Research Article

A Study on Characteristic Roots of Lattice Matrices

Kapiarumalayil Varkey Thomas¹ and Geena Joy²

¹Department of Mathematics, Bharata Mata College, Thrikkakara, Kochi, Kerala, India

²Department of Mathematics, Union Christian College, Aluva, Kochi, Kerala, India

Correspondence should be addressed to Geena Joy; geenamaryjoy@gmail.com

Received 31 July 2016; Revised 5 September 2016; Accepted 15 September 2016

Academic Editor: Basil K. Papadopoulos

Copyright © 2016 K. Varkey Thomas and G. Joy. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with the characteristic roots of different types of lattice matrices and proves that a matrix and its transpose have the same characteristic roots. Also this paper introduces the concept of similar lattice matrices and proves that similar lattice matrices have the same characteristic roots.

1. Introduction

The notion of lattice matrices appeared firstly in the work “Lattice Matrices” [1] by Givón in 1964. A matrix is called a lattice matrix if its entries belong to a distributive lattice. All Boolean matrices and fuzzy matrices are lattice matrices. Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory, and the theory of finite graphs [1].

In classical theory of matrices over a field, the characteristic equation of the matrix concerned can be defined determinantly. In the case of lattice matrices, this is not possible. In 1994, Zhang [2] proved the generalization of the Cayley-Hamilton theorem for lattice matrices. Corresponding lattice equation is taken as the characteristic equation of lattice matrix. For matrices over a field, the eigenvector-eigenvalue problems [3] are closely associated with the characteristic equation of the matrix concerned. However, this relationship is vague for lattice matrices.

The roots of characteristic equations of matrices over lattices were studied firstly in the work of Tan [4]. In [4] Tan proved the existence of the least element and the greatest element for the set of all roots of the characteristic equation of a matrix over complete and completely distributive lattices with 0 and 1.

In the present work, we continue to discuss roots of characteristic equations of matrices (based on [4] by Tan) over a class of complete and completely distributive lattices with 0 and 1. Such lattices are abundant; for example, every

finite distributive lattice, a complete chain, and, especially, the real interval $[0, 1]$ are all such kind of lattices. In Section 3, we study characteristic roots of different types of lattice matrices. We also introduce the concept of similar lattice matrices and prove that similar lattice matrices have the same characteristic roots.

2. Preliminaries

We recall some basic definitions and results on lattice theory and lattice matrices which will be used in the sequel. For details see [1, 3–7].

A partially ordered set $(L, \leq)$ is a *lattice* if for all $a, b \in L$, the least upper bound of $\{a, b\}$ and the greatest lower bound of $\{a, b\}$ exist in L . For any $a, b \in L$, the least upper bound and the greatest lower bound are denoted by $a \vee b$ and $a \wedge b$ (or ab), respectively.

A nonvoid subset K of a lattice L is a *sublattice* of L , if for any $a, b \in K$, $a \vee b, a \wedge b \in K$. A lattice L is called a *complete lattice* if for any $H \subseteq L$, both the least upper bound $\vee\{y \mid y \in H\}$ and the greatest lower bound $\wedge\{y \mid y \in H\}$ of H exist in L . A lattice $(L, \leq, \vee, \wedge)$ is a *distributive lattice* if the operations $\vee$ and $\wedge$ are distributive with respect to each other.

An element $a \in L$ is called *greatest element* of L if $x \leq a$, $\forall x \in L$. An element $b \in L$ is called *least element* of L if $b \leq x$, $\forall x \in L$. We use 1 and 0 to denote the greatest element and the least element of L , respectively.

For any $a, b \in L$, the least element $x \in L$ satisfying the inequality $b \vee x \geq a$ is called the *relative lower*

pseudocomplement of b in a and is denoted by $a - b$. If for any $a, b \in L$, $a - b$ exists, then L is said to be a *dually Brouwerian lattice*.

A lattice L is said to be *completely distributive* if for any $x \in L$ and any $\{y_i \mid i \in I\} \subseteq L$, where I is an index set,

- (a) $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$,
- (b) $x \vee (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \vee y_i)$ holds.

It is known [5] that a complete lattice L is dually Brouwerian if and only if (b) is satisfied in L . Therefore, a complete and completely distributive lattice L is dually Brouwerian.

In this paper, L denotes a complete and completely distributive lattice with the greatest element 1 and the least element 0, respectively.

Lemma 1 (see [7]). *Let L be a complete and completely distributive lattice with 1 and 0. Then for any $a, b, c \in L$, one has*

- (a) $a - b \leq a$,
- (b) $a \leq b \Leftrightarrow a - b = 0$,
- (c) $(a - b) \vee (b - c) = (a \vee b) - (bc)$; in particular, $(a - b) \vee (b - a) = (a \vee b) - (ab)$.

