Research Article

Nonsolvable Subalgebras of $\mathfrak{gl}(4, \mathbb{R})$

Ryad Ghanam¹ and Gerard Thompson²

¹Department of Mathematics, Virginia Commonwealth University in Qatar, P.O. Box 8095, Doha, Qatar

²Department of Mathematics, University of Toledo, Toledo, OH 43606, USA

Correspondence should be addressed to Ryad Ghanam; raghanam@vcu.edu

Received 17 June 2016; Accepted 26 July 2016

Academic Editor: Rutwig Campoamor-Stursberg

Copyright © 2016 R. Ghanam and G. Thompson. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

All the simple and then semisimple subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ are found. Each such semisimple subalgebra acts by commutator on $\mathfrak{gl}(4, \mathbb{R})$. In each case the invariant subspaces are found and the results are used to determine all possible subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ that are not solvable.

1. Introduction

In this paper we consider the problem of classifying the Lie subalgebras of $\mathfrak{gl}(4, \mathbb{R})$. In Section 2 we will explain in what sense we classify the Lie subalgebras. It is a well known fact in Lie theory that every Lie algebra $\mathfrak{g}$ admits a semidirect Levi decomposition; that is to say, $\mathfrak{g} \approx \sigma \rtimes \rho$, where σ is a semisimple subalgebra and ρ is a solvable ideal. We will divide the project into two parts. First of all we will find all the Levi algebras where σ is nonzero. In a separate venue we will examine the solvable algebras.

Details about the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ and $\mathfrak{gl}(3, \mathbb{R})$ may be found in [1]. One could of course opt to classify the Lie subalgebras of $\mathfrak{sl}(4, \mathbb{R})$ instead of $\mathfrak{gl}(4, \mathbb{R})$, which has the advantage of avoiding many trivial and obvious subalgebras; however, the drawback is that in many other cases one has to impose a somewhat arbitrary condition on the parameters in order to obtain a subalgebra of $\mathfrak{sl}(4, \mathbb{R})$ rather than $\mathfrak{gl}(4, \mathbb{R})$ so we have opted to stick with the latter.

Every abstract semisimple subalgebra is a direct sum of simple subalgebras. By “abstract” here we mean that the given subalgebra does not necessarily appear as a subalgebra of $\mathfrak{gl}(k, \mathbb{R})$ for some k , although Ado’s theorem informs us that such k must exist. For small values of k it is feasible to find all the irreducible representations of such a simple subalgebra in $\mathfrak{gl}(k, \mathbb{R})$ and hence, up to isomorphism, all the representations of the semisimple subalgebra, which are direct sums of irreducible representations.

Suppose that we start with a certain semisimple subalgebra $\sigma \subset \mathfrak{gl}(n, \mathbb{R})$. We will work with that particular representation and not change the semisimple part of the algebra until we have found all possible subalgebras that have σ in that representation as its semisimple part. Then σ acts on $\mathfrak{gl}(n, \mathbb{R})$ as a Lie algebra by $s(g) = [s, g]$; that is, commutator defines a Lie algebra homomorphism from σ into $\text{End}(\mathfrak{gl}(n, \mathbb{R}))$, as readily follows from the Jacobi identity. Since σ is semisimple, this representation must be completely reducible. To construct Levi subalgebras of $\mathfrak{gl}(n, \mathbb{R})$ we have to find the irreducible submodules or invariant subspaces of this representation. A particular submodule may or may not define a solvable subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. In order to find all possible Levi subalgebras corresponding to a particular σ we have to see if it is possible to add any of the submodules together so as to define a larger solvable subalgebra. However, it is not true that every submodule is a sum of the basic submodules; see, for example, the six-dimensional subalgebra $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3,1}$ where two submodules are “correlated.”

Thus we have the following four-step procedure:

- (i) Find all simple subalgebras of $\mathfrak{gl}(n, \mathbb{R})$ for the given value of n .
- (ii) Find all possible representations (up to change of basis) of these simple subalgebras of $\mathfrak{gl}(n, \mathbb{R})$.
- (iii) Find all semisimple (not simple) subalgebras of $\mathfrak{gl}(n, \mathbb{R})$ and their representations.

- (iv) Find Levi algebras by adding together solvable invariant submodules if possible.

In Section 2 we will give some results that apply to $\mathfrak{gl}(n, \mathbb{R})$ in general. In Section 3 we discuss the semisimple algebras of $\mathfrak{gl}(4, \mathbb{R})$ and their representations. In Section 4 we make some remarks about the classification of the low-dimensional Lie algebras in general which serve to identify the radicals of the Levi algebras. In Section 5 we consider the submodules of $\mathfrak{gl}(4, \mathbb{R})$ that occur when a certain semisimple algebra in a particular representation acts on $\mathfrak{gl}(4, \mathbb{R})$ by commutator and how they may be combined so as to give subalgebras of $\mathfrak{gl}(4, \mathbb{R})$. In Section 6 we list all representations of the semisimple algebras of $\mathfrak{gl}(4, \mathbb{R})$ and in Section 7 all the Levi subalgebras with the understanding that the companion subalgebras (see Section 2) are not listed: the subalgebras are grouped together accordingly as they determine the same abstract subalgebra. Section 8 lists three seven-dimensional solvable Lie algebras called B_1, B_2, B_3 , respectively, that occur in the classification and finally Section 9 gives the proof of a Lemma.

One of the compelling reasons to construct the subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ is to continue the program of finding minimal dimensional representations of all the low-dimensional Lie algebras: for more about this subject see [2]. Finally in the near future we hope to report in a suitable venue on similar results for the Lie algebras $\mathfrak{gl}(5, \mathbb{R})$ and $\mathfrak{gl}(6, \mathbb{R})$.

2. Some General Results

We quote the following result from [3].

Proposition 1. *Let V be a finite-dimensional vector space over a closed field of characteristic zero and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a nonzero Lie algebra acting irreducibly on V . Then $\mathfrak{g} \approx \sigma \oplus \rho$, where σ is semisimple and ρ consists of multiples of the identity.*

We can also apply this result to real Lie algebras provided that in tensoring with $\mathbb{C}$ the original representation does not become reducible.

Proposition 2. *Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a nonzero Lie algebra of the form $\mathfrak{g} \approx \sigma \rtimes \rho$, where σ is semisimple and ρ is solvable. If σ acts irreducibly on ρ then ρ must be abelian.*

Indeed, σ acts on the derived algebra $[\rho, \rho]$ and if σ acts irreducibly it can only be that $[\rho, \rho] = 0$ since $[\rho, \rho] \neq \rho$ because ρ is solvable. Thus, we can be sure that when we decompose the action of σ on $\mathfrak{gl}(n, \mathbb{R})$ into irreducible submodules, those submodules which correspond to subalgebras must be abelian.

We remark that from work of Dynkin [4] it follows that the maximal dimensional proper subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ is of dimension $n(n-1) + 1$ and is isomorphic to a semidirect product of $\mathfrak{sl}(n, \mathbb{R})$ and an n -dimensional solvable subalgebra. In the case at hand we obtain a subalgebra of dimension 13.

Now we will address the issue of in what sense we classify the nonsolvable subalgebras of $\mathfrak{gl}(4, \mathbb{R})$. We will certainly want to say that two subalgebras that are conjugate are

equivalent; that is to say, all the matrices in one subalgebra are conjugate via a matrix P to the matrices in the second subalgebra. We note also that for any matrix Lie algebra $\mathfrak{g}$ the map $\theta(X) = -X^t$ (negative transpose) defines an automorphism of $\mathfrak{g}$ of period two although two such subalgebras related in this way are not necessarily conjugate. As a result of combining these two notions of equivalence, as is explained in [2], if one transposes about the antidiagonal and takes a negative, one obtains a second representation of a given abstract Lie algebra, which again may not be conjugate to the first. We will refer to both of these representations as *companions* of the first one and, in the interests of efficiency, we will not write them down as separate representations. We will list subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ that are block upper triangular to the extent possible.

