Research Article

Differentiation Theory over Infinite-Dimensional Banach Spaces

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We study, for any positive integer \( k \) and for any subset \( I \) of \( \mathbb{N}^* \), the Banach space \( E_I \) of the bounded real sequences \( \{x_n\}_{n \in I} \) and a measure over \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \) that generalizes the \( k \)-dimensional Lebesgue one. Moreover, we expose a differentiation theory for the functions defined over this space. The main result of our paper is a change of variables’ formula for the integration of the measurable real functions on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \). This change of variables is defined by some infinite-dimensional functions with properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms.

1. Introduction

The aim of this paper is to generalize the results of article [1], where, for any positive integer \( k \) and for any subset \( I = \{1, \ldots, k\} \), coincides with the \( k \)-dimensional Lebesgue one on \( \mathbb{R}^k \). The measure \( \lambda_{N,a,v}^{(k)} \) is a product indexed by \( I \) of \( \sigma \)-finite measures on the Borel \( \sigma \)-algebra \( \mathcal{B} \) on \( \mathbb{R} \) (by using a generalization of the Jessen theorem), and it is defined over the measurable space \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R})) \), and in particular over \( (E_I, \mathcal{B}_I) \), where \( \mathcal{B}(\mathbb{R}^k) \) is the product indexed by \( I \) of the same \( \sigma \)-algebra \( \mathcal{B} \), \( E_I \subset \mathbb{R}^k \) is the Banach space of the bounded real sequences \( \{x_n\}_{n \in I} \) on \( \mathbb{R}^k \), and \( \mathcal{B}_I \) is the restriction to \( E_I \) of \( \mathcal{B}(\mathbb{R}^k) \).

In the mathematical literature, some articles introduced infinite-dimensional measures analogue of the Lebesgue one (see, e.g., the paper of Léandre [2], in the context of the noncommutative geometry, that one of Tsilevich et al. [3], which studies a family of \( \sigma \)-finite measures on \( \mathbb{R}^\infty \), and that one of Baker [4], which defines a measure on \( \mathbb{R}^\infty \) that is not \( \sigma \)-finite).

In paper [1], the main result is a change of variables’ formula for the integration of the measurable real functions on the space \( (E_I, \mathcal{B}_I) \). This change of variables is defined by a particular class of linear functions over \( E_I \), called \((m, \sigma)\)-standard. A related problem is studied in the paper of Accardi et al. [5], where the authors describe the transformations of general measures on locally convex spaces under smooth transformations of these spaces.

In this paper, we prove that the change of variables’ formula given in [1] can be extended by defining some infinite-dimensional functions with properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms.

In Section 2, we construct the infinite-dimensional Banach space \( E_I \), and we define the continuous functions and the homeomorphisms over the open subsets of this space. Moreover, we recall some results about the integration of the measurable real functions defined on a measurable product space. In Section 3, we expose a differentiation theory in the infinite-dimensional context, and in particular we define the functions \( C^1 \) and the diffeomorphisms. Moreover, we introduce a class of functions, called \((m, \sigma)\)-standard, that generalizes the set of the linear \((m, \sigma)\)-standard functions given in [1], and we expose some properties of these functions. In Section 4, we present the main result of our paper, that is, a change of variables’ formula for the integration of the measurable real functions on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \); this change of variables is defined by the biunique, \( C^1 \) and \((m, \sigma)\)-standard functions, with further properties (Theorem 47). This result agrees with the analogous finite-dimensional result. In Section 5, we expose some ideas for further study in the probability theory.
2. Construction of an Infinite-Dimensional Banach Space

Henceforth, we will indicate by $N^*$ and $R^*$ the sets $N \setminus \{0\}$ and $R \setminus \{0\}$, respectively. Let $I \neq \emptyset$ be a set and let $k \in N^*$; indicate by $\tau$, by $\tau^{(k)}$, by $\tau^{(I)}$, by $\vartheta$, by $\vartheta^{(k)}$, by $\vartheta^{(I)}$, by $\vartheta_0$, by $\vartheta_0^{(k)}$, by $\vartheta_0^{(I)}$, respectively, the Euclidean topology on $R$, the Euclidean topology on $R^*$, the topology $\otimes_{i \in I} \tau_i$, the Borel $\sigma$-algebra on $R$, the Borel $\sigma$-algebra on $R^*$, the $\sigma$-algebra $\otimes_{i \in I} \vartheta_i$, the Lebesgue measure on $R$, and the Lebesgue measure on $R^*$. Moreover, for any set $A \subset R$, indicate by $\vartheta(A)$ the $\sigma$-algebra induced by $\vartheta$ on $A$, and by $\tau(A)$ the topology induced by $\tau$ on $A$; analogously, for any set $A \subset R^*$, define the $\sigma$-algebra $\vartheta^{(I)}(A)$ and the topology $\tau^{(I)}(A)$. Finally, if $S = \prod_{i \in I} S_i$ is a Cartesian product, for any $(x_i : i \in I) \in S$ and for any $\emptyset \neq H \subset I$, define $x_H = (x_i : i \in H) \in \prod_{i \in H} S_i$, and define the projection $\pi_{I,H}$ on $\prod_{i \in I} S_i$ as the function $\pi_{I,H} : S \to \prod_{i \in I} S_i$ given by $\pi_{I,H}(x_i) = x_H$.

**Theorem 1.** Let $I \neq \emptyset$ be a set and, for any $i \in I$, let $(S_i, \Sigma_i, \mu_i)$ be a measure space such that $\mu_i$ is finite. Moreover, suppose that, for some countable subset $J \subset I$, $\mu_i$ is a probability measure for any $i \in J$ and $\prod_{i \in I} \mu_i(S_i) \in R^*$. Then, over the measurable space $(\prod_{i \in I} S_i, \otimes_{i \in I} \Sigma_i)$, there is a unique finite measure $\mu$, indicated by $\otimes_{i \in I} \mu_i$, such that, for any $H \subset I$ such that $|H| < \infty$ and for any $A = \prod_{i \in H} S_i \times \prod_{i \in I \setminus H} S_i \in \otimes_{i \in I} \Sigma_i$, where $A_i \in \Sigma_i$, $\forall i \in H$, and $\nu(A) = \prod_{i \in H} \mu_i(A_i) \prod_{i \in I \setminus H} \mu_i(S_i)$. In particular, if $I$ is countable, then $\mu(A) = \prod_{i \in I} \mu_i(A_i)$ for any $A = \prod_{i \in I} A_i \in \otimes_{i \in I} \Sigma_i$.

**Proof.** See the proof of Corollary 4 in Asci [1].

Henceforth, we will suppose that $I, J$ are sets such that $\emptyset \neq I, J \subset N^*$; moreover, for any $k \in N^*$, we will indicate by $I_k = \{i_1, \ldots, i_k\}$ the set of the $k$ first elements of $I$ (with the natural order and with the convention $I_k = I$ if $|I| < k$) and analogously $J_k$; finally, for any $j = i_n \in J$, set $j = n$.

**Theorem 2.** Let $(S_i, \Sigma_i, \mu_i)$ be a probability space, for any $i \in I$, let $(S, \Sigma, \mu) = (\prod_{i \in I} S_i, \otimes_{i \in I} \Sigma_i, \otimes_{i \in I} \mu_i)$, and let $f \in L^1(S, \Sigma, \mu)$. Moreover, for any $H \subset I$ such that $\emptyset \neq H \subset I < \infty$, define the measurable function $f_{I,H} : (S, \Sigma) \to (R, \vartheta)$ by

$$f_{I,H}(x) = \int_{\mu_I} f(x_{I,H}, x_{I,H}) d\mu_I(x_{I,H}),$$

where $(S_{I,H}, \Sigma_{I,H}, \mu_I)$ is the probability space $(\prod_{i \in I} S_i, \otimes_{i \in I} \Sigma_i, \otimes_{i \in I} \mu_i)$. Then, $\mu$-a.e. one has

$$\lim_{n \to +\infty} f_{I^n} = \int f \ d\mu.$$

**Proof.** See, for example, Theorem 3, page 349, in Rao [6].

**Corollary 3.** Let $(S_i, \Sigma_i, \mu_i)$ be a measure space such that $\mu_i$ is finite, for any $i \in I$, and $\prod_{i \in I} \mu_i(S_i) \in [0, +\infty)$; moreover, let $(S, \Sigma, \mu) = (\prod_{i \in I} S_i, \otimes_{i \in I} \Sigma_i, \otimes_{i \in I} \mu_i)$, let $f \in L^1(S, \Sigma, \mu)$, and let the measurable function $f_{I^n} : (S, \Sigma) \to (R, \vartheta)$ defined by (1), for any $n \in N^*$. Then, $\mu$-a.e. one has

$$\lim_{n \to +\infty} f_{I^n} = \int f \ d\mu.$$

**Proof.** Suppose that $\mu_i(S_i) \neq 0$, $\forall i \in I$; then, $\overline{\mu}_i = \mu_i/\mu(S_i)$ is a probability measure, and also $\overline{\mu} = \otimes_{i \in I} \overline{\mu}_i$, and $\overline{\mu}_n = \otimes_{i \in I} \overline{\mu}_i$, $\forall n \in N^*$; then, define the function $f_{I^n} : (S, \Sigma) \to (R, \vartheta)$ by

$$f_{I^n}(x) = \int_{\overline{\mu}_n} f(x_{I,H}, x_{I,H}) d\overline{\mu}_n(x_{I,H}).$$

From Theorem 2, $\overline{\mu}$-a.e., we have

$$\lim_{n \to +\infty} f_{I^n} = \lim_{n \to +\infty} \left( \prod_{i \in I} \mu_i(S_i) \right) f_{I^n} = \prod_{i \in I} \mu_i(S_i) \int f \ d\overline{\mu} = \int f \ d\mu.$$

Conversely, if $\mu_i(S_i) = 0$, for some $i \in I$, then

$$\lim_{n \to +\infty} f_{I^n} = 0 = \int f \ d\mu.$$

Thus, since $\mu = (\prod_{i \in I} \mu_i(S_i))\overline{\mu}$, we obtain the statement. \hfill $\square$

**Definition 4.** For any set $I \neq \emptyset$, define the function $\| \cdot \|_I : R^I \to [0, +\infty)$ by

$$\|x\|_I = \sup_{i \in I} |x_i|, \quad \forall x = (x_i : i \in I) \in R^I,$$

and define the vector space

$$E_I = \{x \in R^I : \|x\|_I < +\infty\}.$$

Moreover, indicate by $\vartheta_I$ the $\sigma$-algebra $\vartheta^{(I)}(E_I)$, by $\tau_I$ the topology $\tau^{(I)}(E_I)$ induced on $E_I$ by the product topology $\tau^{(I)}(R^I)$, and by $\pi_{I,H}$ the topology induced on $E_I$ by the distance $d : E_I \times E_I \to [0, +\infty)$ defined by $d(x, y) = \|x - y\|_I$, $\forall x, y \in E_I$; furthermore, for any set $A \subset E_I$, indicate by $\tau_{I,H}(A)$ the topology induced by $\tau_{I,H}$ on $A$. Finally, for any $x_0 \in E_I$ and for any $\delta > 0$, indicate by $B_I(x_0, \delta)$ the set $\{x \in E_I : \|x - x_0\|_I < \delta\}$.

Observe that $\tau^{(I)}(A) \subset \tau_{I,H}(A), \forall A \subset E_I$; moreover, $E_I$ is a Banach space, with the norm $\| \cdot \|_I$ (see, e.g., the proof of Remark 2 in [1]).

**Proposition 5.** Let $H$ and $I$ be sets such that $\emptyset \neq H \subset I$; then, the function $\pi_{I,H} : (R^I, \tau^{(I)}) \to (R^H, \tau^{(H)})$ is continuous and open.

