Hindawi Publishing Corporation
Journal of Mathematics
Volume 2016, Article ID 2872185, 10 pages
http://dx.doi.org/10.1155/2016/2872185

Research Article

Generalized Fractional Integral Operators and M-Series

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Received 25 November 2015; Revised 7 February 2016; Accepted 18 February 2016

Academic Editor: Tepper L. Gill

Two fractional integral operators associated with Fox H-function due to Saxena and Kumbhat are applied to M-series, which is an extension of both Mittag-Leffler function and generalized hypergeometric function ${}_pF_q$. The Mellin and Whittaker transforms are obtained for these compositional operators with M-series. Further some interesting properties have been established including power function and Riemann-Liouville fractional integral operators. The results are expressed in terms of H-function, which are in compact form suitable for numerical computation. Special cases of the results are also pointed out in the form of lemmas and corollaries.

1. Introduction and Preliminaries

The subject of fractional calculus, which deals with investigations of integrals and derivatives, has gained importance and popularity during the last four decades. It is mainly due to its vast potential demonstrated applications in fields of science and engineering. Different extensions of various fractional integrations operators are studied by Kalla [1, 2], McBride [3], Kilbas [4, 5], Kiryakova [6], Purohit and Kalla [7], Kumbhat and Khan [8], and so forth.

Saxena and Kumbhat [9] defined the fractional integration operators by means of the following equations:

$$R_{x,r}^{\eta,\alpha}\left[f(x)\right] = rx^{-\eta-r\alpha-1}\int_0^x t^{\eta}\left(x^r-t^r\right)^{\alpha} f(t) \cdot H_{P,Q}^{M,N}\left[kU \mid \begin{matrix} a_P, A_P \\ b_Q, B_Q \end{matrix}\right] dt, \tag{1}$$

$$K_{x,r}^{\delta,\alpha}\left[f(x)\right] = rx^{\delta}\int_x^{\infty} t^{-\delta-r\alpha-1}\left(t^r-x^r\right)^{\alpha} f(t) \cdot H_{P,Q}^{M,N}\left[kV \mid \begin{matrix} a_P, A_P \\ b_Q, B_Q \end{matrix}\right] dt, \tag{2}$$

where U and V represent the expressions $(t^r/x^r)^m(1-t^r/x^r)^n$ and $(x^r/t^r)^m(1-x^r/t^r)^n$, respectively. M, N, P, and Q are positive integers such that $1 \le M \le Q, 0 \le N \le P, |\arg k| < (1/2)\lambda\pi, \lambda > 0$, for $a_P, b_Q \in C, A_P, B_Q \in (0,\infty)$, and r, m, and n are positive numbers.

The conditions of the validity of these operators are as follows:

(i) $1 \le p, q < \infty, p^{-1}+q^{-1} = 1$.

(ii) $\mathrm{Re}(\eta + rm(b_j/B_j)) > -1/q$, $\mathrm{Re}(\alpha + rn(b_j/B_j)) > -1/q$, $\mathrm{Re}(\delta+\alpha+mr(b_j/B_j)) > -1/p$, $j = 1,\ldots,M$.

(iii) $f(x) \in L(0,\infty)$.

The last condition ensures that $R_{x,r}^{\eta,\alpha}[f(x)]$ and $K_{x,r}^{\delta,\alpha}[f(x)]$ both exist and also ensures that both belong to $L(0,\infty)$. In (1) and (2) $H_{P,Q}^{M,N}(x)$ denotes H-function introduced and defined by Fox [10] via a Mellin-Barnes type integral as

$$H_{P,Q}^{M,N}(z) \equiv H_{P,Q}^{M,N}\left[z \mid \begin{matrix} (a_i,\alpha_i)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathscr{L}} \mathscr{H}_{P,Q}^{M,N}(s)\, z^{-s} ds \tag{3}$$

with

$$\mathscr{H}_{P,Q}^{M,N}(s) \equiv \mathscr{H}_{P,Q}^{M,N}\left[\begin{array}{c} (a_i,\alpha_i)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{array} \mid s\right] = \frac{\prod_{j=1}^{M}\Gamma(b_j+\beta_j s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=N+1}^{P}\Gamma(a_i+\alpha_i s)\prod_{j=M+1}^{Q}\Gamma(1-b_j-\beta_j s)}. \tag{4}$$

Asymptotic expansions and analytic continuations together with the convergence conditions of H-function have been discussed by Braaksma [11].