As an illustration we give below several companion representations of the smallest dimensional Lie algebra $A_{5,40}$ [5] that has a nontrivial Levi decomposition, where $a, b, c, d, e \in \mathbb{R}$ are arbitrary:

$$\begin{aligned} & \begin{bmatrix} a & b & d \\ c & -a & e \\ 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ d & e & 0 \end{bmatrix} \\ & \begin{bmatrix} 0 & d & e \\ 0 & a & b \\ 0 & c & -a \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 & 0 \\ d & a & b \\ e & c & -a \end{bmatrix}. \end{aligned} \tag{1}$$

3. Semisimple Algebras of $\mathfrak{gl}(4, \mathbb{R})$ and Their Representations

The real simple subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ consist, as abstract Lie algebras, of $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{so}(3)$, $\mathfrak{so}(3, 1)$, $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{sp}(4)$, and $\mathfrak{sl}(4, \mathbb{R})$ and any semisimple subalgebra must be a direct sum of these six. This list of subalgebras is dictated by the requirement that the dimension of such an algebra, as an abstract Lie algebra, should have dimension less than seventeen and should have a representation in $\mathfrak{gl}(4, \mathbb{R})$. In terms of finding representations in $\mathfrak{gl}(4, \mathbb{R})$, the minimal dimension representations of $\mathfrak{so}(3, 1)$, $\mathfrak{sp}(4)$, and $\mathfrak{sl}(4, \mathbb{R})$ are the definition representations and are unique up to isomorphism in $\mathfrak{gl}(4, \mathbb{R})$.

As regards $\mathfrak{sl}(3, \mathbb{R})$, it can only occur, up to isomorphism, as a subalgebra of $\mathfrak{gl}(4, \mathbb{R})$ in its definition representation augmented by an extra row and column of zeros. The cases $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$ remain to be discussed. In Section 6 the reader will find a list of all possible representations in $\mathfrak{gl}(4, \mathbb{R})$ corresponding to the simple subalgebras. In that context the

lower Roman letters are arbitrary and parameterize the space of matrices for each representation. Up to isomorphism there are four representations of $\mathfrak{sl}(2, \mathbb{R})$ and two of $\mathfrak{so}(3)$.

Let us turn now to *semisimple* algebras of $\mathfrak{gl}(4, \mathbb{R})$ noting that they are direct sums of simple subalgebras. Immediately, Schur's lemma shows that each of $\mathfrak{so}(3, 1)$, $\mathfrak{sp}(4)$, and $\mathfrak{sl}(4, \mathbb{R})$ cannot be extended to a semisimple not simple subalgebra. Furthermore, the centralizer of $\mathfrak{sl}(3, \mathbb{R})$ in the representation described above is two-dimensional and so it too cannot be similarly extended. As regards $\mathfrak{so}(3)$, the only possible extension to either of its two inequivalent irreducible representations in $\mathfrak{gl}(4, \mathbb{R})$ gives $\mathfrak{so}(4)$, the full space of skew-symmetric matrices. Finally, for $\mathfrak{sl}(2, \mathbb{R})$ there are two possibilities, one being the block-diagonal representation of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and a second one that can be described as $\begin{bmatrix} A+dI_2 & eI_2 \\ fI_2 & A-dI_2 \end{bmatrix}$, where $A \in \mathfrak{sl}(2, \mathbb{R})$ and I_2 is the 2×2 identity matrix and $d, e, f \in \mathbb{R}$.

All thirteen of the semisimple algebras of $\mathfrak{gl}(4, \mathbb{R})$ and their faithful representations are summarized in Section 6.

4. Classification of the Low-Dimensional Lie Algebras

Having obtained all the semisimple algebras of $\mathfrak{gl}(4, \mathbb{R})$ the task now is to find in each case the possible solvable radicals that can be added so as to produce a Levi subalgebra. In order to describe these radicals it will be helpful to describe the state of the art in the classification of the low-dimensional Lie algebras, a program that can be traced back to Lie himself [6]. Such classifications have been attempted many times but are bedeviled by the fact that Lie algebras in general belong to continuous families and so the situation is much more complicated than for the semisimple algebras. We will follow the classifications obtained by Mubarakzyanov [7, 8], who produced a list of the isomorphism classes of the real indecomposable Lie algebras up to dimension five that is generally accepted as being correct. That list is readily accessible in [5].

The situation in dimension six is altogether more complicated. As regards solvable indecomposable Lie algebras there are three classes, the nilpotent Lie algebras, and the algebras that have nilradicals of dimensions five and four, respectively. Morozov classified the six-dimensional nilpotent Lie algebras, a first version of which was produced by Umlauf [9] as early as 1891. In fact Morozov's list was improved in [10] and it was shown that the six-dimensional nilpotents do not belong to continuous families. Mubarakzyanov [11] himself classified the solvable indecomposable Lie algebras of dimension six that have a five-dimensional nilradical and [12] attempts to correct the obvious flaws in his work. Turkowski [13] classified the solvable indecomposable Lie algebras of dimension six that have a four-dimensional nilradical. Finally in another article Turkowski [14] classified the solvable indecomposable Levi algebras of dimensions between five and eight that have a nontrivial Levi decomposition. Following the original notation of the cited authors [7, 8, 11, 13, 14], the algebras of dimension up to and including five will be denoted by $A_{k,m}$ where k denotes the dimension of the algebra and m the

number in the author's list. A similar notation applies to the six-dimensional Mubarakzyanov algebras [11] and $N_{k,m}$ for the solvable six-dimensional algebras classified by Turkowski [13]. The Levi algebras classified by Turkowski [14] will be denoted by $L_{k,m}$: the algebras $A_{5,40}$ and $L_{5,1}$ coincide. We will not need to refer to the six-dimensional nilpotent algebras. Finally, the nonabelian two-dimensional algebra is denoted by $A_{2,1}$. We must mention also the recent encyclopedic book by Šnobl and Winternitz [15] on the subject of the classification of the low-dimensional Lie algebras for many more references. For lack of space in Section 7 we are not able to give the specific isomorphism that identifies the radical; moreover, when we refer to an algebra such as $A_{4,5a=b=c=1}$ the letters pertain to the classifications obtained in the references given in this paragraph and *not* to the same letters that serve as coordinates for a particular subalgebra that occurs in our list. In the construction of the Levi algebras, three seven-dimensional solvable Lie algebras occur that are called B_1, B_2, B_3 , respectively, that do not belong to established lists of algebras and therefore we give their structure equations in Section 8.

5. Finding Subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ with a Nontrivial Levi Decomposition

Now we consider in turn each of the semisimple subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ and their particular representations that are given in Section 6; find the invariant submodules and see which of them may be combined so as to produce a Levi subalgebra. In the case where the representation is irreducible we find an *invariant* complement.

5.1. $\mathfrak{sl}(4, \mathbb{R})$. For completeness we note the obvious fact that multiples of the identity provide a complement in $\mathfrak{gl}(4, \mathbb{R})$ to $\mathfrak{sl}(4, \mathbb{R})$ and yields the only possible nonzero radical.

5.2. $\mathfrak{sp}(4)$. The definition of $\mathfrak{sp}(4)$ can be found in Section 6. Besides multiples of the identity that span the kernel of the representation of $\mathfrak{sp}(4)$ in $\mathfrak{gl}(4, \mathbb{R})$, it is easily checked that there is a unique five-dimensional irreducible invariant subspace given by $\begin{bmatrix} d & \beta J \\ \gamma J & d^t \end{bmatrix}$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $d \in \mathfrak{sl}(2, \mathbb{R})$ and $\beta, \gamma \in \mathbb{R}$. Since this submodule is not a subalgebra of $\mathfrak{gl}(4, \mathbb{R})$ the only possible nonzero radical consists of multiples of the identity. Alternatively we can use Proposition 1.

5.3. $\mathfrak{so}(4)$. The kernel of the representation consists of multiples of the identity and the other invariant subspace is given by the space of symmetric matrices of trace zero which is not a subalgebra. Again, the only possible nonzero radical consists of multiples of the identity.

5.4. $\mathfrak{so}(3, 1)$. Define the matrix g by $g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ so that $A \in \mathfrak{so}(3, 1)$ if and only if $A^t = -gAg$. Consider next a matrix B that satisfies $B^t = gBg$; then $[A, B]^t = [B^t, A^t] = -[gBg, gAg] = -gBAg + gABg = g[A, B]g$. Hence the space

of matrices in $\mathfrak{gl}(4, \mathbb{R})$ that satisfy $B^t = gBg$ produces an invariant complement to $\mathfrak{so}(3, 1)$. Once again we conclude that the only possible nonzero radical consists of multiples of the identity.