Let $M_n(L)$ be the set of all $n \times n$ matrices over L (lattice matrices). We shall denote by A_{ij} or a_{ij} the element of L which stands in the (i, j) th entry of $A \in M_n(L)$.

For $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij}) \in M_n(L)$, define

$$\begin{aligned} A \leq B &\Leftrightarrow a_{ij} \leq b_{ij} \quad (i, j = 1, 2, \dots, n), \\ A \vee B &= C \Leftrightarrow c_{ij} = a_{ij} \vee b_{ij} \quad (i, j = 1, 2, \dots, n), \\ AB &= C \Leftrightarrow c_{ij} = \bigvee_{k=1}^n a_{ik} b_{kj} \quad (i, j = 1, 2, \dots, n), \\ A \wedge B &= C \Leftrightarrow c_{ij} = a_{ij} \wedge b_{ij} \quad (i, j = 1, 2, \dots, n), \\ A^T &= C \Leftrightarrow c_{ij} = a_{ji} \quad (i, j = 1, 2, \dots, n), \\ aA &= C \Leftrightarrow c_{ij} = aa_{ij} \text{ for } a \in L, (i, j = 1, 2, \dots, n), \end{aligned}$$

$$\begin{aligned} (I)_{ij} &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases} \\ (\mathbf{0})_{ij} &= 0, \\ (E)_{ij} &= 1, \end{aligned} \tag{1}$$

$$(i, j = 1, 2, \dots, n),$$

$A^0 = I$, $A^{k+1} = A^k A$ for $k \geq 0$, k is an integer. $(A^k)_{ij} = (a_{ij}^{(k)})$ ($i, j = 1, 2, \dots, n$).

Lemma 2 (see [1]). *For any $A, B, C \in M_n(L)$, one has the following.*

- (a) *The multiplication in $M_n(L)$ has the following properties:*
 - (i) $A(BC) = (AB)C$,

- (ii) $AI = IA = A$,
- (iii) $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$.

(b) *The transposition in $M_n(L)$ has the following properties:*

- (i) $(AB)^T = B^T A^T$,
- (ii) $(A^T)^T = A$.

Let $A \in M_n(L)$. If $A^2 = A$, then A is called *idempotent*; if there exists some integer $k \geq 1$ such that $A^k = \mathbf{0}$, then A is called *nilpotent*; if $a_{ij} = 0$, $i > j$, $i, j = 1, 2, \dots, n$, then A is called *upper triangular*; if $a_{ij} = 0$, $i < j$, $i, j = 1, 2, \dots, n$, then A is called *lower triangular*; if A is both upper triangular and lower triangular, then A is called *diagonal*: that is, $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$; if A is diagonal and $a_{ii} = k \in L$, $i = 1, 2, \dots, n$, then A is called a *scalar matrix*.

Theorem 3 (see [1]). *Let $A \in M_n(L)$. Then A is nilpotent if and only if $A^n = \mathbf{0}$.*

Let $A \in M_n(L)$. Then A is said to be *invertible* if there is a $B \in M_n(L)$ such that $AB = BA = I$. Here B is called the *inverse* of A and is denoted by A^{-1} . Also, A is called *orthogonal* if and only if $AA^T = A^T A = I$.

Theorem 4 (see [1]). *Let $A \in M_n(L)$. Then A is invertible if and only if A is orthogonal. Hence, $A^{-1} = A^T$.*

A set $S = \{a_1, a_2, \dots, a_m\}$ of elements of L is a *decomposition* of 1 in L if and only if $\bigvee_{1 \leq i \leq m} a_i = 1$. The set S is called *orthogonal* if and only if $a_i a_j = 0$, for all $i, j = 1, 2, \dots, m$, $i \neq j$. Hence, S is called an *orthogonal decomposition* of 1 in L if and only if it is orthogonal and a decomposition of 1 in L .

Theorem 5 (see [1]). *Let $A \in M_n(L)$. Then A is invertible if and only if each row and each column of A is an orthogonal decomposition of 1 in L .*

For $A \in M_n(L)$, the *permanent* $|A|$ of A is defined as

$$|A| = \bigvee_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}, \tag{2}$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \dots, n\}$.