Sharma and Jain [12] introduced the generalized M-series as the function defined by means of the power series:

$$\begin{aligned} {}_p^{\alpha}M_q^{\beta}(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z) &= {}_p^{\alpha}M_q^{\beta}(z) \\ &= {}_p^{\alpha}M_q^{\beta}\left((a_j)_1^p;(b_j)_1^q;z\right) \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta)}, \\ &\qquad z,\alpha,\beta\in C,\ R(\alpha)>0, \end{aligned} \tag{5}$$

where $(a_j)_n$, and $(b_j)_n$ are the known Pochhammer symbols. Series (5) is defined when none of the parameters b_j's, $j = 1,2,\ldots,q$, is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (5) is convergent for all z if $p \le q$, it is convergent for $|z| < \delta$, if $p = q+1$, and it is divergent, if $p > q+1$. When $p = q+1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters. Properties of M-series are further studied by Saxena [13], Chouhan and Saraswat [14], and so forth.

The generalized Mittag-Leffler function [15] is obtained from (5) for $p = q = 1$; $a = \gamma \in C$; $b = 1$, as

$$\begin{aligned} E_{\alpha,\beta}^{\gamma}(z) &= \sum_{m=0}^{\infty}\frac{(\gamma)_m}{\Gamma(\alpha m+\beta)}\frac{z^m}{m!} \\ &= \sum_{m=0}^{\infty}\frac{(\gamma)_m}{(1)_m}\frac{z^m}{\Gamma(\alpha m+\beta)} = {}_1^{\alpha}M_1^{\beta}(\gamma;1;z). \end{aligned} \tag{6}$$

The right sided Riemann-Liouville fractional integral operator I_{a+}^{λ} is defined by Samko et al. [16] for $\mathrm{Re}(\lambda) > 0$, as

$$\left(I_{a+}^{\lambda}f\right)(x) = \frac{1}{\Gamma(\lambda)}\int_a^x (x-t)^{\lambda-1}f(t)\,dt. \tag{7}$$

Particularly due to Chouhan and Saraswat [14] we have

$$\left(I_{0+}^{\lambda}t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\lambda)}x^{\rho+\lambda-1}. \tag{8}$$

Chouhan and Saraswat [14] established the following relation for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a \in R$:

$$\begin{aligned} &\left(I_{0+}^{\alpha}\left[t^{\gamma-1}\,{}_p^{\beta}M_q^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{\beta}\right)\right]\right)(x) \\ &\quad = x^{\alpha+\gamma-1}\,{}_p^{\beta}M_q^{\alpha+\gamma}\left((a_j)_1^p;(b_j)_1^q;ax^{\beta}\right). \end{aligned} \tag{9}$$

Due to Whittaker transform (Whittaker and Watson [17]), the following result holds:

$$\begin{aligned} &\int_0^{\infty} e^{-t/2}t^{\sigma-1}W_{\lambda,\mu}(t)\,dt \\ &\quad = \frac{\Gamma(1/2+\mu+\sigma)\Gamma(1/2-\mu+\sigma)}{\Gamma(1-\lambda+\sigma)}, \end{aligned} \tag{10}$$

where $\mathrm{Re}(\mu\pm\sigma) > -1/2$ and $W_{\lambda,\mu}(\cdot)$ is the Whittaker function [17, 18] defined as

$$\begin{aligned} &W_{\lambda,\mu}(t) \\ &\quad = e^{-t/2}\frac{\Gamma(-2\mu)t^{1/2+\mu}}{\Gamma(1/2-\mu-\lambda)}\,{}_1F_1\left(\frac{1}{2}+\mu-\lambda;1+2\mu;t\right) \\ &\qquad + e^{-t/2}\frac{\Gamma(2\mu)t^{1/2-\mu}}{\Gamma(1/2+\mu-\lambda)}\,{}_1F_1\left(\frac{1}{2}-\mu-\lambda;1-2\mu;t\right). \end{aligned} \tag{11}$$

2. Operators and M-Series

In this section, we established image formulas for M-series (5), involving R-K operators (1) and (2), in terms of Fox H-function. The results are shown in Theorems 3 and 5. We first derive the following two lemmas in order to prove Theorems 3 and 5.