5.5. $\mathfrak{sl}(3, \mathbb{R})$. The action of $\mathfrak{sl}(3, \mathbb{R})$ on $\mathfrak{gl}(4, \mathbb{R})$ consists of the sum of the adjoint representation, two copies of the standard representation, and a two-dimensional kernel. The invariant subspaces are as follows:

$$\begin{aligned} & \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & -(a+e) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & j \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & q & r & 0 \end{bmatrix}. \end{aligned} \quad (2)$$

Of these four subspaces the first corresponds to $\mathfrak{sl}(3, \mathbb{R})$ itself and the second to the kernel of the representation. The third or fourth (but not both) leads to the Lie algebra of the special affine group of $\mathbb{R}^3$. We cannot include both the third and the fourth subspaces or else we will obtain the whole of $\mathfrak{gl}(4, \mathbb{R})$ (see Lemma 3 in Section 9). Besides $\mathfrak{sl}(3, \mathbb{R})$ itself we obtain variously subalgebras of dimensions 9, 10, 11, 12, and 13.

5.6. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. We consider first the block-diagonal direct sum representation. The invariant subspaces are given by

$$\begin{aligned} & \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ & \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix} \\ & \begin{bmatrix} aI_2 & 0 \\ 0 & bI_2 \end{bmatrix}, \end{aligned} \quad (3)$$

where $A, B \in \mathfrak{sl}(2, \mathbb{R})$, $C, D \in \mathfrak{gl}(2, \mathbb{R})$, $a, b \in \mathbb{R}$ and the fourth matrix spans the kernel of the representation.

Besides the subalgebras of dimension seven and eight coming from the kernel, the possible Levi subalgebras are given by $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ or $\begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$ to which may be added elements of the kernel giving variously subalgebras of dimensions ten, eleven, or twelve.

In the second representation of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ given by $\begin{bmatrix} A+dI_2 & eI_2 \\ fI_2 & A-dI_2 \end{bmatrix}$, besides multiples of the identity we find that an invariant complement is given by the space of matrices of the form $\begin{bmatrix} D & B \\ C & -D \end{bmatrix}$ where each of $B, C, D \in \mathfrak{sl}(2, \mathbb{R})$.

The representation is irreducible and the only nonzero radical consists of multiples of the identity.

5.7. $\mathfrak{so}(3)$. Suppose that we have a representation of $\mathfrak{so}(3)$ of type (i) that appears in Section 6. The invariant subspaces of $\mathfrak{gl}(4, \mathbb{R})$ are as follows:

$$\begin{aligned} & \begin{bmatrix} 0 & a & b & 0 \\ -a & 0 & c & 0 \\ -b & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 & 0 & f \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & j & k & 0 \end{bmatrix} \\ & \begin{bmatrix} m & n & p & 0 \\ n & q & r & 0 \\ p & r & -(m+q) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (4)$$

Of these five subspaces the first corresponds to $\mathfrak{so}(3)$ itself, the second to the kernel of the representation and the fifth provides a complement to $\mathfrak{sl}(3, \mathbb{R})$ and must be discounted or else the semisimple part will not be $\mathfrak{so}(3)$. For the same reason as for $\mathfrak{sl}(3, \mathbb{R})$ it is not possible to include both the third and fourth invariant subspaces or else we will obtain $\mathfrak{sl}(4, \mathbb{R})$. We can obtain extensions of $\mathfrak{so}(3)$ by adding a subspace of the kernel and either the third or fourth invariant subspaces leading to the Lie algebra of the Euclidean group of $\mathbb{R}^3$ and algebras of dimension seven and eight.

For a representation of $\mathfrak{so}(3)$ of type (ii), the invariant subspaces are

$$\begin{aligned}
 & \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{bmatrix} \\
 & \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & g \end{bmatrix} \\
 & \begin{bmatrix} 0 & d & e & f \\ -d & 0 & -f & e \\ -e & f & 0 & -d \\ -f & -e & d & 0 \end{bmatrix} \\
 & \begin{bmatrix} h & i & j & k \\ i & m & n & p \\ j & n & q & r \\ k & p & r & -(h+m+q) \end{bmatrix}.
 \end{aligned} \tag{5}$$

The first type of invariant subspace corresponds to the representation of $\mathfrak{so}(3)$ of type (ii) itself, the second and third to the kernel of the representation, and the fourth to an irreducible invariant subspace. In this case the representation does become reducible when we complexify and it is possible to have a two-dimensional abelian radical. This extension can be simplified by realizing that the third invariant subspace can be identified with $\mathfrak{su}(2)$: as such the Lie group $SU(2)$ acts on $\mathfrak{su}(2)$ via the adjoint representation and for matrix of the type in the third invariant subspace e and f may be reduced to zero.

5.8. $\mathfrak{sl}(2, \mathbb{R})$. We have four cases according to Section 6:

(i) In this case we obtain an invariant irreducible complement to the representation of $\mathfrak{sl}(2, \mathbb{R})$ as follows:

$$\begin{bmatrix} -2e-j-(2f+3k) & b & c \\ d & 3e & f & g \\ h & i & 3j & k \\ m & p & -(2i+3d) & -e-2j \end{bmatrix}. \tag{6}$$

There is no nontrivial Levi subalgebra except for adding multiples of I_4 .

(ii) The invariant subspaces are given by

$$\begin{aligned}
 & \begin{bmatrix} 2a & 2b & 0 & 0 \\ c & 0 & b & 0 \\ 0 & 2c & -2a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 & \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}, \\
 & \begin{bmatrix} f & -2g & h & 0 \\ i & -2f & g & 0 \\ j & -2i & f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 & \begin{bmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & q & r & 0 \end{bmatrix}.
 \end{aligned} \tag{7}$$

The first kind of invariant subspace corresponds to the given representation of $\mathfrak{sl}(2, \mathbb{R})$, the second corresponds to its kernel as it acts on $\mathfrak{gl}(4, \mathbb{R})$, and the third provides a complement to $\mathfrak{sl}(3, \mathbb{R})$. The possible subalgebras are given by $\begin{bmatrix} A & a \\ 0 & b \end{bmatrix}$ or $\begin{bmatrix} A & 0 \\ 0 & b \end{bmatrix}$, together with elements in the kernel, and are of dimension four, five, six, seven, or eight where $A \in \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{gl}(2, \mathbb{R})$, $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$.

(iii) The invariant subspaces are given by

$$\begin{aligned}
 & \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix} \\
 & \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
& \begin{bmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n & p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ q & r & 0 & 0 \end{bmatrix},
\end{aligned} \tag{8}$$

where each of d, e, f, g, h are arbitrary, which is to say that each of them induces a one-dimensional invariant subspace and together they span the kernel of the representation. Here $\mathfrak{sl}(2, \mathbb{R})$ acts on $\mathfrak{gl}(4, \mathbb{R})$ by a sum of the adjoint representation and four copies of the standard representation and a five-dimensional kernel.

If we consider two matrices $\begin{bmatrix} \lambda & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} \lambda' & B' \\ C' & D' \end{bmatrix}$ in 2×2 block form in the putative complement then we must have $BC' - B'C$ as a multiple of I_2 . Using the same argument as in Lemma 3 of Section 9, we deduce that if the four entries in B are independent then $C = 0$, whose case will be discussed presently. Similarly if the four entries in C are independent then $B = 0$ although this latter case will be picked up as a companion algebra. In the intermediate cases we will have $k = m = 0 = n = p = 0$ or $i = j = q = r = 0$ but again this former case will be considered to be the companion of the latter. In this latter case then we have to include a new matrix in the kernel that has a nonzero entry only in the $(3, 4)$ -position. We obtain an eight-dimensional algebra that after suitable change of basis may be identified as $L_{8,13}$ in Turkowski's list [14]. This subalgebra may be extended by adding in certain elements in the kernel without changing the semisimple part: in fact such elements have to be diagonal and the $(1, 1)$ and $(2, 2)$ entries must be equal which lead variously to subalgebras of dimensions nine, ten, and eleven.

If there is dependence between the columns with entries i, j and row q, r we may obtain a six-dimensional algebra that is eventually found to be $L_{6,2}$ in Turkowski's list after a suitable change of basis corresponding to a nonsingular matrix in the kernel of the representation. To this subalgebra a diagonal matrix may be added where the $(1, 1)$ and $(2, 2)$ entries are equal to λ , the $(3, 3)$ entry is μ , and the $(4, 4)$ entry is $2\lambda - \mu$ giving algebras of dimensions seven and eight.