Example 6. For $I \in M_n(L)$,

$$|I| = \bigvee_{\sigma \in S_n} I_{1\sigma(1)} I_{2\sigma(2)} \cdots I_{n\sigma(n)} = I_{11} I_{22} \cdots I_{nn} = 1. \tag{3}$$

Theorem 7 (see [2]). *Let $A \in M_n(L)$. Then $|A^T| = |A|$.*

Theorem 8 (see [8]). *Let $A, P, Q \in M_n(L)$ such that P and Q are invertible. Then $|PAQ| = |P||A||Q|$.*

Let $V_n(L)$ denote the set of all column vectors (n -vectors) over L . It can be considered as an $n \times 1$ lattice matrix over L . Denote $e = (1, 1, \dots, 1)^T \in V_n(L)$.

Let $A \in M_n(L)$. An *eigenvector* of A is a vector ξ in $V_n(L)$ such that $A\xi = \lambda\xi$, for some scalar λ in L . The element λ is called the associated *eigenvalue*.

3. Characteristic Roots of Matrices over L

In this section, we discuss characteristic roots of different types of matrices over a complete and completely distributive lattice with 0 and 1.

Theorem 9 (see [2]). *Let $A \in M_n(L)$. Then $A^n \vee q_1(A)A^{n-1} \vee p_2(A)A^{n-2} \vee q_3(A)A^{n-3} \vee p_4(A)A^{n-4} \vee \dots \vee bI = p_1(A)A^{n-1} \vee q_2(A)A^{n-2} \vee p_3(A)A^{n-3} \vee \dots \vee cI$, where*

$$p_k(A) = \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is even}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_k \sigma(r_k)} \right), \quad (4)$$

$$q_k(A) = \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is odd}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_k \sigma(r_k)} \right).$$

$S(r_1, r_2, \dots, r_k)$ is the symmetric group of all permutations of the set $\{r_1, r_2, \dots, r_k\}$, $k = 1, 2, \dots, n$, $b = p_n(A)$, and $c = q_n(A)$, when n is even; $b = q_n(A)$ and $c = p_n(A)$, when n is odd.

In 1994, Zhang [2] proved Theorem 9, the generalization of the Cayley-Hamilton theorem for matrices over distributive lattice with 0 and 1. In other words, Zhang [2] proved that the lattice equation

$$\begin{aligned} & \lambda^n \vee q_1(A) \lambda^{n-1} \vee p_2(A) \lambda^{n-2} \vee q_3(A) \lambda^{n-3} \\ & \vee p_4(A) \lambda^{n-4} \vee \dots \vee b \\ & = p_1(A) \lambda^{n-1} \vee q_2(A) \lambda^{n-2} \vee p_3(A) \lambda^{n-3} \vee \dots \vee c \end{aligned} \quad (5)$$

is satisfied by the matrix $A \in M_n(L)$.

Since, in the lattice case, a characteristic equation cannot be defined determinantly it is natural to choose (5) as the characteristic equation of $A \in M_n(L)$.

Definition 10 (see [4]). Let $A \in M_n(L)$. Then the characteristic equation of A is defined as

$$\begin{aligned} & \lambda^n \vee q_1(A) \lambda^{n-1} \vee p_2(A) \lambda^{n-2} \vee q_3(A) \lambda^{n-3} \\ & \vee p_4(A) \lambda^{n-4} \vee \dots \vee b \\ & = p_1(A) \lambda^{n-1} \vee q_2(A) \lambda^{n-2} \vee p_3(A) \lambda^{n-3} \vee \dots \vee c, \end{aligned} \quad (6)$$

where

$$p_k(A) = \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is even}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_k \sigma(r_k)} \right), \quad (7)$$

$$q_k(A) = \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is odd}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_k \sigma(r_k)} \right).$$

$S(r_1, r_2, \dots, r_k)$ is the symmetric group of all permutations of the set $\{r_1, r_2, \dots, r_k\}$, $k = 1, 2, \dots, n$, $b = p_n(A)$ and $c = q_n(A)$, when n is even; $b = q_n(A)$ and $c = p_n(A)$, when n is odd.

The following notations are used:

$$\begin{aligned} S_n^+ &= \{\sigma \in S_n \mid \sigma \text{ is even}\}, \\ S_n^- &= \{\sigma \in S_n \mid \sigma \text{ is odd}\}. \end{aligned} \quad (8)$$

For $k = n$, we have

$$\begin{aligned} p_n(A) &= \bigvee_{1 \leq r_1 < r_2 < \dots < r_n \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_n), \sigma \text{ is even}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_n \sigma(r_n)} \right) \\ &= \bigvee_{\sigma \in S_n, \sigma \text{ is even}} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \\ &= \bigvee_{\sigma \in S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}, \end{aligned} \quad (9)$$

$$\begin{aligned} q_n(A) &= \bigvee_{1 \leq r_1 < r_2 < \dots < r_n \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_n), \sigma \text{ is odd}} a_{r_1 \sigma(r_1)} a_{r_2 \sigma(r_2)} \dots a_{r_n \sigma(r_n)} \right) \\ &= \bigvee_{\sigma \in S_n, \sigma \text{ is odd}} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \\ &= \bigvee_{\sigma \in S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}. \end{aligned}$$