Lemma 1. *If $f(x) \in L_P(0,\infty)$, $1 \le p \le 2$, $|\arg k| < (1/2)\lambda\pi$, $\lambda > 0$, $\mathrm{Re}(\eta + rm(b_j/B_j)) > q^{-1}$, $\mathrm{Re}(\alpha + rn(b_j/B_j)) > -q^{-1}$, $j = 1,2,\ldots,M$, $p^{-1}+q^{-1} = 1$, $\rho \in C$, $\mathrm{Re}(\rho) > 0$, $x > 0$, then*

$$\begin{aligned} &R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\right](x) \\ &\quad = x^{\rho-1}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{array}{c} \left(1-\frac{(\eta+\rho)}{r},m\right),(-\alpha,n),(a_P,A_P) \\ \left(\frac{(-\eta-\rho)}{r}-\alpha,m+n\right),(b_Q,B_Q) \end{array}\right]. \end{aligned} \tag{12}$$

Proof. By (1), we have

$$\begin{aligned} R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\right](x) &= rx^{-\eta-r\alpha-1}\int_0^x t^{\eta}\left(x^r-t^r\right)^{\alpha} \\ &\quad \cdot t^{\rho-1}H_{P,Q}^{M,N}\left[kU \mid \begin{array}{c} a_P,A_P \\ b_Q,B_Q \end{array}\right]dt \end{aligned} \tag{13}$$

using (3); then, by changing the order of integration valid under the conditions stated with the theorem and solving inner integral with respect to "t," we get

$$\begin{aligned} R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\right](x) &= \frac{x^{\rho-1}}{2\pi i}\int_{\mathscr{L}}\mathscr{H}_{P,Q}^{M,N}(s) \\ &\quad \cdot \frac{\Gamma((\eta+\rho)/r-ms)\Gamma(\alpha+1-ns)}{\Gamma((\eta+\rho)/r+\alpha+1-(m+n)s)}k^{-s}ds; \end{aligned} \tag{14}$$

hence, by virtue of H-function definition (3) and (4), we finally obtain RHS of (12). □

Lemma 2. *If* $f(x) \in L_P(0,\infty)$, $1 \le p \le 2$, $|\arg k| < (1/2)\lambda\pi$, $\lambda > 0$, $\mathrm{Re}(\alpha + rn(b_j/B_j)) > -q^{-1}$, $\mathrm{Re}(\delta+\alpha+rm(b_j/B_j)) > -p^{-1}$, $p^{-1}+q^{-1} = 1$, $j = 1,2,\ldots,M$, $\rho \in C$, $\mathrm{Re}(\rho) < 1$, $x > 0$, *then*

$$K_{x,r}^{\delta,\alpha}\left[t^{\rho-1}\right](x) = x^{\rho-1} H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1-\dfrac{(\delta-\rho+1)}{r}, m\right), (-\alpha, n), (a_P, A_P) \\ \left(\dfrac{(\rho-\delta-1)}{r} - \alpha, m+n\right), (b_Q, B_Q) \end{matrix} \right]. \tag{15}$$

Proof. By (2), we have

$$K_{x,r}^{\delta,\alpha}\left[t^{\rho-1}\right](x) = rx^{\delta}\int_x^{\infty} t^{-\delta-r\alpha-1}\left(t^r - x^r\right)^{\alpha} \cdot t^{\rho-1} H_{P,Q}^{M,N}\left[kV \mid \begin{matrix} (a_P, A_P) \\ (b_Q, B_Q) \end{matrix} \right] dt \tag{16}$$

using (3); then, by changing the order of integration valid under the conditions stated with the theorem and solving inner integration with respect to "t," we get

$$K_{x,r}^{\delta,\alpha}\left[t^{\rho-1}\right](x) = \frac{x^{\rho-1}}{2\pi i}\int_{\mathscr{L}} \mathscr{H}_{P,Q}^{M,N}(s) \cdot \frac{\Gamma\left((\delta-\rho+1)/r - ms\right)\Gamma(\alpha+1-ns)}{\Gamma\left((\delta-\rho+1)/r + \alpha + 1 - (m+n)s\right)} k^{-s} ds; \tag{17}$$

hence, by virtue of H-function definition (3) and (4), we finally obtain RHS of (15). □