Now consider a complement to the representation of $\mathfrak{sl}(2, \mathbb{R})$ of the form $\begin{bmatrix} aI_2 & B \\ 0 & C \end{bmatrix}$. Then $\mathfrak{sl}(2, \mathbb{R})$ acts on the rows of B whereas C acts on the columns of B . In fact, since the complement is assumed to be solvable C may be assumed to be upper triangular or else to have complex eigenvalues. We may invoke the classification of the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$,

bases for which is given by the following list, where ϵ takes the value 0 or 1, the parameter λ can assume any real value, and x, y, z denote variously coordinates on the algebras,

$$\begin{aligned}
& \begin{bmatrix} \epsilon & 1 \\ 0 & \epsilon \end{bmatrix} \\
& \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \\
& \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \\
& \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \\
& \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \\
& \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \\
& \begin{bmatrix} (\lambda + 1)x & y \\ 0 & \lambda x \end{bmatrix} \\
& \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}.
\end{aligned} \tag{9}$$

We also have to add $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{gl}(2, \mathbb{R})$ themselves to obtain the complete list; see [1], but they are not immediately relevant because the complement is assumed to be solvable. We work through each case in the list above and investigate the possibilities for the matrix B introduced at the start of the current paragraph. We find that B may be either of dimension four or two and, in the latter case, we may change basis so that the first column of B is zero.

In the construction of the algebras of type (iii), three seven-dimensional solvable Lie algebras occur that are called B_1, B_2, B_3 , respectively. They do not as yet belong to established lists of algebras and therefore we give their structure equations in Section 8.

For case (iv) invariant subspaces are given by

$$\begin{aligned}
& \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\
& \begin{bmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{bmatrix},
\end{aligned} \tag{10}$$

where each of $A, B, C, D \in \mathfrak{sl}(2, \mathbb{R})$ and a, b, c, d correspond to the kernel of the representation. In addition to the kernel, the action of $\mathfrak{sl}(2, \mathbb{R})$ on $\mathfrak{gl}(4, \mathbb{R})$ consists of four copies of the adjoint representation. We must include the first of these invariant subspaces since it is the representation that we started with but we cannot include the fourth or else the semisimple part will not be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The kernel itself gives a copy of $\mathfrak{sl}(2, \mathbb{R})$ and so we must assume that either $b = 0$ or $c = 0$. As regards the second and third invariant subspaces are now in the same situation as we were in case (iii) of $\mathfrak{sl}(2, \mathbb{R})$. In that case we found first of all four linearly independent matrices among the matrices B and C that led to the existence of an eight-dimensional subalgebra. However, in case (iv) at the same stage we now generate the subalgebra $\mathfrak{sl}(4, \mathbb{R})$. At the next stage where there are two linearly independent matrices among the matrices B and C we obtain $\mathfrak{sp}(4)$, albeit in a nonstandard basis.

The conclusion of the previous paragraph is that we may assume that the radical does not contain the third and fourth invariant subspaces. The next question is what conditions are needed to ensure that the first, second, and fifth subspaces define a subalgebra. To this end let us note the commutators $[[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}], [\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}]] = [\begin{smallmatrix} 0 & [A, B] \\ 0 & 0 \end{smallmatrix}]$ and $[[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}], [\begin{smallmatrix} aI & bI \\ cI & dI \end{smallmatrix}]] = [\begin{smallmatrix} cB & (d-a)B \\ 0 & -cB \end{smallmatrix}]$. It follows that when we combine the second and fifth invariant subspaces the matrix in the kernel must satisfy $c = 0$; otherwise the semisimple part will not be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ since $B \in \mathfrak{sl}(2, \mathbb{R})$, or else $B = 0$. If either of these conditions is satisfied we will obtain subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ and it remains only to normalize such algebras.

We are still at liberty to normalize the subalgebras by using a nonsingular matrix of the form $[\begin{smallmatrix} \alpha I_2 & \beta I_2 \\ 0 & \delta I_2 \end{smallmatrix}]$. As such, consulting the list of matrices that appear at the end of case (iii) for $\mathfrak{sl}(2, \mathbb{R})$, we see that there are seven cases for which the reduced kernel, where $c = 0$, may be put into upper triangular form. To each of these cases the second invariant subspace may be added with $B \in \mathfrak{sl}(2, \mathbb{R})$. We will obtain subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ of dimensions between four and nine.

6. Semisimple Subalgebras of $\mathfrak{gl}(4, \mathbb{R})$

6.1. Dimension 3

6.1.1. $\mathfrak{sl}(2, \mathbb{R})$. Consider the following:

$$(i) \begin{bmatrix} 3a & 3b & 0 & 0 \\ c & a & 2b & 0 \\ 0 & 2c & -a & b \\ 0 & 0 & 3c & -3a \end{bmatrix}$$

$$(ii) \begin{bmatrix} 2a & 2b & 0 & 0 \\ c & 0 & b & 0 \\ 0 & 2c & -2a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iii) \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iv) \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{bmatrix}.$$

(11)

6.1.2. $\mathfrak{so}(3)$. Consider the following:

$$(i) \begin{bmatrix} 0 & c & -b & 0 \\ -c & 0 & a & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{bmatrix}.$$

(12)

6.2. Dimension 6

6.2.1. $\mathfrak{so}(3, 1)$. Consider the following:

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ c & e & f & 0 \end{bmatrix}.$$

(13)

6.2.2. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Consider the following:

$$(i) \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & d & e \\ 0 & 0 & f & -d \end{bmatrix}$$

(14)

$$(ii) \begin{bmatrix} a+d & b & e & 0 \\ c & d-a & 0 & e \\ f & 0 & a-d & b \\ 0 & f & c & -a-d \end{bmatrix}.$$

6.2.3. $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \approx \mathfrak{so}(4)$. Consider the following:

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

(15)

6.3. Dimension 8

6.3.1. $\mathfrak{sl}(3, \mathbb{R})$. Consider the following:

$$\begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & -a-e & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (16)$$

6.4. Dimension 10

6.4.1. $\mathfrak{sp}(4)$. Consider the following:

$$\begin{bmatrix} a & b & e & f \\ c & d & f & g \\ h & i & -a & -c \\ i & j & -b & -d \end{bmatrix}. \quad (17)$$

6.5. Dimension 15

6.5.1. $\mathfrak{sl}(4, \mathbb{R})$. Consider the following:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \\ n & p & q & -(a+f+k) \end{bmatrix}. \quad (18)$$

7. Subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ with a Nontrivial Levi Decomposition by Dimension

λ, μ, ν denote fixed but arbitrary sets of values and ϵ can only assume the value 0 or 1. Letters $a, b, c, d, \dots$, are coordinates on the subalgebras of $\mathfrak{gl}(4, \mathbb{R})$ that are given case by case.

7.1. Dimension 4

7.1.1. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. Consider the following:

$$\begin{aligned} \text{(i)} \quad & \begin{bmatrix} 3a+d & 3b & 0 & 0 \\ c & a+d & 2b & 0 \\ 0 & 2c & d-a & b \\ 0 & 0 & 3c & d-3a \end{bmatrix} \\ \text{(ii)} \quad & \begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 \\ c & \lambda d & b & 0 \\ 0 & 2c & \lambda d-2a & 0 \\ 0 & 0 & 0 & \mu d \end{bmatrix} \quad \lambda^2 + \mu^2 \neq 0 \end{aligned}$$

$$\text{(iii)} \quad \begin{bmatrix} a+\lambda d & b & 0 & 0 \\ c & \lambda d-a & 0 & 0 \\ 0 & 0 & \mu d & 0 \\ 0 & 0 & 0 & \nu d \end{bmatrix} \quad \lambda^2 + \mu^2 + \nu^2 \neq 0$$

$$\text{(iv)} \quad \begin{bmatrix} a+\lambda d & b & 0 & 0 \\ c & \lambda d-a & 0 & 0 \\ 0 & 0 & \epsilon d & d \\ 0 & 0 & 0 & \epsilon d \end{bmatrix}$$

$$\epsilon = 0, \lambda = 1 \text{ or } \epsilon = 1$$

$$\text{(v)} \quad \begin{bmatrix} \lambda d+a & b & 0 & 0 \\ c & \lambda d-a & 0 & 0 \\ 0 & 0 & \mu d & d \\ 0 & 0 & -d & \mu d \end{bmatrix}$$

$$\text{(vi)} \quad \begin{bmatrix} d+a & b & 0 & 0 \\ c & d-a & 0 & 0 \\ 0 & 0 & \mu d+a & b \\ 0 & 0 & c & \mu d-a \end{bmatrix}$$

$$\text{(vii)} \quad \begin{bmatrix} \epsilon d+a & b & d & 0 \\ c & \epsilon d-a & 0 & d \\ 0 & 0 & \epsilon d+a & b \\ 0 & 0 & c & \epsilon d-a \end{bmatrix}$$

$$\text{(viii)} \quad \begin{bmatrix} \lambda d+a & b & d & 0 \\ c & \lambda d-a & 0 & d \\ -d & 0 & \lambda d+a & b \\ 0 & -d & c & \lambda d-a \end{bmatrix}.$$