**Proof.** See, for example, the theory of the product spaces in Weidmann’s book [7].

**Proposition 6.** Let $H$ and $I$ be sets such that $\emptyset \neq H \subset I$, and let $\pi_{I,H} : E_I \to E_H$ be the function given by $\pi_{I,H}(x) = \pi_{I,H}(x)$, for any $x \in E_I$; then,

1. $\pi_{I,H} : (E_I, \tau_I) \to (E_H, \tau_H)$ is continuous and open;
2. $\pi_{I,H} : (E_I, \tau_{I,H}) \to (E_H, \tau_{I,H})$ is continuous and open.
Proof of (1). \( \forall A \in \tau_H \), there exists \( B \in \tau(I) \) such that \( A = B \cap E_H \), and so \( \pi_{I,H}^{-1}(A) = \pi_{I,H}(B \cap E_I) \cap E_I = \pi_{I,H}(B) \cap E_I \); thus, \( \pi_{I,H}^{-1}(B) \in \tau(I) \) by Proposition 5, we have \( \pi_{I,H}^{-1}(A) \in \tau_I \), and so \( \pi_{I,H}^{-1} \) is a continuous function. Moreover, \( \forall A \in \tau_I \), there exists \( B \in \tau(I) \) such that \( A = B \cap E_I \), and so \( \pi_{I,H}^{-1}(A) = \pi_{I,H}(B) \cap E_I \). Then, since \( \pi_{I,H}^{-1}(B) \in \tau(I) \) by Proposition 5, we have \( \pi_{I,H}^{-1}(A) \in \tau_I \), and so \( \pi_{I,H}^{-1} \) is a continuous function.

Proof of (2). \( \forall A \in \tau_{I,H} \), and \( \forall x = (x_i : i \in I) \in \pi_{I,H}^{-1}(A) \), we have \( (x_i : i \in H) \in A \), and so there exists \( a \in \mathbb{R}^I \) and \( (y_i : i \in I) \in \pi_{I,H}^{-1}(A) \) such that \( y_i = x_i \), \( \forall i \in I \setminus H \), let \( z = (y_i : i \in I) \in \pi_{I,H}^{-1}(A) \) such that \( z_i - a_i = y_i, \forall i \in I \setminus H \), and \( z_i = y_i, \forall i \in H \); then, \( x = (z_i : i \in I) \in \pi_{I,H}^{-1}(A) \), and so \( \pi_{I,H}^{-1} \) is a continuous function. Moreover, \( \forall A \in \tau_{I,H} \), and \( \forall x = (x_i : i \in I) \in \pi_{I,H}(A) \), let \( y = (y_i : i \in I) \in A \) such that \( y_i = x_i, \forall i \in I \setminus H \), and since \( A \in \tau_{I,H} \), there exists \( a \in \mathbb{R}^I \) and \( (z_i : i \in I) \in A \) such that \( y = (z_i : i \in I) \in \pi_{I,H}(A) \), and so \( \pi_{I,H}^{-1} \) is an open function.

Definition 7. Let \( U \in \tau_{I,H} \), let \( x_0 \in U \), let \( I \in E_I \), and let \( \varphi : U \to E_I \) be a function; we say that \( \varphi \) is continuous in \( x_0 \in U \) if \( \lim_{x \to x_0} \varphi(x) = \varphi(x_0) \), and we say that \( \varphi \) is continuous in \( U \) if \( \varphi \) is continuous in \( x_0 \) for any \( x_0 \in U \).

Definition 8. Let \( U \in \tau_{I,H} \), and let \( \varphi : U \to E_I \) be a function; we say that \( \varphi \) is continuous in \( x_0 \in U \) if \( \lim_{x \to x_0} \varphi(x) = \varphi(x_0) \), and we say that \( \varphi \) is continuous in \( U \) if \( \varphi \) is continuous in \( x_0 \) for any \( x_0 \in U \).

Remark 9. Let \( U \in \tau_{I,H} \), let \( V \in \tau_{I,H} \), and let \( \varphi : U \to V \) be a function; then, \( \varphi : (U, \tau_{I,H}(U)) \to (V, \tau_{I,H}(V)) \) is continuous if and only if \( \varphi \) is continuous in \( U \).

Remark 10. Let \( H, I \), and \( J \) be sets such that \( \emptyset \neq H \subseteq I \), let \( U \in \tau_{I,H} \), and let \( \varphi : U \to E_I \) be a function continuous in \( U \); then, the function \( \varphi_{I,H} : U \to E_I \) is continuous in \( U \).

Proof. The statement follows from Proposition 6.

\[
\lim_{h \to 0} \frac{\| (\alpha \varphi + \beta \psi)(x_0 + h) - (\alpha \varphi + \beta \psi)(x_0) - (\alpha d\varphi(x_0) + \beta d\psi(x_0)) h \|_I}{\|h\|_J} = \lim_{h \to 0} \frac{\| \alpha \varphi(x_0 + h) - \alpha \varphi(x_0) - \alpha d\varphi(x_0) h + \beta \psi(x_0 + h) - \beta \psi(x_0) - \beta d\psi(x_0) h \|_I}{\|h\|_J} \leq \lim_{h \to 0} \frac{\| \alpha \varphi(x_0 + h) - \alpha \varphi(x_0)\|_I}{\|h\|_J} + \lim_{h \to 0} \frac{\| \beta \psi(x_0 + h) - \beta \psi(x_0)\|_I}{\|h\|_J} = 0 \implies d(\alpha \varphi + \beta \psi)(x_0) = \alpha d\varphi(x_0) + \beta d\psi(x_0).
\]
Remark 15. A linear and continuous function $A = (a_{ij})_{i \in I, j \in J} : E_J \to E_I$, defined by
\[
(Ax)_i = \sum_{j \in J} a_{ij} x_j, \quad \forall x \in E_J, \forall i \in I, \tag{10}
\]
is differentiable and $d\Phi(x_0) = A$, for any $x_0 \in E_J$.

Remark 16. Let $U \in \tau_{E_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, for any $i \in I$, the component $\varphi_i : U \to \mathbb{R}$ is differentiable in $x_0$, and $d\varphi_i(x_0)$ is the matrix $A_i$, given by the $i$th row of $A = d\Phi(x_0)$. Moreover, if $|I| < +\infty$ and $\varphi_i : U \subset E_J \to \mathbb{R}$ is differentiable in $x_0$, for any $i \in I$, then $\varphi : U \subset E_J \to E_I$ is differentiable in $x_0$.

Remark 17. Let $U \in \tau_{E_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, $\varphi$ is continuous in $x_0$.

Proof. $\forall x \in U$, set
\[
\sigma(x, x_0) = f(x) - f(x_0) - df(x_0)(x - x_0). \tag{11}
\]
From (8), we have $\lim_{x \to x_0} \sigma(x, x_0) = 0$; moreover,
\[
\|f(x) - f(x_0)\|_I = \|df(x_0)(x - x_0) + \sigma(x, x_0)\|_I \leq \|df(x_0)\|_I \|x - x_0\|_I + \|\sigma(x, x_0)\|_I, \tag{12}
\]
from which $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 18. Let $U \in \tau_{E_J}$ and let $v \in E_J$ such that $\|v\|_I = 1$; a function $\varphi : U \subset E_J \to \mathbb{R}$ is called derivable in $x_0 \in U$ in the direction $v$ if there exists the limit
\[
\lim_{t \to 0} \frac{\varphi(x_0 + tv) - \varphi(x_0)}{t}. \tag{13}
\]

This limit is indicated by $(\partial \varphi / \partial v)(x_0)$, and it is called derivative of $\varphi$ in $x_0$ in the direction $v$. If for some $j \in J$ one has $v = e_j$, where $(e_j)_k = \delta_{jk}$, for any $k \in J$, denote $(\partial \varphi / \partial v)(x_0)$ by $(\partial \varphi / \partial x_j)(x_0)$, and call it partial derivative of $\varphi$ in $x_0$, with respect to $x_j$. Moreover, if there exists the linear function defined by the matrix $J_{\varphi}(x_0)$, where $(J_{\varphi}(x_0))_{ij} = (\partial \varphi / \partial x_i)(x_0)$, for any $i \in I$ and $j \in J$, then $J_{\varphi}(x_0)$ is called Jacobian matrix of the function $\varphi$ in $x_0$.

Remark 19. Let $U \in \tau_{E_J}$ and suppose that a function $\varphi : U \subset E_J \to E_I$ is derivable in $x_0 \in U$; then, for any $v \in E_J$ such that $\|v\|_I = 1$ and for any $i \in I$, the function $\varphi_i : U \subset E_J \to \mathbb{R}$ is derivable in $x_0$ in the direction $v$, and one has
\[
\frac{\partial \varphi_i}{\partial v}(x_0) = d\varphi_i(x_0) v. \tag{14}
\]

Proof. If $h \in E_J$, by setting $h = tv$, for some matrix $A = (a_{ij})_{i \in I, j \in J}$ we have
\[
\lim_{t \to 0} \left| \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{t} - A_i(tv) \right| = \lim_{t \to 0} \left| \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{tv} - A_i v \right| = 0, \tag{15}
\]
where $A_i = (a_{ij})_{j \in J}$ is a matrix in $E_J$.

Then, since $A_i(tv) = tA_i v$, we have
\[
\lim_{t \to 0} \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{t} = A_i v, \tag{16}
\]
from which
\[
\frac{\partial \varphi_i}{\partial v}(x_0) = A_i v = d\varphi_i(x_0) v. \tag{17}
\]

Corollary 20. Let $U \in \tau_{E_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, the linear function $J_{\varphi}(x_0) : E_J \to E_I$ is defined and continuous; moreover, for any $h \in E_I$, one has $d\varphi(x_0)(h) = J_{\varphi}(x_0) h$.

Proof. From Remark 19, $\forall i \in I$ and $j \in J$, we have $(d\varphi(x_0))_{ij} = d\varphi_i(x_0)(e_j) = (\partial \varphi_i / \partial x_j)(x_0) = (J_{\varphi}(x_0))_{ij}$. 

Theorem 21. Let $U \in \tau_{E_J}$, let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$, let $V \in \nu_{E_I}$ such that $V \supset \varphi(U)$, and let $\psi : V \subset E_I \to E_J$ be a function differentiable in $y_0 = \varphi(x_0)$.

Then, the function $\psi \circ \varphi$ is differentiable in $x_0$, and one has $d(\psi \circ \varphi)(x_0) = d\psi(y_0) \circ d\varphi(x_0)$.

Proof. The proof is analogous to that one true in the particular case $|H| < +\infty, |I| < +\infty, |J| < +\infty$ (see, e.g., the Lang's book [9]).

Definition 22. Let $U \in \tau_{E_J}$, let $i, j \in J$, and let $\varphi : U \subset E_J \to \mathbb{R}$ be a function derivable in $x_0 \in U$ with respect to $x_j$, such that the function $d\varphi / dx_j$ is derivable in $x_0$ with respect to $x_j$. Indicate $(d^2 \varphi / dx_j^2)(x_0)$ by $\varphi_{ii}(x_0)$ and call it second partial derivative of $\varphi$ in $x_0$ with respect to $x_i$ and $x_j$. If $i = j$, it is indicated by $(d^2 \varphi / dx_j^2)(x_0)$.

Analogously, for any $k \in \mathbb{N}^*$ and for any $j_1, \ldots, j_k \in J$, define $(d^k \varphi / dx_{j_1} \cdots dx_{j_k})(x_0)$ and call it $k$th partial derivative of $\varphi$ in $x_0$ with respect to $x_{j_1}, \ldots, x_{j_k}$.