Theorem 3. *With all assumptions and conditions on parameters, as stated in Lemma 1 with* $v, \sigma \in C$ $\mathrm{Re}(v) > 0, a > 0$, *the following property holds true:*

$$R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} x^{\rho+vi-1} H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1-\dfrac{(\eta+\rho+vi)}{r}, m\right), (-\alpha, n), (a_P, A_P) \\ \left(\dfrac{(-\eta-\rho-vi)}{r} - \alpha, m+n\right), (b_Q, B_Q) \end{matrix} \right]. \tag{18}$$

Proof. Using (1) and (5), we get

$$R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)}\left(R_{x,r}^{\eta,\alpha}\left[t^{\rho-1+vi}\right]\right)(x); \tag{19}$$

finally, by virtue of (12), we obtained RHS of (18). □

Corollary 4. *With all assumptions and conditions on parameters, as stated in Theorem 3 with* $\gamma \in C$, *the following result holds:*

$$R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}E_{v,\sigma}^{\gamma}\left(at^{v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(\gamma)_i}{\Gamma(vi+\sigma)}\frac{a^i}{(i!)} x^{\rho+vi-1} H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1-\dfrac{(\eta+\rho+vi)}{r}, m\right)(-\alpha, n)(a_P, A_P) \\ \left(\dfrac{(-\eta-\rho-vi)}{r} - \alpha, m+n\right)(b_Q, B_Q) \end{matrix} \right]. \tag{20}$$

Theorem 5. *With all assumptions and conditions on parameters, as stated in Lemma 2 with $v, \sigma \in C$ $\mathrm{Re}(v) > 0, a > 0$, the following property holds true:*

$$K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{-v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} \cdot x^{-\rho-vi}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix}\left(1-\frac{(\delta+\rho+vi)}{r},m\right)(-\alpha,n)(a_P,A_P)\\ \left(\frac{(-\rho-\delta-vi)}{r}-\alpha,m+n\right)(b_Q,B_Q)\end{matrix}\right]. \tag{21}$$

Proof. By using (2) and (5) and then changing the order of summation, we get

$$K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{-v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)}K_{x,r}^{\delta,\alpha}\left[t^{-\rho-vi}\right](x); \tag{22}$$

finally, by virtue of (15), we arrived at RHS of (21). □

Corollary 6. *With all assumptions and conditions on parameters, as stated in Theorem 5 with $\gamma \in C$, the following result holds:*

$$K_{x,r}^{\delta,\alpha}\left[t^{-\rho}E_{v,\sigma}^{\gamma}\left(at^{-v}\right)\right](x) = \sum_{i=0}^{\infty}\frac{(\gamma)_i}{\Gamma(vi+\sigma)}\frac{a^i}{i!}x^{-\rho-vi}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix}\left(1-\frac{(\delta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\rho-\delta-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]. \tag{23}$$

3. Mellin and Whittaker Transforms

In this section Mellin and Whittaker transforms of the results established in Theorems 3 and 5 have been obtained.

Theorem 7. *If $f(x) \in L_P(0,\infty), 1 \le p \le 2, |\arg k| < (1/2)\lambda\pi, \lambda > 0$, $\mathrm{Re}(\eta+rm(b_j/B_j)) > q^{-1}$, $\mathrm{Re}(\alpha+rn(b_j/B_j)) > -q^{-1}$, $j = 1,2,\ldots,M, p^{-1}+q^{-1} = 1, \mathrm{Re}(s) > -\mathrm{Re}(v)$ and $0 < x < 1$, $\mathrm{Re}(\rho) > 0, \mathrm{Re}(v) > 0, a > 0$, then*

$$\mathscr{M}\left\{R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{v}\right)\right]\right\}(s) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix}\left(1-\frac{(\eta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\eta-\rho-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]\frac{1}{(s+\rho+vi-1)}, \tag{24}$$

where $\mathscr{M}[f(x)]$ is Mellin transform of $f(x)$.