(19)

7.1.2. $\mathfrak{so}(3) \oplus \mathbb{R}$. Consider the following:

$$\begin{aligned} \text{(i)} \quad & \begin{bmatrix} \lambda d & c & -b & 0 \\ -c & \lambda d & a & 0 \\ b & -a & \lambda d & 0 \\ 0 & 0 & 0 & \mu d \end{bmatrix} \quad \lambda^2 + \mu^2 \neq 0 \\ \text{(ii)} \quad & \begin{bmatrix} \lambda d & \mu d-a & -b & -c \\ a-\mu d & \lambda d & -c & b \\ b & c & \lambda d & -a-\mu d \\ c & -b & a+\mu d & \lambda d \end{bmatrix} \quad \lambda^2 + \mu^2 \neq 0. \end{aligned} \quad (20)$$

7.2. Dimension 5

7.2.1. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2 : [e_1, e_3] = -2e_2, [e_1, e_2] = e_1, [e_2, e_3] = e_3$. Consider the following:

$$(i) \begin{bmatrix} 2a+d & 2b & 0 & 0 \\ c & d & b & 0 \\ 0 & 2c & d-2a & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$$

$$(ii) \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$$

$$(iii) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \lambda e & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$$

$$(iv) \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 \\ 0 & 0 & \epsilon d & e \\ 0 & 0 & 0 & \epsilon d \end{bmatrix}$$

$$\epsilon = 0, \lambda = 1 \text{ or } \epsilon = 1$$

$$(v) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \epsilon e & e \\ 0 & 0 & 0 & \epsilon e \end{bmatrix}$$

$$(vi) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \lambda e & e \\ 0 & 0 & -e & \lambda e \end{bmatrix}$$

$$(vii) \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 \\ 0 & 0 & d & e \\ 0 & 0 & -e & d \end{bmatrix}$$

$$(viii) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & b \\ 0 & 0 & c & d+e-a \end{bmatrix}$$

$$(ix) \begin{bmatrix} a & b & e & 0 \\ c & d & 0 & e \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

$$(x) \begin{bmatrix} a & b & e & 0 \\ c & d & 0 & e \\ -e & 0 & a & b \\ 0 & -e & c & d \end{bmatrix}$$

7.2.2. $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^2 = A_{5,40} = L_{5,1}$. Consider the following:

$$\begin{bmatrix} a & b & d & 0 \\ c & -a & e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (22)$$

7.2.3. $\mathfrak{sl}(2, \mathbb{R}) \oplus A_{2,1}$. Consider the following:

$$(i) \begin{bmatrix} a+\lambda d & b & 0 & 0 \\ c & \lambda d-a & 0 & 0 \\ 0 & 0 & (\mu+1)d & e \\ 0 & 0 & 0 & \mu d \end{bmatrix} \quad (23)$$

$$(ii) \begin{bmatrix} (\lambda+1)d+a & b & e & 0 \\ c & (\lambda+1)d-a & 0 & e \\ 0 & 0 & \lambda d+a & b \\ 0 & 0 & c & \lambda d-a \end{bmatrix}$$

7.2.4. $\mathfrak{so}(3) \oplus \mathbb{R}^2$. Consider the following:

$$(i) \begin{bmatrix} d & c & -b & 0 \\ -c & d & a & 0 \\ b & -a & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \quad (24)$$

$$(ii) \begin{bmatrix} e & -a & -b & -c \\ a & e & -c & b \\ b & c & e & -d \\ c & -b & d & e \end{bmatrix}$$

7.3. Dimension 6

7.3.1. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^3$. Consider the following:

$$(i) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$$

$$(ii) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & 0 & e \end{bmatrix} \quad (25)$$

$$(iii) \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & -f & e \end{bmatrix}$$

7.3.2. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^2) \oplus \mathbb{R} = L_{5,1} \oplus \mathbb{R}$. Consider the following:

$$\begin{aligned}
 \text{(i)} \quad & \begin{bmatrix} a + \lambda f & b & d & 0 \\ c & \lambda f - a & e & 0 \\ 0 & 0 & \lambda f & 0 \\ 0 & 0 & 0 & \mu f \end{bmatrix} \quad \lambda^2 + \mu^2 \neq 0 \\
 \text{(ii)} \quad & \begin{bmatrix} a + f & b & 0 & d \\ c & f - a & 0 & e \\ 0 & 0 & f & f \\ 0 & 0 & 0 & f \end{bmatrix} \\
 \text{(iii)} \quad & \begin{bmatrix} a & b & e & 0 \\ c & d & 0 & e \\ 0 & 0 & f & b \\ 0 & 0 & c & d + f - a \end{bmatrix}.
 \end{aligned} \tag{26}$$

7.3.3. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3,3a=1} = L_{6,3}$. Consider the following:

$$\begin{aligned}
 \text{(i)} \quad & \begin{bmatrix} a + \lambda f & b & d & 0 \\ c & \lambda f - a & e & 0 \\ 0 & 0 & \mu f & 0 \\ 0 & 0 & 0 & \nu f \end{bmatrix} \quad \lambda \neq \mu \\
 \text{(ii)} \quad & \begin{bmatrix} a + \lambda f & b & 0 & d \\ c & \lambda f - a & 0 & e \\ 0 & 0 & \epsilon f & f \\ 0 & 0 & 0 & \epsilon f \end{bmatrix} \quad \lambda \neq \epsilon.
 \end{aligned} \tag{27}$$

7.3.4. $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^3 = L_{6,4}$. Consider the following:

$$\begin{aligned}
 \text{(i)} \quad & \begin{bmatrix} 2a & 2b & 0 & d \\ c & 0 & b & e \\ 0 & 2c & -2a & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 \text{(ii)} \quad & \begin{bmatrix} a & b & d & e \\ c & -a & f & -d \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{bmatrix}.
 \end{aligned} \tag{28}$$

7.3.5. $\mathfrak{sl}(2, \mathbb{R}) \oplus A_{2,1} \oplus \mathbb{R}$. Consider the following:

$$\begin{aligned}
 \text{(i)} \quad & \begin{bmatrix} a + \lambda d + \mu f & b & 0 & 0 \\ c & \lambda d + \mu f - a & 0 & 0 \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \end{bmatrix} \\
 \text{(ii)} \quad & \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & (\lambda + 1)e & f \\ 0 & 0 & 0 & \lambda e \end{bmatrix}.
 \end{aligned} \tag{29}$$

7.3.6. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3,1} = L_{6,2}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & d \\ c & -a & 0 & e \\ e & -d & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

7.3.7. $\mathfrak{so}(3) \rtimes \mathbb{R}^3 = L_{6,1}$. Consider the following:

$$\begin{bmatrix} 0 & c & -b & d \\ -c & 0 & a & e \\ b & -a & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

7.4. Dimension 7

7.4.1. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3,1}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a + g & b & 0 & d \\ c & g - a & 0 & e \\ e & -d & g & f \\ 0 & 0 & 0 & g \end{bmatrix}. \tag{32}$$

7.4.2. $\lambda - \mu \neq 0$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{4,9b=1} = L_{7,4}$. Consider the following:

$$\begin{bmatrix} a + \lambda g & b & 0 & d \\ c & \lambda g - a & 0 & e \\ e & -d & \mu g & f \\ 0 & 0 & 0 & (2\lambda - \mu)g \end{bmatrix}. \tag{33}$$

7.4.3. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^3) \oplus \mathbb{R}$. Consider the following:

$$(i) \begin{bmatrix} 2a+g & 2b & 0 & d \\ c & g & b & e \\ 0 & 2c & g-2a & f \\ 0 & 0 & 0 & g \end{bmatrix}$$

$$(ii) \begin{bmatrix} \epsilon d + a & b & d+e & f \\ c & \epsilon d - a & g & d-e \\ 0 & 0 & \epsilon d + a & b \\ 0 & 0 & c & \epsilon d - a \end{bmatrix}.$$
(34)

7.4.4. $L_{5,1} \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a+f+\epsilon g & b & 0 & d \\ c & f+\epsilon g-a & 0 & e \\ 0 & 0 & f & g \\ 0 & 0 & 0 & f \end{bmatrix}.$$
(35)