Definition 23. Let $U \in \tau_{E_J}$ and let $k \in \mathbb{N}^*$; a function $\varphi : U \subset E_J \to E_I$ is called $C^k$ in $x_0 \in U$ if, in a neighbourhood $V \subset \tau_{E_I}(U)$ of $x_0$, for any $i \in I$ and for any $j_1, \ldots, j_k \in J$, there exists the function defined by $x \mapsto (d^k \varphi_i / dx_{j_1} \cdots dx_{j_k})(x)$, and this function is continuous in $x_0$; that is, $C^k$ in $U$ if, for any $x_0 \in U$, $\varphi$ is $C^k$ in $x_0$. Moreover, $\varphi$ is called strongly $C^1$ in $x_0 \in U$ if, in a neighbourhood $V \subset \tau_{E_I}(U)$ of $x_0$, there exists the function defined by $x \mapsto J_{\varphi}(x)$, and this function is continuous in $x_0$, with $\|J_{\varphi}(x_0)\| < +\infty$. Finally, $\varphi$ is called strongly $C^1$ in $U$ if, for any $x_0 \in U$, $\varphi$ is strongly $C^1$ in $x_0$.

Definition 24. Let $U \in \tau_{E_J}$, and let $V \in \tau_{E_J}$; a function $\varphi : U \subset E_J \to E_I$ is called diffeomorphism if $\varphi$ is biunique and $C^1$ in $U$, and the function $\varphi^{-1} : V \subset E_I \to U \subset E_J$ is $C^1$ in $V$.

Remark 25. Let $U \in \tau_{E_J}$, and let $\varphi : U \subset E_J \to E_I$ be a function $C^1$ in $x_0 \in U$, where $|I| < +\infty, |J| < +\infty$; then, $\varphi$ is strongly $C^1$ in $x_0$. 

Theorem 26. Let $U \in \tau_{J_1}$, let $\varphi : U \subset E \rightarrow R$ be a function $C^k$ in $x_0 \in U$, let $i_1, \ldots, i_k \in J$, and let $j_1, \ldots, j_k \in J$ be a permutation of $i_1, \ldots, i_k$. Then, one has

$$\frac{\partial^k \varphi}{\partial x_{i_1} \cdots \partial x_{i_k}}(x_0) = \frac{\partial^k \varphi}{\partial x_{j_1} \cdots \partial x_{j_k}}(x_0).$$

(18)

Proof. The proof is analogous to that one true in the particular case $|J| < +\infty$ (see, e.g., the Langs’s book [9]).

Proposition 27. Let $U = \prod_{j \in J} U_j \cap E_j \in \tau_{J_1}$, where $U_j \in \tau$, for any $j \in J$, and let $\varphi : U \subset E \rightarrow E_1$ be a function $C^1$ in $x_0 \in U$, such that

$$\varphi_1(x) = \sum_{j \in J} \varphi_{ij}(x_j), \quad \forall x = (x_j : j \in J) \in U, \forall i \in I,$n (19)

where $\varphi_{ij} : U_j \rightarrow R$, $\forall i \in I$ and $\forall j \in J$; moreover, suppose that there exists a neighbourhood $V$ in $\tau_{J_1}(U)$ of $x_0$ such that $\sup_{x \in V} \|f_0(x)\| < +\infty$. Then, $\varphi$ is continuous in $x_0$; in particular, if $\varphi$ is strongly $C^1$ in $x_0$ and $|I| < +\infty$, $\varphi$ is differentiable in $x_0$.

Proof. Since $\varphi$ is $C^1$ in $x_0$, from (19) there exists a neighbourhood of $x_0 = (x_{o_j} : j \in J)$ that we can suppose to be $V$, such that $V = \prod_{j \in J} V_j \subset U$, where $V_j = (x_{o_j} - \delta_j, x_{o_j} + \delta_j)$, $\forall j \in J$, and such that, $\forall i \in I$ and $\forall j \in J$, $\varphi_{ij}$ exists in $V_j$. Let $x = (x_j : j \in J) \in V$; from the Lagrange theorem, $\forall i \in I$ and $\forall j \in J$, there exists $\xi_j \in (\min\{x_{o_j}, x_j\}, \max\{x_{o_j}, x_j\}) \subset V_j$ such that $\varphi_{ij}(x_j) - \varphi_{ij}(x_{o_j}) = \frac{\partial \varphi}{\partial x_j}(\xi_j)(x_j - x_{o_j})$. Then,

$$\varphi(x) - \varphi(x_0) = A_\varphi(x)(x, x_0) + \varphi_0(x_0),$$

where $A_\varphi(x)(x, x_0) = \sum_{i \in I} \sum_{j \in J} \varphi_{ij}(x_j)(x_j - x_{o_j})$. Thus, if $\sup_{x \in \bar{V}} \|f_0(x)\| = c < +\infty$, $\forall x \in \bar{V}$, we have

$$\|A_\varphi(x, x_0)\| = \sup_{i \in I} \|A_\varphi(x, x_0)\|_i \leq \sup_{i \in I} \|f_0(\xi_{ij} : j \in J)\| \leq c,$$

from which

$$\|\varphi(x) - \varphi(x_0)\|_i \leq c \|x - x_0\|_i;$$

then, $\varphi$ is continuous in $x_0$. Moreover, we have

$$\|\varphi(x) - \varphi(x_0) - J_\varphi(x)(x-x_0)\|_i \leq \|A_\varphi(x, x_0) - J_\varphi(x_0)\|_i \leq \sup_{i \in I} \|f_0(\xi_{ij} : j \in J) - J_\varphi(x_0)\|_i.$$

Then, if $\varphi$ is strongly $C^1$ in $x_0$ and $|I| < +\infty$, we obtain

$$\lim_{x \to x_0} \frac{\|\varphi(x) - \varphi(x_0) - J_\varphi(x)(x-x_0)\|_i}{\|x - x_0\|_i} = 0,$$

(24)

and so $\varphi$ is differentiable in $x_0$, with $d\varphi(x_0) = J_\varphi(x_0)$. \qed

Definition 28. Let $m \in N^*$, let $U = (U^{(m)} \times \prod_{j \in J_m} A_j) \cap E_j \in \tau_{J_1}$, where $L^{(m)} \in \tau_{J_1}^m$, $A_j \subset R$ is an open interval, for any $j \in J_m$, and let $\sigma : \cap^{m_0} I \mapsto \cap^{m_0} I$ be an increasing function. A function $\varphi : U \subset E_1 \rightarrow E_1$ is called $(m, \sigma)$-standard if

\begin{enumerate}
  \item $\forall i \in I$, there exist some functions $\varphi_1^{(m,\sigma)} : U^{(m)} \rightarrow R$ and $\{\varphi_{ij}^{(m,\sigma)} : A_j \rightarrow R\}$ such that, $\forall x \in U^{(m)}$, one has $\varphi_i(x) = \varphi_1^{(m,\sigma)}(x_{i_1}, \ldots, x_{i_m}) + \sum_{j \in J_m} \varphi_{ij}^{(m,\sigma)}(x_j)$;
  \item $\forall i \in J_m$, $\forall x \in U$, one has $\varphi_i^{(m,\sigma)}(x_{i_1}, \ldots, x_{i_m}) = 0$ and $\varphi_{ij}^{(m,\sigma)}(x_j) = 0, \forall j \neq \sigma(i)$;
  \item $\forall i \in J_m \setminus J_m$, the function $g_i = \varphi_1^{(m,\sigma)}$ is constant or injective derivable; moreover, $\forall x = (x_j : j \in J_m) \in \prod_{j \in J_m} A_j$, there exists $\prod_{i \in J_1} (-1)^{[i]+[\sigma(i)]} g_i(x_{\sigma(i)}) \in R^*$, where $\prod_{i \in J_1} f_i = \{i \in J_1 : f_i : g_i \text{ is injective derivable}\}$.
\end{enumerate}

If $\sigma(i) = i, \forall i \in J_m$, $\varphi$ is called $m$-standard; moreover, if the sequence $\{(1)^{[i]+[\sigma(i)]} g_i(x_{\sigma(i)})\}_{i \in J_m}$ converges uniformly to $1$, with respect to $x = (x_j : j \in J_m) \in \prod_{j \in J_m} A_j$, then $\varphi$ is called strongly $(m, \sigma)$-standard.

Furthermore, define the $(m, \sigma)$-standard function $\varphi^{(n)} : U \subset E_1 \rightarrow E_1$ in the following manner:

$$\varphi_i^{(n)}(x) = \sum_{j \in J_n} \varphi_{ij}^{(m_n)}(x_{i_1}, \ldots, x_{i_m}), \quad \forall x = (x_j : j \in I) \in U, \forall i \in I,$n (26)

Finally, define the $(m, \sigma)$-standard function $\varphi^{(n)} : U \subset E_1 \rightarrow E_1$ in the following manner:

$$\varphi_i^{(n)}(x) = \sum_{j \in J_n} \varphi_{ij}^{(m_n)}(x_{i_1}, \ldots, x_{i_m}), \quad \forall x = (x_j : j \in I) \in U, \forall i \in I,$n (26)

and indicate $\varphi^{(m)}$ by $\tilde{\varphi}$. 

Remark 29. Let $\varphi : U \subset E_1 \rightarrow E_1$ be a $(m, \sigma)$-standard function. Then,

\begin{enumerate}
  \item if $\sigma$ is injective, for any $n \in N, n \geq m$, $\varphi$ is $(n, \sigma)$-standard;
  \item $\sigma$ is biunique if and only if $\sigma(i) = i, \forall i \in J_m$;
  \item if $\prod_{j \in J_m} A_j \subset E_1^{(m_0)}$, there exist $a \in R$ and $m_0 \in N$, $m_0 \geq m$, such that, for any $j \in J_m$ and $a_j$ one has $A_j \subset (-a, a)$.
\end{enumerate}
Proof. Points (1) and (2) follow from the fact that \( \sigma \) is increasing. Moreover, the proof of the point (3) is trivial. \( \square \)

Remark 30. Let \( \varphi : U \subset E_1 \to E_1 \) be a \((m, \sigma)\)-standard function such that \( \prod_{j \in I_m} A_j \subset E_{\varphi(j)} \), and \( \sigma \) is injective; then, there exists \( m_1 \in \mathbb{N}, m_1 \geq m \), such that, for any \( i \in I \setminus I_m, g_i \) is bounded. In particular, if \(|I| = +\infty\), \( \varphi \) is not surjective.

Proof. Since \( \prod_{j \in I_m} A_j \subset E_{\varphi(j)} \) from Remark 29, there exists \( m_1 \in \mathbb{N}, m_1 \geq m \), such that, \( \forall j \in I \setminus I_m \), the set \( A_j \) is bounded; then, let \( \mathcal{H} \equiv \{ i \in I \setminus I_m : g_i \) is not bounded \( \} \subset \mathcal{F} \); since \( \sigma \) is injective and increasing, we have \( \sigma(\mathcal{H}) \subset I \setminus I_m \). Moreover, we have \(|\mathcal{H}| < +\infty\); indeed, \( \forall i \in \mathcal{H}, \) the set \( A_{\sigma(i)} \) is bounded, and so there exists \( t_i \in A_{\sigma(i)} \) such that \( |g_i'(t_i)| \geq 2 \); by supposing by contradiction \( |\mathcal{H}| = +\infty \), \( \forall x \in \mathcal{H}, \) we would obtain \([ \prod_{j \in \mathcal{H}} g_j(x) |] = [ \prod_{j \in \mathcal{H}} g_j(x) |] = +\infty \) (a contradiction). Then, there exists \( m_1 \in \mathbb{N}, m_1 \geq m \), such that, \( \forall i \in I \setminus I_m, g_i \) is bounded. In particular, \( \forall i \in I \setminus I_m, \) the function \( g_i \) is not surjective; then, if \(|I| = +\infty\), \( \varphi \) is not surjective. \( \square \)

Proposition 31. Let \( \varphi : U \subset E_1 \to E_1 \) be a \((m, \sigma)\)-standard function; then,

1. Suppose that \( \varphi \) is injective, \( \pi_{1,2}^H(\varphi(U)) \in \tau^H \), for any \( H \subset I \setminus I_m \) such that \( 0 < |H| \leq 2 \), the function \( \varphi_1 : U \to \mathbb{R} \) is \( C^1 \), for any \( i \in I_m, \) and let \( \det \varphi_{(m,n)(x)}(x) \neq 0 \), \( \forall x \in U^{(m)}; \) then, the functions \( g_i \), for any \( i \in I \setminus I_m, \) are not injective, and \( \sigma \) is biunique.

