$ and all vertices in V_1 are adjacent, then $d(u, v) \leq 3$ for all $u, v \in V(\mu)$. On the other hand, set $\mu_1 := \delta_{a_1 a_2 \dots a_{n-1}}$ and $\mu_2 := \delta_{a_2 a_3 \dots a_n}$. Clearly, the path

$$\mu_1 \mapsto \delta_{a_n} \mapsto \delta_{a_1} \mapsto \mu_2 \quad (14)$$

is of minimum length; therefore $d(\mu_1, \mu_2) = 3$; consequently

$$\text{diam}(\Gamma(\mu)) = \max \{d(u, v) : u, v \in \Gamma(\mu)\} = 3. \quad (15)$$

$\square$

The following theorem determines the automorphism group of $\Gamma(\mu)$, where μ is an atomic measure.

Theorem 8. For any atomic measure $\mu = \sum_{i=1}^n \lambda_i \delta_{a_i}$ one has

$$\text{Aut}(\Gamma(\mu)) \cong S_n, \quad (16)$$

where S_n is the group of permutations on n .

Proof. Let $\sigma \in S_n$. Define the mapping

$$\sigma^* : V(\mu) \longrightarrow V(\mu) \quad (17)$$

by

$$\sigma^* \left(\delta_{a_{i_1} \dots a_{i_k}} \right) := \delta_{a_{\sigma(i_1)} \dots a_{\sigma(i_k)}}. \quad (18)$$

Clearly σ^* is a bijection; so $\sigma^* \in \text{Aut}(\Gamma(\mu))$. Now, define

$$\Phi : S_n \longrightarrow \text{Aut}(\Gamma(\mu)) \quad (19)$$

by

$$\Phi(\sigma) := \sigma^*. \quad (20)$$

It is easily seen that Φ is an injective group homomorphism. To complete the proof, it is enough to show that Φ is surjective.

To do this, let $\beta \in \text{Aut}(\Gamma(\mu))$. For any given $i \in \{1, 2, \dots, n\}$, in light of Theorem 6, since the number of adjacent measures should remain the same by β we have

$$\beta(\delta_i) = \delta_j \quad (21)$$

for some $j \in \{1, 2, \dots, n\}$.

Let $\sigma \in S_n$ be defined by $\sigma(i) = j$. So

$$\beta(\delta_{a_i}) = \delta_{a_{\sigma(i)}}. \quad (22)$$

We show that $\beta = \sigma^*$. Let $\delta_{a_{i_1} \dots a_{i_k}}$ be any given element of $V(\mu)$. Let also

$$\beta(\delta_{a_{i_1} \dots a_{i_k}}) = \delta_{a_{j_1} \dots a_{j_k}}. \quad (23)$$

We should show that

$$\delta_{a_{j_1} \dots a_{j_k}} = \delta_{a_{\sigma(i_1)} \dots a_{\sigma(i_k)}} \quad (24)$$

or equivalently

$$\{j_1, j_2, \dots, j_k\} = \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}. \quad (25)$$

Let $j \notin \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}$. Since σ is surjective, there exists a natural number $i \notin \{i_1, i_2, \dots, i_k\}$ such that $\sigma(i) = j$. Note that, since $i \notin \{i_1, i_2, \dots, i_k\}$ then the vertices δ_{a_i} and

$\delta_{a_{i_1} \dots a_{i_k}}$ are adjacent. Since $\beta \in \text{Aut}(\Gamma(\mu))$, the vertices $\beta(\delta_{a_i})$ and $\beta(\delta_{a_{i_1} \dots a_{i_k}})$ are adjacent which means that the vertices $\delta_{a_{\sigma(i)}}$ and $\delta_{a_{j_1} \dots a_{j_k}}$ are adjacent; therefore $j = \sigma(i) \notin \{j_1, j_2, \dots, j_k\}$.

The previous discussion results in

$$\{j_1, j_2, \dots, j_k\} \subseteq \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\}, \quad (26)$$

but the two sets have the same cardinality, namely, k ; therefore

$$\{j_1, j_2, \dots, j_k\} = \{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)\} \quad (27)$$

which completes the proof. $\square$

Theorem 9. Let $\mu = \sum_{i=1}^n \lambda_i \delta_{a_i}$ be an atomic measure; then $\alpha(\Gamma(\mu)) = 2^{n-1} - 1$.

Proof. For $0 \leq k, s \leq n-2$ and $2 \leq i_a, j_b \leq n$, two vertices $x = \delta_{1a_{i_1} \dots a_{i_k}}$ and $y = \delta_{1a_{j_1} \dots a_{j_s}}$ are not adjacent, since 1 is a common index for both x and y . Therefore, the set

$$I = \{\delta_{1a_{i_1} \dots a_{i_k}} \mid 0 \leq k \leq n-2, 2 \leq i_1, \dots, i_k \leq n\} \quad (28)$$

is an independent set. By elementary combinatorial principles of counting we have

$$|I| = \binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{n-2} = 2^{n-1} - 1. \quad (29)$$

Thus we have an independent set with cardinality $2^{n-1} - 1$.