Proof. From (18), it follows that

$$\mathscr{M}\left\{R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ v}M_{q_1}^{\sigma}\left(at^{v}\right)\right]\right\}(s) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix}\left(1-\frac{(\eta+\rho+vi)}{r},m\right)(-\alpha,n)(a_P,A_P)\\ \left(\frac{(-\eta-\rho-vi)}{r}-\alpha,m+n\right)(b_Q,B_Q)\end{matrix}\right]\mathscr{M}\left(t^{\rho+vi-1}\right); \tag{25}$$

the theorem readily follows on evaluating the Mellin transform of $t^{\rho+vi-1}$ by means of the formula given by Erdelyi [18]. □

Theorem 8. *If* $f(x) \in L_P(0,\infty), 1 \le p \le 2, |\arg k| < (1/2)\lambda\pi, \lambda > 0, \mathrm{Re}(\alpha + rn(b_j/B_j)) > -q^{-1}$, $\mathrm{Re}(\delta + \alpha + rm(b_j/B_j)) > -p^{-1}, p^{-1} + q^{-1} = 1, j = 1,2,\ldots,M; \rho,\nu,\sigma \in C, a > 0, \mathrm{Re}(\nu) > 0$, $\mathrm{Re}(1-\rho) < 1, \mathrm{Re}(s) > -\mathrm{Re}(\nu), 0 < x < 1$, *then*

$$\mathscr{M}\left\{K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}{}^{\nu}M_{q_1}^{\sigma}\left(at^{-\nu}\right)\right]\right\}(s) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \;\middle|\; \begin{matrix}\left(1-\frac{(\delta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\rho-\delta-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right] \cdot \frac{1}{(s-\rho-\delta-vi)}, \tag{26}$$

where $\mathscr{M}[f(x)]$ *is Mellin transform of* $f(x)$.

Proof. From (21), it follows that

$$\mathscr{M}\left\{K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}{}^{\nu}M_{q_1}^{\delta}\left(at^{-\nu}\right)\right]\right\}(s) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \;\middle|\; \begin{matrix}\left(1-\frac{(2\delta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\rho-2\delta-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]\mathscr{M}\left[t^{-\rho-\delta-vi}\right]. \tag{27}$$

Theorem 8 readily follows on evaluating the Mellin transform of $t^{-\rho-\delta-vi}$ by means of the formula given by Erdelyi [18]. □

Theorem 9. *With all assumptions and conditions on parameters, as stated in Theorem 3 with* $\mathrm{Re}[\mu\pm(\sigma+\rho+vi-1)] > -1/2$, *the following result holds:*

$$\int_0^{\infty} e^{-\phi t/2}t^{\sigma-1}w_{\lambda,\mu}(\phi t)\left[R_{x,r}^{\eta,\alpha}\left\{t^{\rho-1}\,{}_{p_1}{}^{\nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right\}\right]dt = \phi^{1-\rho-\sigma}\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{(a\phi^{-\nu})^i}{\Gamma(vi+\sigma)} \cdot \frac{\Gamma(1/2+\mu+\sigma+\rho+vi-1)\Gamma(1/2-\mu+\sigma+\rho+vi-1)}{\Gamma(vi+\sigma+\rho-\lambda)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \;\middle|\; \begin{matrix}\left(1-\frac{(\eta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\eta-\rho-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]. \tag{28}$$