2. If \( \varphi \) is biunique, then the functions \( g_i \), for any \( i \in I \setminus I_m, \) \( \varphi_{(m,n)} \), and \( \sigma \) are biunique.

Proof of (I). Suppose that \( \varphi \) is injective, \( \pi_{1,2}^H(\varphi(U)) \in \tau^H \), for any \( H \subset I \setminus I_m \) such that \( 0 < |H| \leq 2 \), and let \( i \in I \setminus I_m; \) we have \( i \in \mathcal{F}_\sigma \), since otherwise we would obtain \( g_i(x) = c, \) \( \forall c \in \mathcal{F}_\sigma, \) for some \( c \in \mathcal{F}, \) from which \( \pi_{ij}^H(\varphi(U)) = |c| \neq \tau = \tau^{(i)} \), and this should contradict the assumption; then, \( g_i \) is injective. Moreover, \( \sigma \) must be injective; by supposing by contradiction that \( \sigma(i_1) = \sigma(i_2), \) for some \( m < i_1 < i_2, \) then

\[
\pi_{1,2}^{(i_1,i_2)}(\varphi(U)) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = g_{i_1}(x), y_2 = g_{i_2}(x)\} = \{(y_1, y_2) \in A_{\sigma(i_1)} : y_1 = g_{i_1}(g_{i_1}^{-1}(y_1))\} \neq \tau \quad \forall x \in U^{(i_1,i_2)} \]  

(a contradiction). Moreover, \( \sigma \) is surjective; in fact, suppose by contradiction that there exists \( n \in (I \setminus I_m) \setminus \sigma(I \setminus I_m); \) since \( \varphi \) is injective, \( \forall y \in \varphi(U), \) there is a unique \( x = (x : j \in I) \in U \) such that \( \varphi(x) = y, \) and so

\[
y_i = \varphi_i(x) = \varphi_{(m,m)}(x_1, \ldots, x_m) + \varphi_m(x_n) + \sum_{j \in I \setminus (I \cup \{n\})} \varphi_{ij}(x_j), \quad \forall i \in I_m \implies \]

\[
\varphi_{(m,m)}(x_1, \ldots, x_m) + \varphi_m(x_n) = 0, \quad \forall i \in I_m.
\]

(28)

Then, consider the function \( F : U^{(m)} \times A_n \to \mathbb{R}^m \) defined by

\[
F_i(x, y) = \varphi_{(m,m)}(x) + \varphi_m(t) - y_i + \sum_{j \in (I \setminus (I \cup \{n\}))} \varphi_{ij}(x_j), \quad \forall i \in I_m, \forall x \in U^{(m)}, \forall y \in A_n,
\]

(29)

From (28) and by assumption, we have

\[
F(x_1, \ldots, x_m, x_n) = 0,
\]

(30)

then, there exist a neighbourhood \( V_x \subset A_n \) of \( x_n \) and a neighbourhood \( V_y \subset U^{(m)} \) of \( (x_1, \ldots, x_m) \) such that, \( \forall i \in V_x, \) there exists a unique \( z = f(t) \) in \( V_y \) such that \( F(z, t) = 0 \) (a contradiction with the uniqueness of \( x = (x_1 : j \in I) \in U \) such that \( \varphi(x) = y \)). Finally, let \( x, y \in U^{(m)} \) be such that \( \varphi_{(m,m)}(x) = \varphi_{(m,m)}(y), \) and let \( \bar{x}, \bar{y} \in U \) be such that \( \bar{x}_i = x_i \) and \( \bar{y}_i = y_i, \forall i \in I \setminus I_m, \bar{x}_m = \bar{y}_m, \forall i \in I \setminus I_m. \) We have

\[
\varphi_i(\bar{x}) = \varphi_{(m,m)}(x) + \sum_{j \in (I \setminus (I \cup \{m\}))} \varphi_{ij}(\bar{x}_j) = \varphi_{(m,m)}(y) + \sum_{j \in (I \setminus (I \cup \{m\}))} \varphi_{ij}(\bar{y}_j) = \varphi_{(m,m)}(y), \quad \forall i \in I_m;
\]

(31)

\[
\varphi_i(\bar{x}) = g_i(\bar{x}_m) = g_i(\bar{y}_m) = \varphi_i(\bar{y}), \quad \forall i \in I \setminus I_m
\]

from which \( \varphi(\bar{x}) = \varphi(\bar{y}); \) then, since \( \varphi \) is injective, we have \( \bar{x} = \bar{y}, \) and so \( x = y. \) Then, the function \( \varphi_{(m,m)} \) is injective. \( \square \)

Proof of (2). Suppose that \( g_i \) is injective, \( \forall x \in I \setminus I_m, \) and let \( x, y \in U \) be such that \( \varphi(x) = \varphi(y); \) then, \( \forall x \in I \setminus I_m, \) we have \( g_i(x_{\sigma(i)}) = \varphi_i(x) = \varphi_i(y) = g_i(y_{\sigma(i)}), \) from which \( x_{\sigma(i)} = y_{\sigma(i)}; \) then, if \( \sigma \) is biunique, from Remark 29, we have \( \sigma(i) = i, \) and so \( \sigma(x : j \in I \setminus I_m) = (y : j \in I \setminus I_m). \) This implies that

\[
\varphi_i^{(m,m)}(x_1, \ldots, x_m) = \varphi_i(x) - \sum_{j \in (I \setminus (I \cup \{m\}))} \varphi_{ij}(x_j) = \varphi_i(x) - \sum_{j \in (I \setminus (I \cup \{m\}))} \varphi_{ij}(y_j) = \varphi_i^{(m,m)}(y_1, \ldots, y_m), \quad \forall i \in I_m
\]

(32)

\[
\varphi_{(m,m)}(x_1, \ldots, x_m) = \varphi_{(m,m)}(y_1, \ldots, y_m),
\]

(33)
and so, if \( \varphi^{(m,m)} \) is injective, we have \((x_1, \ldots, x_m) = (y_1, \ldots, y_m)\); then, \( x = y \); that is, \( \varphi \) is injective.

**Proposition 32.** Let \( \varphi : U \subset E \rightarrow E \) be a \((m, \sigma)\)-standard function; then:

1. Suppose that \( \varphi \) is biunique, the function \( \varphi_i : U \rightarrow R \) is \( C^1 \), for any \( i \in I_m \) and \( \det I_{\text{dim}(x)}(x) \neq 0 \), for any \( x \in U^{(m)} \), then, the functions \( g_i \), for any \( i \in I \setminus I_m, \sigma \) and \( \varphi^{(m,m)} \) are biunique.

2. If the functions \( g_i, \) for any \( i \in I \setminus I_m, \sigma \) and \( \varphi^{(m,m)} \) are biunique, then \( \varphi \) is biunique.

**Proof of (1).** If \( \varphi \) is biunique, \( \forall H \subset I_m \) such that \( 0 < |H| \leq 2 \); we have \( \pi_{I,H}(\varphi(U)) = R^H \in \tau^{(H)} \); then, from Proposition 31, the functions \( g_i, \forall i \in I \setminus I_m, \) and \( \varphi^{(m,m)} \) are injective, and \( \sigma \) is biunique; thus, \( \forall i \in I \setminus I_m, \) we have \( \sigma(i) = i \). Moreover, \( \forall i \in I \setminus I_m, g_i \) is surjective, since \( \varphi \) is surjective. Furthermore, \( \forall y = (y_1, \ldots, y_n) \in R^n \), let \( x_i \in I \setminus I_m, \) \( \varphi \in \Pi_{i \in I_m} A_j \in E_{I \setminus I_m} \) and let \( \tilde{y} \in E_i \) be such that

\[
\tilde{y}_i = y_i + \sum_{j \in I \setminus I_m} q_{ij} (x_j), \quad \forall i \in I \setminus I_m,
\]

\[
\tilde{y} = g_i (x_i), \quad \forall i \in I \setminus I_m;
\]

moreover, let \( \tilde{x} = \varphi^{-1}(\tilde{y}) \) \( \in U \), from which \( \tilde{y}_i = g_i(\tilde{x}_i), \forall i \in I \setminus I_m \). Since \( g_i \) is injective, we have \( \tilde{x}_i = x_i, \forall i \in I \setminus I_m \); then, \( \forall i \in I_m \), we have

\[
q_i^{(m,m)} (x_i, \ldots, x_m) + \sum_{j \in I \setminus I_m} q_{ij} (x_j) = \tilde{y}_i
\]

\[
= y_i + \sum_{j \in I \setminus I_m} q_{ij} (x_j) \implies
\]

\[
y = \varphi^{(m,m)} (x),
\]

where \( x = (x_i, \ldots, x_m) \in U^{(m)} \); then, \( \varphi^{(m,m)} \) is surjective.

**Proof of (2).** If the functions \( g_i, \forall i \in I \setminus I_m, \) and \( \varphi^{(m,m)} \) are biunique, from Proposition 31, we obtain that \( \varphi \) is injective. Moreover, \( \forall y \in E_i \), define \( x \in U^{(m)} \times \Pi_{j \in I_m} A_j \) in the following manner:

\[
x_i = g_i^{-1} (y_i) \in A_j, \quad \forall i \in I \setminus I_m,
\]

\[
(x_i, \ldots, x_m) = \left( \varphi^{(m,m)} \right)^{-1} (z_i, \ldots, z_m) \in U^{(m)},
\]

where

\[
z_i = y_i - \sum_{j \in I \setminus I_m} q_{ij} (x_j), \quad \forall i \in I_m.
\]

Let \( x_0 = (x_{0i} : i \in I) \in U; \forall i \in I \setminus I_m \), we have

\[
|x_i| = \left| g_i^{-1} (y_i) - x_{0i} + x_{0i} \right|
\]

\[
\leq \left| g_i^{-1} (y_i) - g_i^{-1} (g_i (x_{0i})) \right| + |x_0|;
\]

moreover, the function \( g_i^{-1} : R \rightarrow A_j \) is derivable, and

\[
\left( g_i^{-1} \right)' (t) = \frac{1}{g_i (g_i^{-1} (t))} \in R^* \quad \forall i \in I \setminus I_m, \forall t \in R;
\]

then, the Lagrange theorem implies that, for some \( \xi_i \in \) \( \text{mid}(y_i, g_i (x_{0i})), \max(y_i, g_i (x_{0i}))), \) we have

\[
\left| \left( g_i^{-1} \right)' (\xi_i) \right| \left| y_i - g_i (x_{0i}) \right|,
\]

the, thus, from (37) and (38), we obtain

\[
|x_i| \leq \left| y_i - g_i (x_{0i}) \right| \left| g_i^{(m,m)} (\xi_i) \right| + |x_0|.
\]

Moreover, we have \( \prod_{i \in I_m} |g_i (g_i^{-1} (\xi_i))| = \prod_{i \in I_m} |g_i (g_i^{-1} (\xi_i))| = R^* \), from which there exists \( i_o \in I \setminus I_m \) such that \( \forall i \in I \setminus I_m, i > i_o \), we have \( |g_i (g_i^{-1} (\xi_i))] > 1/2 \); then, there exists \( c \in \) \( R^* \) such that \( \sup_{i \in I_m} |g_i (g_i^{-1} (\xi_i))] \leq c \), and so formula (40) implies

\[
\sup_{i \in I_m} \left| |x_i| \leq c (\left| y_i \right| + \left| g_i (x_{0i}) \right|) + \left| x_0 \right| \right| < +\infty.
\]

then, we have \( x \in E_i \), from which \( x \in U \). Finally, it is easy to prove that \( \varphi(x) = y \), and so \( \varphi \) is surjective.