Observe that $|V| = 2^n - 2$, since any vertex can be written as $\delta_{a_{i_1} \dots a_{i_k}}$, where $\{a_{i_1}, \dots, a_{i_k}\}$ is a nonempty proper subset of $\{a_1, \dots, a_n\}$.

Now suppose that A is an independent set of $\Gamma(\mu)$. For every $\delta_{a_{i_1} \dots a_{i_k}} \in A$, define $\delta'_{a_{i_1} \dots a_{i_k}} = \delta_{a_{j_1} \dots a_{j_s}}$, where $\{i_1, \dots, i_k\}^c = \{j_1, \dots, j_s\}$. Obviously for every $\delta_{a_{i_1} \dots a_{i_k}} \in A$, $\delta'_{a_{i_1} \dots a_{i_k}}$ is adjacent to $\delta_{a_{i_1} \dots a_{i_k}}$; hence $\delta'_{a_{i_1} \dots a_{i_k}} \notin A$.

For distinct δ 's, relative δ 's are distinct; thus we will have $|A| \leq |A^c|$. Finally, we can say

$$|A| \leq \frac{|V|}{2} = \frac{2^n - 2}{2} = 2^{n-1} - 1. \quad (30)$$

$\square$

4. General Measures

In the previous section, we discussed the graph assigned to atomic measures. In this section we study the general case.

Theorem 10. For a given measure μ , if α and β are two vertices of $\Gamma(\mu)$, one has the following.

- (1) $d(\alpha, \beta) = 1$ iff α and β are mutually singular.
- (2) $d(\alpha, \beta) = 2$ iff α and β are not mutually singular, but for some measurable set E with positive measure, one has $\alpha(E) = \beta(E) = 0$.
- (3) $d(\alpha, \beta) = 3$ iff α and β are not mutually singular, but for no measurable set E with positive measure, one has $\alpha(E) = \beta(E) = 0$.

Proof. (1) It is done by definition.

(2) If the conditions hold, then by first part $d(\alpha, \beta) > 1$. Define the measure γ by $\gamma(A) = \mu(E \cap A)$ for every $A \in \mathfrak{B}(X)$. Obviously $\gamma \ll \mu$. Since $\mu(E) > 0$, then $\gamma \neq 0$. By the definition of E , one can obtain $\mu \neq \gamma$. We also have

$$\begin{aligned} \alpha(E) &= \beta(E) = 0, \\ \gamma(E^c) &= 0. \end{aligned} \quad (31)$$

That is, $\gamma \perp \alpha$ and $\gamma \perp \beta$. Thus $\alpha \sim \gamma \sim \beta$ is a path and $d(\alpha, \beta) \leq 2$; therefore we have $d(\alpha, \beta) = 2$.

Now suppose that $d(\alpha, \beta) = 2$; thus for some $\gamma \ll \mu$, we may have $\gamma \perp \alpha$ and $\gamma \perp \beta$. Therefore, for some nonzero μ -measurable sets G and F , we will have

$$\begin{aligned} \alpha(G) &= \beta(F) = 0, \\ \gamma(G^c) &= \gamma(F^c) = 0; \end{aligned} \quad (32)$$

thus we have

$$\gamma(G^c \cup F^c) = 0. \quad (33)$$

If $\gamma(G \cap F) = 0$ then $\gamma((G^c \cup F^c)^c) = \gamma(G \cap F) = 0$; thus $\gamma = 0$, a contradiction. Thus $\gamma(G \cap F) \neq 0$; hence $\mu(G \cap F) \neq 0$. It is enough to take $E = G \cap F$.

(3) If $d(\alpha, \beta) = 3$, none of the conditions of part (1) and (2) hold; thus the condition mentioned in this part will occur.

Now if the conditions of part (3) hold, by definition of α and β , since $V(\mu)$ does not contain μ (indeed $[\mu]$), for some nonzero μ -measurable sets E and F , we have

$$\begin{aligned} \alpha(E) &= \beta(F) = 0, \\ \alpha(F) &> 0, \\ \beta(E) &> 0, \end{aligned} \quad (34)$$

where $\mu(E \cap F) = 0$. Now we can define two measures γ and δ by

$$\begin{aligned} \gamma(A) &= \mu(A \cap E), \quad \forall A \in \mathfrak{B}(X), \\ \delta(A) &= \mu(A \cap F), \quad \forall A \in \mathfrak{B}(X). \end{aligned} \quad (35)$$

Thus we will have $\alpha \sim \gamma \sim \delta \sim \beta$, which shows $d(\alpha, \beta) \leq 3$.