Proof. From (18), it follows that

$$I \equiv \int_0^{\infty} e^{-\phi t/2}t^{\sigma-1}w_{\lambda,\mu}(\phi t)\left[R_{x,r}^{\eta,\alpha}\left\{t^{\rho-1}\,{}_{p_1}{}^{\nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right\}\right]dt = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \;\middle|\; \begin{matrix}\left(1-\frac{(\eta+\rho+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\eta-\rho-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]\int_0^{\infty} e^{-\phi t/2}t^{(\sigma+\rho+vi-1)-1}w_{\lambda,\mu}(\phi t)\,dt; \tag{29}$$

letting $\phi t = z \Rightarrow dt = dz/\phi$, we obtained

$$I = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_{p_1})_i}{(b_1)_i \cdots (b_{q_1})_i} \frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1 - \frac{(\eta+\rho+vi)}{r}, m\right), (-\alpha, n), (a_P, A_P) \\ \left(\frac{(-\eta-\rho-vi)}{r} - \alpha, m+n\right), (b_Q, B_Q) \end{matrix} \right] \phi^{1-\rho-\sigma-vi} \int_0^{\infty} e^{-z/2} z^{(\sigma+\rho+vi-1)-1} w_{\lambda,\mu}(z)\, dz; \tag{30}$$

finally by virtue of Whittaker transform (10) we obtained (28). □

Theorem 10. *With all assumptions and conditions on parameters, as stated in Theorem 3 with* $\mathrm{Re}[\mu \pm (\sigma - \rho - vi)] > -1/2$, *the following result holds:*

$$\int_0^{\infty} e^{-\phi t/2} t^{\sigma-1} w_{\lambda,\mu}(\phi t) \left[K_{x,r}^{\delta,\alpha} \left\{ t^{-\rho} {}_{p_1}^{\ \nu} M_{q_1}^{\sigma}(at^{-\nu}) \right\} \right] dt = \phi^{\rho-\sigma} \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_{p_1})_i}{(b_1)_i \cdots (b_{q_1})_i} \frac{(a\phi^{\nu})^i}{\Gamma(vi+\sigma)} \cdot \frac{\Gamma(1/2+\mu+\sigma-\rho-vi)\Gamma(1/2-\mu+\sigma-\rho-vi)}{\Gamma(1-\lambda+\sigma-\rho-vi)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1 - \frac{(\eta+\rho+vi)}{r}, m\right), (-\alpha, n), (a_P, A_P) \\ \left(\frac{(-\eta-\rho-vi)}{r} - \alpha, m+n\right), (b_Q, B_Q) \end{matrix} \right]. \tag{31}$$

Proof. From (21), it follows that

$$I \equiv \int_0^{\infty} e^{-\phi t/2} t^{\sigma-1} w_{\lambda,\mu}(\phi t) \left[K_{x,r}^{\delta,\alpha} \left\{ t^{-\rho} {}_{p_1}^{\ \nu} M_{q_1}^{\sigma}(at^{-\nu}) \right\} \right] dt = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_{p_1})_i}{(b_1)_i \cdots (b_{q_1})_i} \frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1 - \frac{(\delta+\rho+vi)}{r}, m\right) (-\alpha, n) (a_P, A_P) \\ \left(\frac{(-\rho-\delta-vi)}{r} - \alpha, m+n\right) (b_Q, B_Q) \end{matrix} \right] \int_0^{\infty} e^{-\phi t/2} t^{(\sigma-\rho-vi)-1} w_{\lambda,\mu}(\phi t)\, dt. \tag{32}$$

Letting $\phi t = z \Rightarrow dt = dz/\phi$, we obtained

$$I = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_{p_1})_i}{(b_1)_i \cdots (b_{q_1})_i} \frac{a^i}{\Gamma(vi+\sigma)} \cdot H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{matrix} \left(1 - \frac{(\eta+\rho+vi)}{r}, m\right), (-\alpha, n), (a_P, A_P) \\ \left(\frac{(-\eta-\rho-vi)}{r} - \alpha, m+n\right), (b_Q, B_Q) \end{matrix} \right] \phi^{\rho-\sigma+vi} \int_0^{\infty} e^{-z/2} z^{(\sigma-\rho-vi-1)-1} w_{\lambda,\mu}(z)\, dz; \tag{33}$$

finally, by virtue of Whittaker transform (10), we obtained (31). □

4. Properties of Integral Operators

Here we give some formal properties of the operators as consequences of Theorems 3 and 5. These properties show

compositions of power function and Riemann-Liouville fractional integral operator (7) with operators (1) and (2).