**Remark 33.** Let \( \varphi : U \subset E \rightarrow E \) be a \((m, \sigma)\)-standard function such that \( \varphi_i(x_i) = 0 \), for any \( x_j \in A_j, \) for any \( i \in I_m \), and for any \( j \in I \setminus I_m \); then,

1. if \( \varphi \) is injective, and \( \sigma_{I,H}(\varphi(U)) \in \tau^{(H)} \), for any \( H \subset I \setminus I_m \) such that \( 0 < |H| \leq 2 \), then the functions \( g_i \), for any \( i \in I \setminus I_m, \), and \( \varphi^{(m,m)} \) are injective, and \( \sigma \) is biunique.

2. if \( \varphi \) is biunique, then the functions \( g_i \), for any \( i \in I \setminus I_m, \), and \( \varphi^{(m,m)} \) and \( \sigma \) are biunique.

**Proof of (1).** Suppose that \( \varphi \) is injective; by proceeding as in the proof of Proposition 31, we have that the functions \( g_i, \forall i \in I \setminus I_m, \) and \( \varphi^{(m,m)} \) are injective, and \( \sigma \) is surjective. Moreover, \( \sigma \) is surjective; in fact, suppose by contradiction that there exists \( n \in (I \setminus I_m) \) \( \setminus \sigma(I \setminus I_m) \), and let \( x = (x_i : i \in I) \); moreover, \( \forall i \in A_n, i \neq n \), let \( \tilde{x} = (x_j : j \in I) \in U \) be the sequence defined by \( \tilde{x}_j = x_j, \forall j \neq n, \) and \( \tilde{x}_n = t \); we have

\[
\varphi_i (x) = \varphi_i^{(m,m)} (x_i, \ldots, x_m) = \varphi_i^{(m,m)} (x_i, \ldots, x_m)
\]

\[
= \varphi_i (\tilde{x}), \quad \forall i \in I_m;
\]

\[
\varphi_i (x) = g_i (x_{0i}) = g_i (\tilde{x}_{0i}) = \varphi_i (\tilde{x}), \quad \forall i \in I \setminus I_m;
\]

thus, we have \( \varphi(x) = \varphi(\tilde{x}) \), and so \( \varphi \) is not injective (a contradiction).

**Proof of (2).** Since \( \varphi \) is surjective, the functions \( g_i, \forall i \in I \setminus I_m, \) and \( \varphi^{(m,m)} \) are surjective; moreover, \( \forall H \subset I \setminus I_m \) such that \( 0 < |H| \leq 2 \), we have \( \pi_{I,H}(\varphi(U)) = R^H \in \tau^{(H)} \); then, since \( \varphi \) is injective, from point 1, the functions \( g_i, \forall i \in I \setminus I_m, \) and \( \varphi^{(m,m)} \) are injective, and \( \sigma \) is biunique.
Corollary 34. Let $\varphi : U \subset E_1 \to E_1$ be a $(m, \sigma)$-standard function such that $\pi_{1,H}(\varphi(U)) \in \mathcal{R}^{(3)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, and $\overline{\varphi}$ is injective, then, the functions $\varphi$, $\varphi^{(n)}$, and $\varphi^{(n,m)}$, for any $n \in \mathbb{N}$, $n \geq m$, are injective.

Proof. Observe that $\overline{\varphi}$ is $(m, \sigma)$-standard, and $\pi_{1,H}(\overline{\varphi}(U)) = \pi_{1,H}(\varphi(U)) \in \mathcal{R}^{(3)}$, $\forall H \subset I \setminus I_m$ such that $0 < |H| \leq 2$; then, from Remark 33, we have that the functions $\varphi_i$, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and $\sigma$ is biunique; then, from Proposition 31, $\varphi$ is injective; analogously, since $\forall n \in \mathbb{N}$, $n \geq m$, the function $\varphi^{(n)}$ is $(n, \sigma)$-standard, from Proposition 31 $\varphi^{(n)}$ is injective, Moreover, we have $\pi_{1,H}(\varphi^{(n)}(U)) = \pi_{1,H}(\varphi(U)) \in \mathcal{R}^{(3)}$, $\forall H \subset I \setminus I_n$ such that $0 < |H| \leq 2$; then, from Remark 33, $\varphi^{(n,m)} = (\varphi^{(n)})^{(m,m)}$ is injective.

Proposition 35. Let $\varphi : U \subset E_1 \to E_1$ be a function $C^1$ in $x_0 = (x_{0,i} : j \in I_0)$ and $(m, \sigma)$-standard. Then, for any $n \geq m$, the function $\varphi^{(n)} : \pi_{1,\mathcal{A}}(U) \to \mathbb{R}^n$ is $C^1$ in $(x_{0,i} : j \in I_n)$, the function $\varphi^{(n)} : U \subset E_1 \to E_1$ is $C^1$ in $x_0$, there exists the function $\varphi^{(n)}(x_0) : E_1 \to E_1$, and it is continuous. Moreover, if $\varphi$ is $C^1$ in $x_0$ and strongly $(m, \sigma)$-standard, then $\varphi^{(n)}$ is differentiable in $x_0$. Finally, if $\varphi$ is strongly $C^1$ in $x_0$ and strongly $(m, \sigma)$-standard, then $\varphi^{(n)}$ is differentiable in $x_0$.

Proof. By assumption, there exists a neighbourhood $V = \bigcap_{j \in I_0} V_j$ of $x_0$ such that, for $i \in I_0$, there exists the function $x_j \to \partial \varphi_i(x) / \partial x_j$ on $V$, and this function is continuous in $x_0$; then, $\forall x \in V$, there exists $(x_{0,i} : j \in I_n) = \bigcap_{j \in I_n} V_j$ such that $(x_{j,i} : j \in I_n) = x$; since $\varphi$ is a $(m, \sigma)$-standard function, $\forall i, j \in I_n$, we have

$$\frac{\partial \varphi^{(n)}(x)}{\partial x_j} = \frac{\partial \varphi_i(x)}{\partial x_j},$$

from which $\varphi^{(n)}$ is $C^1$ in $(x_{0,i} : j \in I_n)$. Moreover, $\forall x \in V$, we have

$$\frac{\partial \varphi^{(n)}(x)}{\partial x_j} = \begin{cases} \frac{\partial \varphi_i(x)}{\partial x_j} & \text{if } (i, j) \in (I \setminus I_m) \times (I \setminus I_m) \\ 0 & \text{if } (i, j) \in I_m \times (I \setminus I_m) \end{cases}$$

and so $\varphi^{(n)}$ is $C^1$ in $x_0$. Furthermore, $\forall i \in I$, the function $\varphi^{(n)} : U \subset E_1 \to \mathbb{R}$ is differentiable in $x_0$, since $\varphi^{(n)}$ depends only on a finite number of variables, and so there exists the function $\varphi^{(n)}(x_0) : E_1 \to \mathbb{R}$, moreover, $\forall i \in I \setminus I_m$ we have $\| \varphi^{(n)}(x_0) \| = \| g_i'(x_{0,i}) \|$ and since the sequence $\{ g_i'(x_{0,i}) \}_{i \in I_0}$ is convergent, this implies that $\sup_{i \in I_0} \| g_i'(x_{0,i}) \| < +\infty$, and so $\sup_{i \in I_0} \| \varphi^{(n)}(x_0) \| < +\infty$; then, there exists the function $\varphi^{(n)}(x_0) : E_1 \to E_1$, and it is continuous.

Moreover, suppose that $\varphi$ is $C^1$ in $x_0$ and strongly $(m, \sigma)$-standard; then, the sequence $\{ (-1)^{|i|+|\sigma(i)|} g_i'(x_{0,i}) \}_{i \in I_0}$ converges uniformly to 1, with respect to $x = (x_j : j \in I \setminus I_m) \in \prod_{j \in I \setminus I_m} A_j$; thus, $\forall \varepsilon > 0$, there exists $\delta > 0$, such that $\forall i \in I \setminus I_m$, we have

$$\frac{|(-1)^{|i|+|\sigma(i)|} g_i'(x_{0,i}) - 1|}{\delta} < \frac{\varepsilon}{2},$$

Observe that, since $\forall i \leq \delta$ the function $g_i$ is differentiable in $x_{0,i}$, there exists a neighbourhood $N \subset \pi_{1,\mathcal{A}}(U)$ of $x_0$ such that, $\forall x \in N \setminus x_0$, we have

$$\frac{\| \varphi^{(n)}(x) - \varphi^{(n)}(x_0) - \varphi^{(n)}(x_0)(x - x_0) \|}{\| x - x_0 \|} < \varepsilon.$$
Then, \( \forall B \in \tau \), we have \((\varphi(\{n\}))^{-1}(B) \in \tau^{(n)} \); moreover, if we consider \(\varphi(\{n\})\) as a function from \(U \to R\), we have \(\pi_{1,\mathbb{R}}((\varphi(\{n\}))^{-1}(B)) = \mathbb{R}^{\tau^{(n)}}\), and so \((\varphi(\{n\}))^{-1}(B) \in \tau^{(1)}\); then, since \(\sigma(\tau) = B, \forall B \in B\), we have \((\varphi(\{n\}))^{-1}(B) \in \tau^{(n)}\), and so \((\varphi(\{n\}))^{-1}(B) \in \tau^{(1)}(U)\). Moreover, since \(\lim_{n \to \infty} \varphi(\{n\}) = \varphi\), the function \(\varphi\) is \((\tau^{(1)}(U), \mathcal{B}(U))\)-measurable. Let

\[
\Sigma(I) = \left\{ B = \bigcap_{i \in I} B_i : B_i \in \mathcal{B}, \forall i \in I \right\}
\]

and

\[
\forall B \in \Sigma(I),\text{ we have } \varphi^{-1}(B) \in \tau^{(I)}(U).
\]

Finally, since \(\sigma(\Sigma(I)) = \mathcal{B}(I), \forall B \in \mathcal{B}(I), \) we have \(\varphi^{-1}(B) \in \tau^{(I)}(U)\).

**Proposition 37.** Let \( \varphi : U \subset E_1 \to E_1 \) be a \((m, \sigma)\)-standard function, such that \( \varphi(U) \to \varphi(U) \) is a homeomorphism. Then, the functions \( \varphi(\{m\}) : U^{(m)} \to \varphi(\{m\}) (U^{(m)}) \) and \( g_i : A_i \to g_i(A_i), \forall i \in I \setminus M, \) are homeomorphisms, and \( \sigma \) is biunique.

Proof. Since \( \varphi \) is a homeomorphism, we have \( \varphi(U) \in \tau_{1U} \); then, from Proposition 6, \( \forall H \subset I \setminus M, \) such that \( 0 < |H| \leq 2, \) we have \( \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau_{1U} = \tau^{(1)} \); thus, since \( \varphi \) is injective, from Remark 33, the functions \( g_i, \forall i \in I \setminus M, \) are bijective, and \( \sigma \) is biunique; then, the functions \( g_i \) are derivable, and so they are continuous. Moreover, we have \( \varphi(\{m\})(U^{(m)}) = \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau^{(m)}, g_i(A_i) = \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau, \forall i \in I \setminus M. \) Finally, from Remark 10, the function \( \pi_{1,\mathbb{R}} \circ \varphi : (U, \tau_{1U}) \to (\mathbb{R}^m, \tau^{(m)}) \) is continuous; thus, \( \forall B \in \tau^{(m)}, \) we have \( \varphi(\{m\})(B) \times \bigcap_{i \in I \setminus M} A_i = \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau_{1U}(U), \) from which \( \varphi(\{m\})(B) \in \tau^{(m)}(U^{(m)}) \); then, \( \varphi(\{m\}) \) is continuous; analogously, we can prove that the function \( \varphi(\{m\}) \) is continuous.