7.4.5. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3,3a=1}) \oplus \mathbb{R}$. Consider the following:

$$(i) \begin{bmatrix} a & b & e & 0 \\ c & d & f & 0 \\ 0 & 0 & \lambda g & 0 \\ 0 & 0 & 0 & \mu g \end{bmatrix} \quad \lambda^2 + \mu^2 \neq 0$$

$$(ii) \begin{bmatrix} a+\lambda f+\epsilon g & b & 0 & d \\ c & \lambda f+\epsilon g-a & 0 & e \\ 0 & 0 & f & g \\ 0 & 0 & 0 & f \end{bmatrix} \quad \lambda \neq 1$$

$$(iii) \begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ 0 & 0 & \epsilon g & g \\ 0 & 0 & 0 & \epsilon g \end{bmatrix}$$

$$(iv) \begin{bmatrix} a+\lambda f+\mu g & b & d & 0 \\ c & \lambda f+\mu g-a & e & 0 \\ 0 & 0 & \nu f & 0 \\ 0 & 0 & 0 & \rho g \end{bmatrix}$$

$\lambda \neq \nu, \rho = 1 \text{ or } \nu = 1.$

(36)

7.4.6. $\lambda = \mu (\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^2) \oplus A_{2,1}; \lambda \neq \mu \mathfrak{sl}(2, \mathbb{R}) \rtimes A_{4,5a=1b=1/(\lambda-\mu)} = L_{7,3p=1/(\lambda-\mu)}$. Consider the following:

$$\begin{bmatrix} a+\lambda d & b & 0 & f \\ c & \lambda d - a & 0 & g \\ 0 & 0 & (\mu+1)d & e \\ 0 & 0 & 0 & \mu d \end{bmatrix}.$$
(37)

7.4.7. $\lambda = \mu \neq 0 (\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^3) \oplus \mathbb{R} = L_{6,4} \oplus \mathbb{R}; \lambda \neq \mu \mathfrak{sl}(2, \mathbb{R}) \rtimes A_{4,5a=b=1}$. Consider the following:

$$(i) \begin{bmatrix} 2a+\lambda g & 2b & 0 & d \\ c & \lambda g & b & e \\ 0 & 2c & \lambda g-2a & f \\ 0 & 0 & 0 & \mu g \end{bmatrix}$$

$$(ii) \begin{bmatrix} a+\lambda d & b & \nu d+e & f \\ c & \lambda d-a & g & \nu d-e \\ 0 & 0 & a+\mu d & b \\ 0 & 0 & c & \mu d-a \end{bmatrix}.$$
(38)

7.4.8. $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^4 = L_{7,7}$. Consider the following:

$$\begin{bmatrix} a & b & d & e \\ c & -a & f & g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
(39)

7.4.9. $\mathfrak{sl}(2, \mathbb{R}) \oplus A_{2,1} \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & 0 & g \end{bmatrix}.$$
(40)

7.4.10. $\lambda = \mu \neq 0 (\mathfrak{so}(3) \rtimes \mathbb{R}^3) \oplus \mathbb{R} = L_{6,1} \oplus \mathbb{R}; \lambda \neq \mu \mathfrak{so}(3) \rtimes A_{4,5a=b=1} = L_{7,1}$. Consider the following:

$$\begin{bmatrix} \lambda g & c & -b & d \\ -c & \lambda g & a & e \\ b & -a & \lambda g & f \\ 0 & 0 & 0 & \mu g \end{bmatrix}.$$
(41)

7.4.11. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. Consider the following:

$$(i) \begin{bmatrix} a+g & b & 0 & 0 \\ c & g-a & 0 & 0 \\ 0 & 0 & d+\lambda g & e \\ 0 & 0 & f & \lambda g-d \end{bmatrix}$$

$$(ii) \begin{bmatrix} a+d+g & b & e & 0 \\ c & d-a+g & 0 & e \\ f & 0 & a-d+g & b \\ 0 & f & c & g-a-d \end{bmatrix}.$$
(42)

7.4.12. $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathbb{R} \approx \mathfrak{so}(4) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} g & a & b & c \\ -a & g & d & e \\ -b & -d & g & f \\ -c & -e & -f & g \end{bmatrix}. \quad (43)$$

7.4.13. $\mathfrak{so}(3, 1) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} g & a & b & c \\ -a & g & d & e \\ -b & -d & g & f \\ c & e & f & g \end{bmatrix}. \quad (44)$$

7.5. Dimension 8

7.5.1. $\lambda \neq \mu$, $\lambda \neq \nu$, $\mu \neq \nu$; $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{5.7a=b=\lambda-\mu c=d=\lambda-\nu})$; $\lambda = \mu \neq \nu$; $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{3.3} \oplus \mathbb{R}^2)$; $\lambda = \mu = \nu \neq 0$ ($\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^4$) $\oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a + \lambda d & b & e & f \\ c & \lambda d - a & g & h \\ 0 & 0 & \mu d & 0 \\ 0 & 0 & 0 & \nu d \end{bmatrix}. \quad (45)$$

7.5.2. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.7a=b=c=1} = L_{8.17p=1}$. Consider the following:

$$(i) \begin{bmatrix} a + \lambda d & b & e & f \\ c & \lambda d - a & g & h \\ 0 & 0 & \mu d & 0 \\ 0 & 0 & 0 & \mu d \end{bmatrix} \quad \lambda \neq \mu \quad (46)$$

$$(ii) \begin{bmatrix} (\lambda + 1)d + a & b & e & f \\ c & (\lambda + 1)d - a & g & h \\ 0 & 0 & \lambda d + a & b \\ 0 & 0 & c & \lambda d - a \end{bmatrix}.$$

7.5.3. $\epsilon = 1$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.15a=1} = L_{8.16}$; $\epsilon = 0$, $\lambda = 1$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.1} = L_{8.14}$. Consider the following:

$$\begin{bmatrix} a + \lambda d & b & e & f \\ c & \lambda d - a & g & h \\ 0 & 0 & \epsilon d & d \\ 0 & 0 & 0 & \epsilon d \end{bmatrix}. \quad (47)$$

7.5.4. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.17p=q=s=1} = L_{8.18}$. Consider the following:

$$(i) \begin{bmatrix} \lambda d + a & b & e & f \\ c & \lambda d - a & g & h \\ 0 & 0 & \mu d & d \\ 0 & 0 & -d & \mu d \end{bmatrix}. \quad (48)$$

7.5.5. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{4.5a=2, b=1}) \oplus \mathbb{R} = L_{7.4} \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a + g & b & 0 & d \\ c & g - a & 0 & e \\ e & -d & g + h & f \\ 0 & 0 & 0 & g - h \end{bmatrix}. \quad (49)$$

7.5.6. $\lambda + \mu \neq 1$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{2.1} \oplus A_{3.3})$; $\mu = 1 - \lambda$, $\lambda \neq 0$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{2.1} \oplus A_{3.5})$; $\mu = 1$, $\lambda = 0$ $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{2.1} \oplus \mathbb{R}^3)$. Consider the following:

$$\begin{bmatrix} a + \lambda d + \mu f & b & 0 & g \\ c & \lambda d + \mu f - a & 0 & h \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \end{bmatrix}. \quad (50)$$

7.5.7. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.4} : L_{8.13}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & d \\ c & -a & 0 & e \\ f & g & 0 & h \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (51)$$

7.5.8. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3.3a=1}) \oplus \mathbb{R}^2 = L_{6.3} \oplus \mathbb{R}^2$. Consider the following:

$$(i) \begin{bmatrix} a & b & e & 0 \\ c & d & f & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{bmatrix} \quad (52)$$

$$(ii) \begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ 0 & 0 & g & h \\ 0 & 0 & 0 & g \end{bmatrix}.$$

7.5.9. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3.3a=1}) \oplus A_{2.1} = L_{6.3} \oplus A_{2.1}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ 0 & 0 & (\lambda + 1)g & h \\ 0 & 0 & 0 & \lambda g \end{bmatrix}. \quad (53)$$

7.5.10. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{4.5a=b=1}) \oplus \mathbb{R}$. Consider the following:

$$(i) \begin{bmatrix} g+2a & 2b & 0 & d \\ c & g & b & e \\ 0 & 2c & g-2a & f \\ 0 & 0 & 0 & h \end{bmatrix} \quad (54)$$

$$(ii) \begin{bmatrix} a & b & e & f \\ c & d & g & -e \\ 0 & 0 & h & b \\ 0 & 0 & c & d+h-a \end{bmatrix}.$$