Theorem 11. *With all assumptions and conditions on parameters, as stated in Theorem 3 along with* $\mathrm{Re}(\beta+\rho)>0$, *the following property holds:*

$$x^{\beta}R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)=R_{x,r}^{\eta-\beta,\alpha}\left[t^{\beta+\rho-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x). \tag{34}$$

Proof. From (18), the LHS of (34) follows as

$$x^{\beta}R_{x,r}^{\eta,\alpha}\left[t^{\rho-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(\nu i+\sigma)}\cdot x^{\beta+\rho+\nu i-1}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\frac{(\eta+\rho+\nu i)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{(-\eta-\rho-\nu i)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]; \tag{35}$$

again by (18) the RHS of (34) follows as

$$R_{x,r}^{\eta-\beta,\alpha}\left[t^{\beta+\rho-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(\nu i+\sigma)}\cdot x^{\rho+\beta+\nu i-1}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\frac{(\eta-\beta+\rho+\beta+\nu i)}{r},m\right)(-\alpha,n)(a_P,A_P)\\ \left(\frac{(-\eta+\beta-\rho-\beta-\nu i)}{r}-\alpha,m+n\right)(b_Q,B_Q)\end{matrix}\right]. \tag{36}$$

Apparently, Theorem 11 readily follows from (35) and (36). □

Theorem 12. *With all assumptions and conditions on parameters, as stated in Theorem 5 along with* $\mathrm{Re}(1-\rho-\beta)<1$, *the following property holds:*

$$x^{-\beta}K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{-\nu}\right)\right](x)=K_{x,r}^{\delta-\beta,\alpha}\left[t^{-\rho-\beta}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{-\nu}\right)\right](x). \tag{37}$$

Proof. From (21), the LHS of (37) follows as

$$x^{-\beta}K_{x,r}^{\delta,\alpha}\left[t^{-\rho}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{-\nu}\right)\right](x)=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(\nu i+\sigma)}\cdot x^{-\rho-\beta-\nu i}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\frac{(\delta+\rho+\nu i)}{r},m\right)(-\alpha,n)(a_P,A_P)\\ \left(\frac{(-\rho-\delta-\nu i)}{r}-\alpha,m+n\right)(b_Q,B_Q)\end{matrix}\right]; \tag{38}$$

again by (21) the RHS of (37) follows as

$$K_{x,r}^{\delta-\beta,\alpha}\left[t^{-\rho-\beta}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{-\nu}\right)\right](x)=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma)}$$
$$\cdot\, x^{-\rho-\beta-vi}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\dfrac{(\delta-\beta+\rho+\beta+vi)}{r},m\right)(-\alpha,n)(a_P,A_P)\\ \left(\dfrac{(-\rho-\beta-\delta+\beta-vi)}{r}-\alpha,m+n\right)(b_Q,B_Q)\end{matrix}\right]. \tag{39}$$

Apparently, Theorem 12 readily follows from (38) and (39). □

Theorem 13. *With all assumptions and conditions on parameters, as stated in Theorem 3 along with $\lambda\in C,\mathrm{Re}(\lambda)>0$, the following property holds:*

$$I_{0+}^{\lambda}R_{x,r}^{\eta,\alpha}\left[t^{\sigma-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)=R_{x,r}^{\eta,\alpha}I_{0+}^{\lambda}\left[t^{\sigma-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x). \tag{40}$$

Proof. Applying right sided Riemann-Liouville fractional integral operator I_{0+}^{λ} to (18) and using (8), we obtained

$$I_{0+}^{\lambda}R_{x,r}^{\eta,\alpha}\left[t^{\sigma-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)$$
$$=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma+\lambda)}x^{\sigma+\lambda+vi-1}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\dfrac{(\eta+\sigma+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\dfrac{(-\eta-\sigma-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]; \tag{41}$$

again, by virtue of relations (9) and (18), we obtained

$$R_{x,r}^{\eta,\alpha}I_{0+}^{\lambda}\left[t^{\sigma-1}\,{}_{p_1}^{\ \nu}M_{q_1}^{\sigma}\left(at^{\nu}\right)\right](x)$$
$$=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(vi+\sigma+\lambda)}x^{\sigma+\lambda+vi-1}H_{P+2,Q+1}^{M,N+2}\left[k\ \middle|\ \begin{matrix}\left(1-\dfrac{(\eta+\sigma+vi)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\dfrac{(-\eta-\sigma-vi)}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]. \tag{42}$$