By parts (1) and (2), we can deduce that $d(\alpha, \beta) = 3$. $\square$

By Result 2, every measure μ can be decomposed into two mutually singular measures: one is atomic and the other is nonatomic. We call the nonatomic measure the nonatomic part of μ and the other one the atomic part.

Theorem 11. If the nonatomic part of μ is nonzero, then $\omega(\Gamma) = \infty$.

Proof. In light of Result 2, suppose that $\mu = \mu_1 + \mu_2$, where μ_1 is the atomic and μ_2 is the nonatomic part of μ .

Since $\mu_1 \perp \mu_2$, for some $E \in \mathfrak{B}(X)$, we will have $\mu_1(E) = 0$ and $\mu_2(E^c) = 0$. So, $\mu(F) = \mu_2(F) \forall F \subseteq E$. Set $F_1 = E$. Since μ_2 is nonatomic, we can split F_1 into two

disjoint measurable sets with positive μ_2 -measure. Let F_2 be one of them. By continuing the previous process, there exists a sequence $E = F_1 \supset F_2 \supset \dots$ of nonzero μ_2 -measurable sets, and consequently a sequence $E_i = F_i - F_{i-1}$ with positive μ_2 -measure and therefore positive μ -measure.

Now define $\nu_i(A) = \mu(A \cap E_i)$ for all $i \in \mathbb{N}$. It is easily seen that ν_i 's are mutually singular and $\nu_i \ll \mu$ for all $i \in \mathbb{N}$, that is, $\{\nu_i \mid i \in \mathbb{N}\}$ is a clique of Γ . $\square$

Corollary 12. *If the nonatomic part of μ is nonzero then $\omega(\Gamma) = \chi(\Gamma) = \infty$.*

In the next theorem, we consider the domination number as well.

Theorem 13. *If the nonatomic part of μ is nonzero then the domination number of $\Gamma = \Gamma(\mu)$ cannot be finite.*

Proof. Let $\mu = \mu_1 + \mu_2$, where μ_1 is the atomic and μ_2 is the nonatomic part of μ . Suppose that $\mathcal{D} = \{\nu_1, \dots, \nu_n\}$ is a dominating set of Γ .

Let $\mathcal{A} = \{A \in \mathfrak{B}(X) : \exists \nu_i \in \mathcal{D}, \nu_i(A) \neq 0, \text{ and } \mu_2(A) \neq 0\}$. $\mathcal{A}$ is a poset with inclusion relation. Set B as the minimal element of $\mathcal{A}$. Suppose that C and D are two subsets of B such that $\mu_2(C) < \mu_2(D) < \mu(B)$ and consequently, $\mu(C) < \mu(D) < \mu(B)$.

For every $1 \leq i \leq n$, we can choose a nonzero measurable set B_i with $\nu_i(B_i) = 0$ and $B_i \cap D = \emptyset$. Let $E = (\cup_{i=1}^n B_i) \cup C$.

Now define the measure η by $\eta(A) = \mu(E \cap A)$ which is a measure different from all of ν_i 's and adjacent to none of them. A contradiction with the hypothesis is that $\mathcal{D}$ is a dominating set. Therefore, every dominating set is infinite. $\square$

Theorem 14. *If the nonatomic part of μ is nonzero, then $\alpha(\Gamma) = \infty$.*

Proof. By the hypothesis of the theorem, there is some measurable set A of positive measure which does not contain any atom and hence there is a sequence of measurable sets

$$A = A_1 \supset A_2 \supset A_3 \supset \dots \quad (36)$$

such that

$$\mu(A) = \mu(A_1) > \mu(A_2) > \mu(A_3) > \dots > 0. \quad (37)$$

Now define the sequence of measures ν_i by $\nu_i(E) = \mu(A_i \cap E)$ for $i \in \mathbb{N}$. By definition of ν_i 's we have $\nu_i \ll \mu$, ($i \in \mathbb{N}$). Now suppose that for some $i < j$ we have $\nu_i \perp \nu_j$. Thus for A_j , there is some $F \subseteq A_j$, where $\nu_j(A_j) = \nu_j(F)$ and $\nu_i(F) = 0$. But by definition $\nu_j \ll \nu_i$ which is a contradiction. Therefore, we conclude that the set $\{\nu_i : i \in \mathbb{N}\}$ is an independent set of Γ , which is not finite and the theorem is proved. $\square$

Applying the material presented in this paper, we will have the following corollary.

Corollary 15. *Given any number n (finite or infinite), there exists a nonregular graph Γ such that*

$$\begin{aligned} \omega(\Gamma) &= \chi(\Gamma) = \gamma(\Gamma) = n, \\ \alpha(\Gamma) &= 2^{n-1} - 1. \end{aligned} \quad (38)$$

Competing Interests

The authors declare that they have no competing interests.

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