**Proposition 38.** Let \( \varphi : U \subset E_1 \to E_1 \) be a \((m, \sigma)\)-standard function such that \( \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau^{(1)}, \) for any \( H \subset I \setminus M, \) such that \( 0 < |H| \leq 2. \) Then, \( \varphi \) is a diffeomorphism, and so it is a biunique function, and \( \sigma \) is biunique.

Proof. We have \( \pi_{1,\mathbb{R}}(\varphi(U)) = \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau^{(1)}, \) for any \( H \subset I \setminus M, \) such that \( 0 < |H| \leq 2. \) If \( \varphi \) is a diffeomorphism, then \( \varphi \) is injective, and so, from Remark 33, the functions \( g_i, \forall i \in I \setminus M, \) are injective, and \( \sigma \) is biunique. Moreover, \( \varphi \) is \( C^1 \) in \( U, \) and so, from Proposition 35, \( \varphi(\{m\}) = \varphi(\{m\})(U^{(m)}) = C^1 \) in \( U^{(m)} \); analogously, since \( \varphi^{-1} : \varphi(U) \to U \) is \( C^1 \) in \( \varphi(U), \) \( \varphi(\{m\})(B) \times \bigcap_{i \in I \setminus M} A_i = \pi_{1,\mathbb{R}}(\varphi(U)) \in \tau_{1U}(U), \) then, \( \varphi(\{m\}) \) is a diffeomorphism. Moreover, \( \forall i \in I \setminus M, \) let \( x_i = g_i^{-1}(y_i); \) then, \( \forall i \in I \setminus M, \)

\[
y_i = \varphi_i \left( x_i, \ldots, x_{i_m} \right) + \sum_{j \in I \setminus M} \varphi_{i_j} (g^{-1} (y_j)) \tag{55}
\]
and so
\[ (x_i, \ldots, x_m) = (\varphi^{(m,m)})^{-1}(z_i, \ldots, z_m), \] (56)
\[ x_i = g_i^{-1}(y_i), \quad \forall i \in I_n \setminus I_m, \]
where
\[ z_i = y_i - \sum_{j \in I_n \setminus I_m} q_{ij} (g_j^{-1}(y_j)), \quad \forall i \in I_n \setminus I_m. \] (57)

Then, since the functions \( (q^{(m,m)}(\cdot))^{-1}, q_{ij}, \) and \( g_i^{-1} \) are continuous, the function \( (\varphi^{(n,n)}(\cdot))^{-1} \) is continuous too, and so \( q^{(n,n)}(\pi_{I_n}(U)) \in \tau^{(n)} ; \) then, \( \varphi^{(n,n)} : \pi_{I_n}(U) \to \varphi^{(n,m)}(\pi_{I_m}(U)) \) is a homeomorphism. Furthermore, since \( \bar{\varphi}(U) \in \tau^{(1)} \), we have \( \pi_{I_n \setminus I_m}(\bar{\varphi}(U)) \in \tau^{(1)} \), and so
\[ \varphi^{(n)}(U) = \varphi^{(n,n)}(\pi_{I_n}(U)) \times \pi_{I_n \setminus I_m}(\bar{\varphi}(U)) \in \tau^{(1)}. \] (58)

Proof of (2). If \( \bar{\varphi} : U \to \bar{\varphi}(U) \) is a diffeomorphism, from Proposition 38, the functions \( q^{(m,m)} : U^{(m)} \to q^{(m,m)}(U^{(m)}) \) and \( g_i : A_i \to g_i(A_i) \), \( \forall i \in I_n \setminus I_m \), are diffeomorphisms; in particular, the functions \( (q^{(m,m)}(\cdot))^{-1} \) and \( g_i^{-1} \), \( \forall i \in I_n \setminus I_m \), are \( C^1 \); then, since \( \forall i \in I_m \), \( j \in I_n \setminus I_m \), the function \( \varphi_{ij} \) is \( C^1 \), from (56) and (57), we obtain that the function \( (\varphi^{(n,n)}(\cdot))^{-1} \) is \( C^1 \) in \( \tau^{(n)}(I_n(U)) \). Moreover, from Proposition 35 and since, \( \forall i \in I_n \setminus I_m \), the functions \( g_i \) are \( C^1 \), \( \varphi^{(n,n)} \) is \( C^1 \) in \( \pi_{I_n}(U) \), and so \( \varphi^{(n,n)}(\cdot) \) is a diffeomorphism. Finally, since \( \bar{\varphi}(U) \in \tau^{(1)} \), \( \forall i \in I_n \setminus I_m \) such that \( 0 < |H| \leq 2 \), we have \( \pi_{I_n}(\bar{\varphi}(U)) \in \tau^{(1)} \), then, from Proposition 38, \( \varphi^{(n)} : U \to \varphi^{(n,m)}(U) \) is a diffeomorphism. \( \square \)

Remark 40. A linear function \( A = \left( a_{ij} \right)_{i,j \in I} : E_i \to E_j \) is \((m,\sigma)\)-standard, where \( m \in \mathbb{N}^* \) and \( \sigma : I \setminus I_m \to I \setminus I_m \) is an increasing function, if
\begin{enumerate}
  \item \( a_{ij} = 0, \forall (i, j) \notin (I_m \times I) \cup \bigcup_{n \in I \setminus I_m} \{(m, \sigma(n))\}; \)
  \item there exists \( \prod_{i \in I \setminus I_m} a_{\sigma(i)} \lambda_i \in \mathbb{R}^* \), where \( \lambda_i = (-1)^{|I_i|+|\sigma(i)|}a_{\sigma(i)} \), \( \forall i \in I \setminus I_m \).
\end{enumerate}

Recall the following concept, defined in [1].

Definition 41. Let \( A = \left( a_{ij} \right)_{i,j \in I} : E_i \to E_j \) be a linear \((m,\sigma)\)-standard function; define the determinant of \( A \), and call it \( \det_{(m,\sigma)}A \), or det \( A \), the real number
\[ \det_{(m,\sigma)}A = \begin{cases} 
\det A^{(m,m)} \prod_{i \in I \setminus I_m} \lambda_i & \text{if } \sigma \text{ is biunique} \\
0 & \text{if } \sigma \text{ is not biunique,} 
\end{cases} \] (59)

where \( A^{(m,m)} \) is the \( m \times m \) real matrix defined by \( A^{(m,m)}(i,j) = a_{ij}, \forall i, j \in I_m \).

Proposition 42. Let \( A = (a_{ij})_{i,j \in I} : E_i \to E_j \) be a linear and injective \((m,\sigma)\)-standard function; then, \( A \) is biunique if and only if \( A \) is \((\tau_{H^1}, \tau_{H^1})\)-open.

Proof. From Remark 9 in [1], \( A \) is continuous; then, if \( A \) is biunique, from the Banach theorem of the open function, \( A \) is \((\tau_{H^1}, \tau_{H^1})\)-open. Conversely, if \( A \) is \((\tau_{H^1}, \tau_{H^1})\)-open, \( \forall i \in I \setminus I_m \) such that \( 0 < |H| \leq 2 \), we have \( \pi_{H_i}(\varphi(A(E_i))) \in \tau^{(H)}_{H_i} = \tau^{(H_i)}_{H_i} \); moreover, since \( A \) is linear, the function \( A_i : E_i \to \mathbb{R} \) is \( C^1 \), \( \forall i \in I_m \); furthermore, by proceeding as in the proof of Proposition 31, we obtain that \( A^{(m,m)} \) is injective, and so det \( A^{(m,m)}(x) \neq 0, \forall x \in E_i \); then, from Proposition 31, the functions \( g_i, \forall i \in I \setminus I_m \), are injective, and \( A \) is biunique. Thus, since the functions \( A^{(m,m)} \) and \( g_i, \forall i \in I \setminus I_m \), are biunique, and so \( A \) is biunique from Proposition 32. \( \square \)

4. Change of Variables’ Formula

Henceforth, we will suppose \( |I| = +\infty \).

Definition 43. Let \( k \in \mathbb{N}^* \), let \( M, N \in \mathbb{R}^* \), let \( a = (a_i : i \in I) \in [0, +\infty)^I \) such that \( \prod_{i \in I \setminus I_0} a_i \in \mathbb{R}^* \), and let \( v = (v_i : i \in I) \in E_i \); define the following sets in \( \mathcal{B}^I \):
\[ E^{(k,l)}_{N,a,v} = \mathbb{R}^k \times \prod_{i \in I \setminus I_k} \left( v_i - Na_i, v_i + Na_i \right); \]
\[ E^{(k,l)}_{M,N,a,v} = (-M, M)^k \times \prod_{i \in I \setminus I_k} \left( v_i - Na_i, v_i + Na_i \right). \] (60)

Moreover, define the \( \sigma \)-finite measure \( \lambda^{(k,l)}_{N,a,v} \) over \((\mathbb{R}^I, \mathcal{B}(\mathbb{R}^I))\) in the following manner:
\[ \lambda^{(k,l)}_{N,a,v} = \left( \prod_{i \in I \setminus I_k} \frac{1}{2N} \right) \mathcal{L} \left( \mathcal{B} \cap \left( v_i - Na_i, v_i + Na_i \right) \right). \] (61)

Proposition 44. Let \( \varphi : U \subset E_i \to E_j \) be a \((m,\sigma)\)-standard function such that \( \bar{\varphi} \) is biunique; moreover, let \( N \in \mathbb{R}^* \), let \( a = (a_i : i \in I) \in [0, +\infty)^I \) such that \( \prod_{i \in I \setminus I_0} a_i \in \mathbb{R}^* \), and let \( v \in E_i \); then,
\begin{enumerate}
  \item there exist \( b = (b_i : i \in I) \in [0, +\infty)^I \) and \( z \in E_j \) such that \( \prod_{i \in I \setminus I_0} b_i \in \mathbb{R}^* \) and such that, for any \( n, k \in \mathbb{N}, n \geq m, k \geq m, \) one has
\[ \varphi^{-1} \left( E^{(k,l)}_{N,a,z} \right) \cap \left( U^{(m)} \times \tau^{(m)}(U^{(m)}) \right) \to \left( \mathbb{R}^m, \tau^{(m)} \right) \]
\[ \varphi^{-1} \left( E^{(k,l)}_{N,a,z} \right) = \left( \prod_{i \in I \setminus I_k} \left( v_i - Na_i, v_i + Na_i \right) \right). \] (62)
  \item suppose that \( \varphi_i : U \to \mathbb{R} \) is \( C^1 \), for any \( i \in I_m \), and the function \( \varphi^{(m,m)} : (U^{(m)}, \tau^{(m)}(U^{(m)}) \to (\mathbb{R}^m, \tau^{(m)}) \).
is open; then, for any \( M_1 \in \mathbb{R}^* \), there exists \( M_2 \in \mathbb{R}^* \) such that, for any \( n,k \in \mathbb{N}, n \geq m, k \geq m \), one has
\[
\phi^{-1}\left(E_{M_1,N,a,v}\right) \subset E_{M_2,N,b,z}^{(k,l)}.
\]
(63)

Proof of (1). Since \( \overline{\phi} \) is biuniqueness, from Remark 33, the functions \( g_1, \forall i \in I \setminus I_m \), and \( \sigma \) are biuniqueness. Set \( b = (b_i : i \in I) \) by definition of \( \phi \). If \( i \in I \), \( z = (z_i : i \in I) \in E_I \) such that \( b_i = z_i = 1 \), \( \forall i \in I_m \); moreover, define
\[
b_i = \frac{\left| g_1^{-1}(v_i + Na_i) - g_1^{-1}(v_i - Na_i) \right|}{2N}\), \( \forall i \in I \setminus I_m \),
(64)