7.5.11. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^3) \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}. \quad (55)$$

7.5.12. $(\mathfrak{so}(3) \rtimes A_{4.5a=b=1}) \oplus \mathbb{R} = L_{7.1} \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} g & c & -b & d \\ -c & g & a & e \\ b & -a & g & f \\ 0 & 0 & 0 & h \end{bmatrix}. \quad (56)$$

7.5.13. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}. \quad (57)$$

7.6. Dimension 9

7.6.1. $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{6.82\delta=(\mu-\nu)/2 \ a=b=\lambda-(\mu+\nu)/2}$. Consider the following:

$$\begin{bmatrix} a+\lambda i & b & 0 & d \\ c & \lambda i-a & 0 & e \\ f & g & \mu i & h \\ 0 & 0 & 0 & \nu i \end{bmatrix}. \quad (58)$$

7.6.2. $((\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3.3a=1}) \oplus A_{2.1}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ 0 & 0 & g & h \\ 0 & 0 & 0 & i \end{bmatrix}. \quad (59)$$

7.6.3. $\lambda = 1 \ (\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.7a=b=c=1}) \oplus \mathbb{R} : \lambda \neq 1 \ \mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.3})$. Consider the following:

$$\begin{bmatrix} a & b & f & g \\ c & d & h & i \\ 0 & 0 & \lambda e & 0 \\ 0 & 0 & 0 & e \end{bmatrix}. \quad (60)$$

7.6.4. $\lambda + \mu \neq 1 \ \mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.3})$. Consider the following:

$$\begin{bmatrix} a+\lambda d+\mu e & b & f & g \\ c & \lambda d+\mu e-a & h & i \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}. \quad (61)$$

7.6.5. $(\lambda - 1)\lambda \neq 0 \ (\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.7a=b=c=1}) \oplus \mathbb{R} : \lambda = 0 \ (\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{3.3} \oplus \mathbb{R}^2)) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a+(1-\lambda)d+\lambda e & b & f & g \\ c & (1-\lambda)d+\lambda e-a & h & i \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}. \quad (62)$$

7.6.6. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^4) \oplus \mathbb{R}^2 = L_{7.7} \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a+d+\epsilon e & b & f & g \\ c & d+\epsilon e-a & h & i \\ 0 & 0 & d & e \\ 0 & 0 & 0 & d \end{bmatrix}. \quad (63)$$

7.6.7. $\lambda \neq 1 \ \mathfrak{sl}(2, \mathbb{R}) \rtimes A_{6.54a=1b=0}$. Consider the following:

$$(i) \begin{bmatrix} a+\lambda d+\epsilon e & b & f & g \\ c & \lambda d+\epsilon e-a & h & i \\ 0 & 0 & d & e \\ 0 & 0 & 0 & d \end{bmatrix} \quad (64)$$

$$(ii) \begin{bmatrix} a & b & f & g \\ c & d & h & i \\ 0 & 0 & \epsilon e & e \\ 0 & 0 & 0 & \epsilon e \end{bmatrix}.$$

7.6.8. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.18q=p \ s=1}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a+d+\mu e & b & f & g \\ c & d+\mu e-a & h & i \\ 0 & 0 & d & e \\ 0 & 0 & -e & d \end{bmatrix}. \quad (65)$$

7.6.9. $\mathfrak{sl}(2, \mathbb{R}) \rtimes N_{6.18\alpha=0\beta=\gamma=1}$. Consider the following:

$$(i) \begin{bmatrix} a & b & f & g \\ c & d & h & i \\ 0 & 0 & \lambda e & e \\ 0 & 0 & -e & \lambda e \end{bmatrix} \quad (66)$$

$$(ii) \begin{bmatrix} a + \lambda d + \mu e & b & f & g \\ c & \lambda d + \mu e - a & h & i \\ 0 & 0 & d & e \\ 0 & 0 & -e & d \end{bmatrix} \quad \lambda \neq 1.$$

7.6.10. $\lambda = \mu \mathfrak{sl}(2, \mathbb{R}) \rtimes A_{6.53} : \lambda \neq \mu \mathfrak{sl}(2, \mathbb{R}) \rtimes A_{6.54a=1b=1/(\lambda-\mu)}$. Consider the following:

$$\begin{bmatrix} a + \lambda d & b & f & g \\ c & \lambda d - a & h & i \\ 0 & 0 & (\mu + 1)d & e \\ 0 & 0 & 0 & \mu d \end{bmatrix}. \quad (67)$$

7.6.11. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.7a=b=c=1}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & b \\ 0 & 0 & c & d + i - a \end{bmatrix}. \quad (68)$$

7.6.12. $\lambda^2 + \mu^2 \neq 0 \mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a + \lambda i & b & c & 0 \\ d & e + \lambda i & f & 0 \\ g & h & \lambda i - (a + e) & 0 \\ 0 & 0 & 0 & \mu i \end{bmatrix}. \quad (69)$$

7.7. Dimension 10

7.7.1. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.3})) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & g & h \\ c & d & i & j \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix}. \quad (70)$$

7.7.2. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{6.54a=1b=0}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & g & h \\ c & d & i & j \\ 0 & 0 & e & f \\ 0 & 0 & 0 & e \end{bmatrix}. \quad (71)$$

7.7.3. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes N_{6.18\alpha=0\beta=\gamma=1}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & g & h \\ c & d & i & j \\ 0 & 0 & e & f \\ 0 & 0 & -f & e \end{bmatrix}. \quad (72)$$

7.7.4. $\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{5.1} \rtimes \mathbb{R}^2)$. Consider the following:

$$\begin{bmatrix} a + \lambda d + \mu f & b & g & h \\ c & \lambda d + \mu f - a & i & j \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \end{bmatrix}. \quad (73)$$

7.7.5. $\mathfrak{sl}(2, \mathbb{R}) \rtimes B_{7.1}$. Consider the following:

$$\begin{bmatrix} a + \lambda i + \mu j & b & 0 & d \\ c & \lambda i + \mu j - a & 0 & e \\ f & g & i & h \\ 0 & 0 & 0 & j \end{bmatrix}. \quad (74)$$

7.7.6. $\lambda^2 + \mu^2 \neq 0 \mathfrak{sl}(2, \mathbb{R}) \rtimes B_{7.2}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ g & h & \lambda j & i \\ 0 & 0 & 0 & \mu j \end{bmatrix}. \quad (75)$$

7.7.7. $(\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})) \rtimes \mathbb{R}^4$. Consider the following:

$$\begin{bmatrix} a & b & g & h \\ c & -a & i & j \\ 0 & 0 & d & e \\ 0 & 0 & f & -d \end{bmatrix}. \quad (76)$$

7.7.8. $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathbb{R}^2$. Consider the following:

$$\begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & j \end{bmatrix}. \quad (77)$$

7.8. Dimension 11

7.8.1. $(\mathfrak{sl}(2, \mathbb{R}) \rtimes (A_{5.1} \rtimes \mathbb{R}^2)) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & h & i \\ c & d & j & k \\ 0 & 0 & e & f \\ 0 & 0 & 0 & g \end{bmatrix}. \quad (78)$$

7.8.2. $\lambda = \mu \neq 0$ ($\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \rtimes \mathbb{R}^4$) $\oplus \mathbb{R} : \lambda \neq \mu \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \rtimes A_{5,7a=b=c=1}$. Consider the following:

$$\begin{bmatrix} \lambda k + a & b & g & h \\ c & \lambda k - a & i & j \\ 0 & 0 & \mu k + d & e \\ 0 & 0 & f & \mu k - d \end{bmatrix}. \quad (79)$$

7.8.3. ($\mathfrak{sl}(2, \mathbb{R}) \rtimes B_{7,3}$) $\oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & 0 & e \\ c & d & 0 & f \\ g & h & i & j \\ 0 & 0 & 0 & k \end{bmatrix}. \quad (80)$$

7.8.4. $\mathfrak{sl}(3, \mathbb{R}) \rtimes \mathbb{R}^3$. Consider the following:

$$\begin{bmatrix} a & b & c & j \\ d & e & f & k \\ g & h & -(a+e) & m \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (81)$$

7.8.5. $\mathfrak{sp}(4) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a+k & b & e & f \\ c & d+k & f & g \\ h & i & k-a & -c \\ i & j & -b & k-d \end{bmatrix}. \quad (82)$$

7.9. Dimension 12

7.9.1. ($\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \rtimes A_{5,7a=b=c=1}$) $\oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & m \end{bmatrix}. \quad (83)$$