Theorem 13 readily follows from (41) and (42). □

Theorem 14. *With all assumptions and conditions on parameters, as stated in Theorem 5 along with $\lambda\in C,\mathrm{Re}(\lambda)>0,\mathrm{Re}(1-\sigma)<1$, the following property holds:*

$$I_{0+}^{\lambda}\left[K_{x,r}^{\delta,\alpha}\left(t^{-\sigma-\nu}\,{}_{p_1}M_{q_1}^{1-\sigma}\left(at^{-\nu}\right)\right)\right](x)=K_{x,r}^{\delta,\alpha}I_{0+}^{\lambda}\left[t^{-\sigma-\nu}\,{}_{p_1}M_{q_1}^{1-\sigma}\left(at^{-\nu}\right)\right](x). \tag{43}$$

Proof. Applying right sided Riemann-Liouville fractional operator I_{0+}^{λ} to (21) after suitable changes on parameters, we get

$$
\begin{aligned}
&I_{0+}^{\lambda}\left[K_{x,r}^{\delta,\alpha}\left(t^{-\sigma-\nu}{}_{p_1}M_{q_1}^{1-\sigma}\left(at^{-\nu}\right)\right)\right](x)\\
&=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(-\nu i+1-\sigma)}H_{P+2,Q+1}^{M,N+2}\left[k\;\middle|\;\begin{matrix}\left(1-\frac{(\delta+\sigma-\nu i)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{-\sigma-\delta+\nu i}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]I_{0+}^{\lambda}\left[x^{-\sigma-\nu i}\right]\\
&=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(-\nu i+1-\sigma+\lambda)}H_{P+2,Q+1}^{M,N+2}\left[k\;\middle|\;\begin{matrix}\left(1-\frac{(\delta+\sigma-\nu i)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{-\sigma-\delta+\nu i}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right]x^{-\sigma-\nu i+\lambda};
\end{aligned}
\tag{44}
$$

again, by virtue of relations (9) and (21), we get

$$
\begin{aligned}
&K_{x,r}^{\delta,\alpha}I_{0+}^{\lambda}\left[t^{-\sigma-\nu}{}_{p_1}M_{q_1}^{1-\sigma}\left(at^{-\nu}\right)\right](x)=K_{x,r}^{\delta,\alpha}\left[x^{\lambda-\sigma-\nu}{}_{p_1}M_{q_1}^{\lambda+1-\sigma}\left(at^{-\nu}\right)\right](x)\\
&=\sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_{p_1})_i}{(b_1)_i\cdots(b_{q_1})_i}\frac{a^i}{\Gamma(-\nu i+\lambda+1-\sigma)}x^{\lambda-\sigma-\nu i}H_{P+2,Q+1}^{M,N+2}\left[k\;\middle|\;\begin{matrix}\left(1-\frac{(\delta+\sigma-\nu i)}{r},m\right),(-\alpha,n),(a_P,A_P)\\ \left(\frac{-\sigma-\delta+\nu i}{r}-\alpha,m+n\right),(b_Q,B_Q)\end{matrix}\right];
\end{aligned}
\tag{45}
$$

hence, from (44) and (45), Theorem 14 readily follows. □

5. Conclusions

Recently, fractional operator's theory was recognized to be good tool for modeling complex problems, kinetic equations, fractional reaction, fractional diffusion equations, and so forth. In this work, the authors investigated and studied two fractional integral operators associated with Fox H-function due to Saxena and Kumbhat which are applied to M-series. We obtain series expansion of the images of the M-series through these fractional operators. Besides some interesting properties of these operators in Section 4, the authors also discussed their behavior under Mellin and Whittaker transforms and results are given in better realistic series solutions which converge rapidly. Results derived in this paper are very significant and may find applications in the solution of fractional order differential equations that are arising in certain areas of turbulence, propagation of seismic waves, and diffusion processes. On account of general nature of H-function and M-series a number of results involving special functions can be obtained merely by specializing the parameters.

Competing Interests

The authors declare that they have no competing interests regarding the publication of this paper.

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