Observe that, \( \forall i \in I \setminus I_m \), we have \( b_i = 0 \) if and only if \( a_i = 0 \); then, by definition of \( b = (b_i : i \in I) \), we have
\[
\prod_{i \in I \setminus I_m, i \neq 0} b_i = \left( \prod_{i \in I \setminus I_m, i \neq 0} \left| g_1^{-1}(v_i + Na_i) - g_1^{-1}(v_i - Na_i) \right| \right)^{\frac{1}{2N}}. \prod_{i \in I \setminus I_m, i \neq 0} a_i.
\]
(65)

Moreover, since \( \forall i \in I \setminus I_m \) the function \( g_1^{-1} \) is derivable on \( \mathbb{R} \), if \( a_i \neq 0 \), the Lagrange theorem implies that, for some \( \xi_i \in (v_i - Na_i, v_i + Na_i) \), we have
\[
g_1^{-1}(v_i + Na_i) - g_1^{-1}(v_i - Na_i) = (g_1^{-1})'(\xi_i)\]
(66)

from which \( \prod_{i \in I \setminus I_m, i \neq 0} b_i \in \mathbb{R}^* \). Furthermore, \( \forall i \in I \setminus I_m \), we have
\[
g_1^{-1}(v_i - Na_i, v_i + Na_i) = \begin{cases} 1 & \text{if } g_1 \text{ is increasing} \\ \left\{(g_1^{-1}(v_i - Na_i), g_1^{-1}(v_i + Na_i)) \right\} & \text{if } g_1 \text{ is decreasing} \end{cases}
\]
(67)

from which
\[
g_1((z_i - Nb_i, z_i + Nb_i)) = (v_i - Na_i, v_i + Na_i).
\]
(68)

Then, from (67) and (68), \( \forall k \in \mathbb{N}, k \geq m \), we obtain
\[
\phi^{-1}\left(E_{M_1,N,a,v}\right) \subset E_{M_2,N,b,z}^{(k,l)}, \quad \phi\left(E_{N,b,z}^{(k,l)}\right) \subset E_{M_2,N,a,v}^{(k,l)} \quad \Rightarrow \quad E_{M_2,N,b,z}^{(k,l)} \subset \phi^{-1}\left(E_{N,b,z}^{(k,l)}\right) \subset \phi^{-1}\left(E_{M_1,N,a,v}\right) \subset E_{M_2,N,b,z}^{(k,l)}.
\]
(69)

Analogously, \( \forall n \in \mathbb{N}, n \geq m \), from (67) and (68), we have
\[
\phi^{(n)}\left(E_{M_1,N,a,v}\right) = E_{M_2,N,b,z}^{(k,l)}.
\]
\( \square \)

Proof of (2). Suppose that \( \phi_i : U \to \mathbb{R} \) is \( C^1 \), for any \( i \in I_m \), and \( \phi^{(m)} : (U^{(m)}, r^{(m)}(U^{(m)})) \to (\mathbb{R}^m, r^{(m)}) \) is an open function; thus, \( \forall k \in \mathbb{N}, k \geq m \), and \( \forall x = (x_i : i \in I) \in \phi^{-1}(E_{M_1,N,a,v}^{(k,l)}) \), let \( y = \phi(x) \in (E_{M_1,N,a,v}^{(k,l)}) \); \( \forall i \in I \setminus I_m \), we have \( x_i = g^{-1}_i(y_i) \); then, \( \forall i \in I_m \),
\[
y_i = \phi^{(m)}(x_i, \ldots, x_i) + \sum_{j \in I \setminus I_m} \phi_j(g^{-1}_j(y_j)),
\]
(70)

and so
\[
(x_i, \ldots, x_i) = \left(\phi^{(m)}(y_i)\right)^{-1}(z_i, \ldots, z_i),
\]
(71)

\[
x_i = g^{-1}_i(y_i), \quad \forall i \in I \setminus I_m,
\]
(72)

Moreover, since \( \forall i \in I \setminus I_m \), the functions \( \phi_j \) and \( g^{-1}_j \) are continuous and \( |y_j| \leq \max\{M_1, \|\phi^{(m)}\|_{I \setminus I_m} + N\|\phi\|_{I_m}\} \), there exists \( M \in \mathbb{R}^* \) such that
\[
\sup_{i \in I \setminus I_m, j \in I \setminus I_m} \left| \phi_j(g^{-1}_j(y_j)) \right| \leq M,
\]
(73)

and so \( \|z_i, \ldots, z_i\|_{I_m} \leq M_1 + M \); then, since the function \( \phi^{(m)}(y) \) is continuous, there exists \( M \in \mathbb{R}^* \) such that \( \|x_i, \ldots, x_i\|_{I_m} \leq M \); finally, if \( b, c \) are the sequences defined by the point (1), by setting \( M_2 = \max\{M, \|\phi^{(m)}\|_{I \setminus I_m} + N\|\phi\|_{I_m}\} \), from (67), we have \( \phi^{-1}(E_{M_1,N,a,v}^{(k,l)}) \subset E_{M_2,N,b,z}^{(k,l)} \). Analogously, \( \forall n \in \mathbb{N}, n \geq m \), we have
\[
\sup_{i \in I \setminus I_m, j \in I \setminus I_m} \left| \phi_j(g^{-1}_j(y_j)) \right| \leq M,
\]
(74)

from which \( \phi^{(n)}(E_{M_1,N,a,v}^{(k,l)}) \subset E_{M_2,N,b,z}^{(k,l)} \). \( \square \)

**Proposition 45.** Let \( (S, \Sigma) \) be a measurable space, let \( \mathcal{F} \) be a \( \pi \)-system on \( S \), and let \( \mu_1 \) and \( \mu_2 \) be two measures on \( (S, \Sigma) \), \( \sigma \)-finite on \( \mathcal{F} \); if \( \sigma(\mathcal{F}) = \Sigma \) and \( \mu_1 \) and \( \mu_2 \) coincide on \( \mathcal{F} \), then \( \mu_1 \) and \( \mu_2 \) coincide on \( \Sigma \).

**Proof.** See, for example, Theorem 10.3 in Billingsley [10]. \( \square \)

**Lemma 46.** Let \( k \in \mathbb{N}^* \), let \( N \in \mathbb{R}^* \), let \( a = (a_i : i \in I) \) in \( [0, +\infty)^I \) such that \( \prod_{i = \mathbb{N}, a, v} \in \mathbb{R}^* \), and let \( v = (v_i : i \in I) \) in \( E_I \); then, for any measurable function \( f : \mathbb{R}^k \to (R, B) \) such that \( f^*(\mathcal{F}) \) or \( f^* \) is \( \lambda_{N,a,v}^{(k,l)} \)-integrable, one has
\[
\int_{\mathbb{R}^k} f d \lambda_{N,a,v}^{(k,l)} = \int_{\mathbb{R}^k} f d \lambda_{N,a,v}^{(k,l)}.
\]
(75)
Proof. Let \( B = \prod_{i \in I} B_i \in \mathcal{B}(I) \); by definition of \( \lambda^{(k,j)}_{N,a,v} \), we have

\[
\int_{R^I} 1_B \, d\lambda^{(k,j)}_{N,a,v} = \int_{E^{(k,j)}_{N,a,v}} 1_B \, d\lambda^{(k,j)}_{N,a,v};
\]

(76)

then, consider the measures \( \mu_1 = \lambda^{(k,j)}_{N,a,v} \) and \( \mu_2 = \lambda^{(k,j)}_{N,a,v} \cap E^{(k,j)}_{N,a,v} \) on \( (R^I, \mathcal{B}(I)) \); from (76), \( \mu_1 \) and \( \mu_2 \) coincide on the set \( \mathcal{F} = \{ B \in \mathcal{B}(I) : B = \prod_{i \in I} B_i \}; moreover, we have \( R^I = \bigcup_{n \in N} [-n,n]^k \times R^{I \setminus I_k} \), where \([-n,n]^k \times R^{I \setminus I_k} \in \mathcal{F} \), \( \mu_1([-n,n]^k \times R^{I \setminus I_k}) = \mu_2([-n,n]^k \times R^{I \setminus I_k}) < +\infty \), \( \forall n \in N \), and so \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-finite on \( \mathcal{F} \). Then, since \( \mathcal{F} \) is a \( \pi \)-system on \( R^I \) such that \( \sigma(\mathcal{F}) = \mathcal{B}(I) \), from Proposition 45, formula (76) is true \( \forall B \in \mathcal{B}(I) \). This implies that, if \( \psi : (R^I, \mathcal{B}(I)) \rightarrow (\{0, +\infty\}, \mathcal{B}(\{0, +\infty\})) \) is a simple function, we have

\[
\int_{R^I} \psi \, d\lambda^{(k,j)}_{N,a,v} = \int_{E^{(k,j)}_{N,a,v}} \psi \, d\lambda^{(k,j)}_{N,a,v};
\]

(77)

Then, if \( f : (R^I, \mathcal{B}(I)) \rightarrow (\{0, +\infty\}, \mathcal{B}(\{0, +\infty\})) \) is a measurable function, \( \psi \in R^I \) is a sequence of increasing positive simple functions over \( (R^I, \mathcal{B}(I)) \) such that \( \lim_{n \rightarrow +\infty} \psi_n = f \), from Beppo Levi theorem we have

\[
\int_{R^I} f \, d\lambda^{(k,j)}_{N,a,v} = \lim_{n \rightarrow +\infty} \int_{R^I} \psi_n \, d\lambda^{(k,j)}_{N,a,v}
= \lim_{n \rightarrow +\infty} \int_{E^{(k,j)}_{N,a,v}} \psi_n \, d\lambda^{(k,j)}_{N,a,v}
= \int_{E^{(k,j)}_{N,a,v}} f \, d\lambda^{(k,j)}_{N,a,v};
\]

(78)

Then, for any measurable function \( f : (R^I, \mathcal{B}(I)) \rightarrow (R, \mathcal{B}) \) such that \( f^+ \) (or \( f^- \)) is \( \lambda^{(k,j)}_{N,a,v} \)-integrable,

\[
\int_{R^I} f \, d\lambda^{(k,j)}_{N,a,v} = \int_{R^I} f^+ \, d\lambda^{(k,j)}_{N,a,v} - \int_{R^I} f^- \, d\lambda^{(k,j)}_{N,a,v}
= \int_{E^{(k,j)}_{N,a,v}} f^+ \, d\lambda^{(k,j)}_{N,a,v} - \int_{E^{(k,j)}_{N,a,v}} f^- \, d\lambda^{(k,j)}_{N,a,v}
= \int_{E^{(k,j)}_{N,a,v}} f \, d\lambda^{(k,j)}_{N,a,v};
\]

(79)

Now, we can prove the main result of our paper, that improves Theorem 29 in [1] and generalizes the change of variables' formula for the integration of a measurable function on \( R^m \) with values in \( R \) (see, e.g., the Lang's book [9]).