7.9.2. $\mathfrak{sl}(3, \mathbb{R}) \rtimes A_{4,5a=b=1} \lambda \neq \mu$. Consider the following:

$$\begin{bmatrix} \lambda m + a & b & c & i \\ d & \lambda m + e & f & j \\ g & h & \lambda m - (a+e) & k \\ 0 & 0 & 0 & \mu m \end{bmatrix}. \quad (84)$$

7.9.3. ($\mathfrak{sl}(3, \mathbb{R}) \rtimes \mathbb{R}^3$) $\oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} m+a & b & c & i \\ d & m+e & f & j \\ g & h & m-(a+e) & k \\ 0 & 0 & 0 & m \end{bmatrix}. \quad (85)$$

7.10. Dimension 13

7.10.1. ($\mathfrak{sl}(3, \mathbb{R}) \rtimes A_{4,5a=b=c=1}$) $\oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & c & j \\ d & e & f & k \\ g & h & i & m \\ 0 & 0 & 0 & n \end{bmatrix}. \quad (86)$$

7.11. Dimension 16

7.11.1. $\mathfrak{gl}(4, \mathbb{R}) \approx \mathfrak{sl}(4, \mathbb{R}) \oplus \mathbb{R}$. Consider the following:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \\ n & p & q & r \end{bmatrix}. \quad (87)$$

8. Three Solvable Classes of Lie Algebras

8.1. $B_{7,1}$. Consider the following:

$$\begin{aligned} [e_1, e_3] &= \lambda e_3, \\ [e_1, e_4] &= \lambda e_4, \\ [e_1, e_5] &= (1 - \lambda) e_5, \\ [e_1, e_6] &= (1 - \lambda) e_6, \\ [e_1, e_7] &= e_7, \\ [e_2, e_3] &= (\mu - 1) e_3, \\ [e_2, e_4] &= (\mu - 1) e_4, \\ [e_2, e_5] &= -\mu e_5, \\ [e_2, e_6] &= -\mu e_6, \\ [e_2, e_7] &= -e_7, \\ [e_3, e_5] &= -e_7, \\ [e_4, e_6] &= -e_7. \end{aligned} \quad (88)$$

8.2. $B_{7,2}$. Consider the following:

$$\begin{aligned} [e_1, e_3] &= e_3, \\ [e_1, e_4] &= e_4, \\ [e_1, e_5] &= -e_5, \\ [e_1, e_6] &= -e_6, \\ [e_2, e_3] &= -\mu e_3, \\ [e_2, e_4] &= -\mu e_4, \end{aligned}$$

$$\begin{aligned}
[e_2, e_5] &= \lambda e_5, \\
[e_2, e_6] &= \lambda e_6, \\
[e_2, e_7] &= (\lambda - \mu) e_7, \\
[e_3, e_5] &= -e_7, \\
[e_4, e_6] &= -e_7.
\end{aligned} \tag{89}$$

8.3. $B_{7,3}$. Consider the following:

$$\begin{aligned}
[e_1, e_5] &= e_5, \\
[e_1, e_6] &= e_6, \\
[e_1, e_7] &= e_7, \\
[e_2, e_3] &= -e_3, \\
[e_2, e_4] &= -e_4, \\
[e_2, e_7] &= -e_7, \\
[e_3, e_5] &= e_7, \\
[e_4, e_6] &= e_7.
\end{aligned} \tag{90}$$

9. Lemma

Lemma 3. A pair of matrices in $\mathfrak{gl}(n+1, \mathbb{R})$ of the form

$$\begin{bmatrix} 0 & a \\ d^t & 0 \end{bmatrix} \tag{91}$$

can never be such that their commutator is of the form

$$\begin{bmatrix} \lambda I_n & 0 \\ 0 & \mu \end{bmatrix} \tag{92}$$

except in the trivial cases where either $d = x = 0$ or $a = u = 0$ and $\lambda = \mu = 0$ or $u = pa$ and $x = pd$.

Proof. The commutator of the two matrices is

$$\begin{bmatrix} ax^t - ud^t & 0 \\ 0 & d^t u - x^t a \end{bmatrix}. \tag{93}$$

We can only have $\mu = -n\lambda$. By multiplying a and u by the same factor we can reduce to the cases $\lambda = \mu = 0$ or $\lambda = 1$ and $\mu = -n$.

In the first case we obtain $x^t da - d^t du = 0$ and $x^t xa - d^t xu = 0$. Unless $d = 0$ and $x = 0$ either a is a multiple of u or u is a multiple of a . Similarly either d is a multiple of x or x is a multiple of d . By symmetry we have to consider the subcases $u = \alpha a$, $x = \beta d$, $u = \alpha a$, and $d = \beta x$. In the first case we find

that $ax^t - ud^t = (\beta - \alpha)ad^t = 0$ and hence either $\beta = \alpha$ or a or $d = x = 0$. In the second case $ax^t - ud^t = (1 - \alpha\beta)ad^t = 0$ and hence either $\beta\alpha = 1$ or a or $d = x = 0$.

In the second case we obtain $ax^t - ud^t = I$ and $x^t a - d^t u = n$ and hence $x^t ua - d^t uu = u$ and $x^t aa - d^t au = a$. Thus a and u are linearly dependent unless $x^t u = 1 + d^t uu = x^t - 1 = d^t = 0$. However, these latter conditions imply that $n = 2$. Similarly d and x are linearly dependent. Thus there exist ρ, σ such that $u = \rho a$ and $x = \sigma d$; but now we find that $(\sigma - \rho)ad^t = I$. In this case either $\sigma = \rho$ or $a = 0$ and hence $u = 0$ or $d = 0$ and hence $x = 0$. $\square$

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The authors thank Qatar Foundation and Virginia Commonwealth University in Qatar for funding this project.

References

- [1] G. Thompson and Z. Wick, "Subalgebras of $\mathfrak{gl}(3, \mathbb{R})$," *Extracta Mathematica*, vol. 27, no. 2, pp. 201–230, 2012.
- [2] R. Ghanam and G. Thompson, "Minimal matrix representations of five-dimensional Lie algebras," *Extracta Mathematicae*, vol. 30, no. 1, pp. 95–133, 2015.
- [3] J. Humphreys, *Lie Algebras and Their Representations*, Springer, 1997.
- [4] E. B. Dynkin, "The maximal subgroups of the classical groups," in *Five Papers on Algebra and Group Theory*, vol. 6 of *American Mathematical Society Translations: Series 2*, pp. 245–378, American Mathematical Society, Providence, RI, USA, 1957.
- [5] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, "Invariants of real low dimension Lie algebras," *Journal of Mathematical Physics*, vol. 17, no. 6, pp. 986–994, 1976.
- [6] S. Lie, *Theorie der Transformationsgruppen*, B.G. Teubner, Leipzig, Germany, 1893.
- [7] G. M. Mubarakzyanov, "On solvable Lie algebras," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, vol. 1, no. 32, pp. 114–123, 1963.
- [8] G. M. Mubarakzyanov, "Classification of real structures of Lie algebras of fifth order," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, vol. 3, no. 34, pp. 99–106, 1963.
- [9] K. A. Umlauf, *Über die Zusammensetzung der endlichen kontinuierlichen Transformationsgruppen insbesondere der Gruppen von Rang null [Ph.D. thesis]*, University of Leipzig, 1891.
- [10] R. Ghanam, G. Thompson, and S. Tonon, "Representations for six-dimensional nilpotent Lie algebras," *Hadronic Journal*, vol. 29, no. 3, pp. 299–317, 2006.
- [11] G. M. Mubarakzyanov, "Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, vol. 4, no. 35, pp. 104–116, 1963.
- [12] A. Shabanskaya and G. Thompson, "Six-dimensional Lie algebras with a five-dimensional nilradical," *Journal of Lie Theory*, vol. 23, no. 2, pp. 313–355, 2013.
- [13] P. Turkowski, "Solvable Lie algebras of dimension six," *Journal of Mathematical Physics*, vol. 31, no. 6, pp. 1344–1350, 1990.

- [14] P. Turkowski, “Low-dimensional real Lie algebras,” *Journal of Mathematical Physics*, vol. 29, no. 10, pp. 2139–2144, 1988.
- [15] L. Šnobl and P. Winternitz, *Classification and Identification of Lie Algebras*, vol. 33 of *CRM Monograph Series*, American Mathematical Society, Providence, RI, USA, 2015.

Hindawi

Submit your manuscripts at
<http://www.hindawi.com>