**Theorem 47** (change of variables' formula). Let \( \varphi : U \subset E_I \rightarrow E_J \) be a \( C^1 \) and \( (m, n) \)-standard function, such that the function \( \tilde{\varphi} : U \rightarrow E_I \) is a diffeomorphism; moreover, let \( N \in R^I \), let \( a = (a_i : i \in I) \in [0, +\infty)^I \) such that \( \prod_{i \in I} a_i > 0 \), let \( v \in E_I \), and let \( b \in [0, +\infty)^I \) and \( z \in E_J \) defined by Proposition 44. Then, for any \( k \in N, k \geq m, \) for any \( B \in \mathcal{B}(I) \), and for any measurable function \( f : (R^I, \mathcal{B}(I)) \rightarrow (R, \mathcal{B}) \) such that \( f^+ \) (or \( f^- \)) is \( \lambda^{(k,j)}_{N,a,v} \)-integrable, one has

\[
\int_B f \, d\lambda^{(k,j)}_{N,a,v} = \left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right)
\]

(80)

\[
\int_B f \, d\lambda^{(k,j)}_{N,a,v} = \int_{B_{I_J} \times \mathbf{R}^{A_J}} f \, d\left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right).
\]

(81)

Moreover, we have

\[
\int_{\prod_{q \in I_J} B_q} d \left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right)
= \lim_{p \rightarrow +\infty} \int_{\prod_{q \in I_J} B_q} d \left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right)
= \lim_{p \rightarrow +\infty} \int_{\prod_{q \in I_J} B_q} d \left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right)
\]

(by Theorem 1)

\[
= \lim_{p \rightarrow +\infty} \int_{\prod_{q \in I_J} B_q} d \left( \prod_{q \in I_J} \frac{1}{2N} \text{Leb} \left[ (\varphi(v_{N,a,v}, v_{N,a,v}) \setminus B_q) \right] \right)
\]

(by Proposition 38)
\[
\lim_{p \to +\infty} \prod_{q \in I^p} \frac{1}{\prod_{q \in I^p}} \left| \det J_{g_q} (x_q) \right|
\]
\[
\cdot \prod_{q \in I^p} \prod_{q \in I^p} \left| \det J_{g_q} (x_q) \right|
\]
\[
\cdot d \left( \bigotimes_{q \in I^p} \frac{1}{2N} \text{Leb} \left| \partial g_q^i (B_q) \right| \right) \left( x_{I^p} \right)
\]
\[
= \prod_{q \in I^p} \prod_{q \in I^p} \left| \det J_{g_q} (x_q) \right|
\]
\[
\cdot d \left( \bigotimes_{q \in I^p} \frac{1}{2N} \text{Leb} \left| \partial g_q^i (B_q) \right| \right) \left( x_{I^p} \right)
\]

(by Corollary 3)

\[
= \prod_{q \in I^p} \prod_{q \in I^p} \left| \det J_{g_q} (x_q) \right|
\]
\[
\cdot d \left( \bigotimes_{q \in I^p} \frac{1}{2N} \text{Leb} \left| \partial (z_q - Nb_q z_q + Nb_q) \right| \right) \left( x_{I^p} \right).
\]

(82)

Then, from Proposition 39, formula (81) implies

\[
\int_B d\lambda^{(k,j)}_{N,\alpha,\nu} = \int_{(|q|^{(k,j)} - (B, \nu - B))} \left| \det J_{g_q} \right| d\lambda^{(k,j)}_{N,\alpha,\nu}.
\]

(83)

Moreover, since \((|q|^{(k,j)} - (B, \nu - B)) \subset E_{M,\alpha,\nu}^{(k,j)}\) from Proposition 44, we have

\[
\int E_{M,\alpha,\nu}^{(k,j)} 1_B d\lambda^{(k,j)}_{N,\alpha,\nu} = \int E_{M,\alpha,\nu}^{(k,j)} 1_B \left| \det J_{g_q} \right| d\lambda^{(k,j)}_{N,\alpha,\nu}.
\]

(87)

This implies that, if \(\psi : (\mathbb{R}^l, \mathcal{B}^{(l)}) \to ([0, +\infty), \mathcal{B}([0, +\infty)))\) is a simple function such that \(\psi(x) = 0, \forall x \notin E_{M,\alpha,\nu}^{(k,j)}\), we have

\[
\int E_{M,\alpha,\nu}^{(k,j)} \psi d\lambda^{(k,j)}_{N,\alpha,\nu} = \int E_{M,\alpha,\nu}^{(k,j)} \psi \left| \det J_{g_q} \right| d\lambda^{(k,j)}_{N,\alpha,\nu}.
\]

(88)

Then, if \(l : (\mathbb{R}^l, \mathcal{B}^{(l)}) \to ([0, +\infty), \mathcal{B}([0, +\infty)))\) is a measurable function such that \(l(x) = 0, \forall x \notin E_{M,\alpha,\nu}^{(k,j)}\) and \(\{\psi_l\}_{l \in \mathbb{N}}\) is a sequence of increasing positive simple functions.
over \((\mathbb{R}^l, \mathcal{B}^{(l)})\) such that \(\lim_{i \to +\infty} \psi_i = 1, \psi_i(x) = 0, \forall x \not\in E_{M_i,N,b,z}, \forall i \in \mathbb{N}\), from Beppo Levi theorem we have

\[
\int_{E^{(l)}_{M_i,N,b,v}} l d\lambda^{(l)}_{N,v} = \lim_{i \to +\infty} \int_{E^{(l)}_{M_i,N,b,v}} \psi_i d\lambda^{(l)}_{N,b,z} = \lim_{i \to +\infty} \int_{E^{(l)}_{M_i,N,b,z}} l(\varphi^n) |\det J\varphi^n| d\lambda^{(l)}_{N,b,z} = \int_{E^{(l)}_{M_i,N,b,z}} l(\varphi^n) |\det J\varphi^n| d\lambda^{(l)}_{N,b,z}.
\]

Moreover, we have \(\lim_{n \to +\infty} \varphi^{(n)} = \varphi\), and so \(\lim_{n \to +\infty} f_n(x) = l(\varphi(x)) \int_{E^{(l)}_{M_i,N,b,z}} d\lambda^{(l)}_{N,b,z}\); then, from the dominated convergence theorem:

\[
\lim_{n \to +\infty} \int_{E^{(l)}_{M_i,N,b,z}} l(\varphi^{(n)}) |\det J\varphi^n| d\lambda^{(l)}_{N,b,z} = \int_{E^{(l)}_{M_i,N,b,z}} l(\varphi) |\det J\varphi| d\lambda^{(l)}_{N,b,z},
\]

consequently, from (89), we have

\[
\int_{E^{(l)}_{M_i,N,b,z}} l(\varphi) |\det J\varphi| d\lambda^{(l)}_{N,b,z}.
\]

Let \(B = \prod_{j \in I} B_j \in \mathcal{B}^{(l)}(E^{(l)}_{M_i,N,b,z}), \) where \(B_j = (a_j, b_j), \) \(\forall j \in I_i; \) moreover, \(\forall n \in \mathbb{N}^*\), consider the function \(l_n : \mathbb{R}^l \to [0, 1]\) defined by

\[
l_n(x) = l_{(k)}(x : j \in I_k) = \prod_{j \in I_k} 1_{B_j}(x_j),
\]

where

\[
l_n(x) = \begin{cases} 1 & \text{if } x \in \prod_{j \in I_k} (a_j + \delta_j, b_j - \delta_j), \\ 0 & \text{otherwise,} \end{cases}
\]

then, by definition of \(E^{(l)}_{M_i,N,b,z}\) and since the functions \(\varphi^{(n,m)}\) and \(g''_j, \forall j \in I \setminus I_l\) are continuous, there exists \(\beta \in \mathbb{R}^*\) such that \(g(x) \leq \beta, \forall x \in E^{(l)}_{M_i,N,b,z}\), and so

\[
\int_{E^{(l)}_{M_i,N,b,z}} g d\lambda^{(l)}_{N,b,z} \leq \beta \lambda^{(l)}_{N,b,z} \left(E^{(l)}_{M_i,N,b,z}\right) = \beta \prod_{p \in I_k} \frac{1}{2N} \cdot \text{Leb}((-M_2, M_2))
\]

\[
\cdot \prod_{q \in I_k} \frac{1}{2N} \cdot \text{Leb}\left(\left[q - Nb_z, q + Nb_z\right]\right) = \frac{\beta M_k^2}{2Nk}
\]

\[
\cdot \prod_{q \in I_k} b_q < +\infty.
\]
and set $B_n = B \cap E(k,I)_n, N, a, V, \forall n \in N^*$ and $\forall B \in B(I)(E(k,I)_N, a, V)$. Since $B_n \subset B_{n+1}$, $A^{-1}(B_n) \subset A^{-1}(B_{n+1})$, from the continuity property of $\mu$ and $\nu$ and (99), we have

$$\int_B d\lambda^{(k,I)}_{N,a,v} = \lim_{n \to \infty} \int_{B_n} d\lambda^{(k,I)}_{N,a,v} = \lim_{n \to \infty} \int_{\varphi^{-1}(B_n)} |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (101)$$

$$= \int_{\varphi^{-1}(B)} |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (102)$$

Then, suppose that $B \in \mathcal{B}(I)$; from Lemma 46, formula (101), and Proposition 44, we have

$$\int_{R^1} 1_B d\lambda^{(k,I)}_{N,a,v} = \int_{R^1} d\lambda^{(k,I)}_{N,a,v}$$

$$= \int_{\varphi^{-1}(B)} |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (103)$$

Thus, by proceeding as in the proof of formula (89), for any measurable function $f : (R^1, \mathcal{B}(I)) \to ([0, +\infty), \mathcal{B}((0, +\infty)))$, we obtain

$$\int_{R^1} f d\lambda^{(k,I)}_{N,a,v} = \int_{R^1} f(\varphi) |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (103)$$

Then, for any measurable function $f : (R^1, \mathcal{B}(I)) \to (R, \mathcal{B})$ such that $f^+$ (or $f^-$) is $\lambda^{(k,I)}_{N,a,v}$-integrable and for any $B \in \mathcal{B}(I)$, we have

$$\int_B f d\lambda^{(k,I)}_{N,a,v} = \int_{R^1} f^+ 1_B d\lambda^{(k,I)}_{N,a,v} - \int_{R^1} f^- 1_B d\lambda^{(k,I)}_{N,a,v}$$

$$= \int_{R^1} (f^+ 1_B)(\varphi) |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z}$$

$$- \int_{R^1} (f^- 1_B)(\varphi) |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (104)$$

$$= \int_{R^1} (f 1_B)(\varphi) |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z}$$

$$= \int_{\varphi^{-1}(B)} f(\varphi) |det f_{\varphi}| d\lambda^{(k,I)}_{N,b,z} \quad (103)$$


5. Problems for Further Study

A natural extension of this paper is the generalization of Theorem 47, by substituting the $(m, \sigma)$-standard functions for more general functions $\varphi : U \subset E_I \to E_J$ such that, for any $i \in I \setminus m$, the function $\varphi_i : U \subset E_I \to R$ depends on a finite number of variables.

Moreover, a natural application of this paper, in the probabilistic framework, is the development of the theory of the infinite-dimensional continuous random elements, defined in the paper [1]. In particular, we can prove the formula of the density of such random elements composed with the $(m, \sigma)$-standard functions given in the change of variables' formula in Theorem 47. Consequently, it is possible to introduce many random elements that generalize the well-known continuous random vectors in $R^m$ (e.g., the Gaussian random elements in $E_I$ defined by the $(m, \sigma)$-standard matrices) and to develop some theoretical results and some applications in the statistical inference. It is possible also to define a convolution between the laws of two independent and infinite-dimensional continuous random elements, as in the finite case.

Furthermore, we can generalize paper [11] by considering the recursion $\{X_n\}_{n \in N}$ on $[0, p)^N$ defined by

$$X_{n+1} = AX_n + B_n \quad (mod p), \quad (105)$$

where $X_0 = x_0 \in E_I, A$ is a biunique, linear, integer and $(m, \sigma)$-standard function, $p \in R^*$, and $\{B_n\}_{n \in N}$ is a sequence of independent and identically distributed random elements on $E_I$. Our target is to prove that, with some assumptions on the law of $B_n$, the sequence $\{X_n\}_{n \in N}$ converges with geometric rate to a random element with law $\mathcal{M}_{\varphi}((1/p)Leb(\mathcal{A}(p)))$. Moreover, we wish to quantify the rate of convergence in terms of $A, p, m$, and the law of $B_n$, and to prove that, if $A$ has an eigenvalue that is a root of 1, then $O(p^2)$ steps are necessary to achieve randomness.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References


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