In [4], Tan proceeds to solve (5), by assuming that the parameter λ in (5) is an element in L . From the idempotency of λ and the absorption law, (5) takes the form

$$\lambda \vee b = \lambda d \vee c, \quad (10)$$

where $d = p_1(A) \vee q_2(A) \vee p_3(A) \vee q_4(A) \vee \dots$, $b = p_n(A)$ and $c = q_n(A)$, when n is even; $b = q_n(A)$ and $c = p_n(A)$, when n is odd.

Lemma 11 (see [4]). $b \leq d$.

Definition 12. Let $A \in M_n(L)$ and $\lambda \in L$. If λ satisfies (10), λ is called a characteristic root or characteristic value of A .

Theorem 13. Let $A \in M_n(L)$ and S be the set of all characteristic roots of A . Then S is a sublattice of L .

Proof. Let $\lambda, \mu \in S$. Then $\lambda \vee b = \lambda d \vee c$ and $\mu \vee b = \mu d \vee c$.
Now

$$\begin{aligned} (\lambda \vee \mu) \vee b &= (\lambda \vee b) \vee (\mu \vee b) = (\lambda d \vee c) \vee (\mu d \vee c) \\ &= (\lambda d \vee \mu d) \vee c = (\lambda \vee \mu) d \vee c, \\ (\lambda \mu) \vee b &= (\lambda \vee b) (\mu \vee b) = (\lambda d \vee c) (\mu d \vee c) \\ &= (\lambda d \mu d) \vee c = (\lambda \mu) d \vee c. \end{aligned} \quad (11)$$

Therefore, $\lambda \vee \mu, \lambda \mu \in S$. Hence, the set of all characteristic roots of A is a sublattice of L . $\square$

Corollary 14. Let $A \in M_n(L)$. Then the set of all characteristic roots of A is a subset of the set of all eigenvalues of A .

In 1998, Tan [4] proved the existence of the least element and the greatest element for the set of all characteristic roots of A by the following theorem.

Theorem 15 (see [4]). Let $A \in M_n(L)$. Then the set of all characteristic roots of A is $[(b - c) \vee (c - b), d \vee c] = \{\lambda \in L \mid (b - c) \vee (c - b) \leq \lambda \leq d \vee c\}$, where $d \vee c = e^T A^n e$.

For simplicity, the greatest and the least characteristic roots of A are denoted by $R^u(A)$ and $r^l(A)$, respectively. Thus, $R^u(A) = d \vee c = e^T A^n e$ and $r^l(A) = (b - c) \vee (c - b) = (p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A))$.

Theorem 16. Let $A \in M_n(L)$. Then the permanent of A is a characteristic root of A .

Proof. We have

$$\begin{aligned} |A| &= \bigvee_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= \left(\bigvee_{\sigma \in S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \right) \\ &\quad \vee \left(\bigvee_{\sigma \in S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \right) \\ &= p_n(A) \vee q_n(A) = \begin{cases} b \vee c, & \text{if } n \text{ is even} \\ c \vee b, & \text{if } n \text{ is odd} \end{cases} \\ &= b \vee c. \end{aligned} \quad (12)$$

FIGURE 1

Now

$$\begin{aligned} r^l(A) &= (b - c) \vee (c - b) = (b \vee c) - (bc) \\ &\quad \text{(by Lemma 1(c))} \\ &\leq b \vee c \quad \text{(by Lemma 1(a))} \\ &= |A|. \end{aligned} \quad (13)$$

Also by Lemma 11, $b \leq d \Rightarrow |A| = b \vee c \leq d \vee c = R^u(A)$.

Therefore, $r^l(A) \leq |A| \leq R^u(A)$. Hence, by Theorem 15, permanent of A is a characteristic root of A . $\square$

Theorem 17. Let $A \in M_n(L)$. Then A is nilpotent if and only if the only characteristic root of A is 0.

Proof. First assume that A is nilpotent. Then by Theorem 3, $A^n = \mathbf{0}$. Therefore, by Theorem 15, $R^u(A) = e^T A^n e = e^T \mathbf{0} e = 0$. Hence, the only characteristic root of A is 0. Conversely assume that the only characteristic root of A is 0. Then by Theorem 15, $R^u(A) = e^T A^n e = \bigvee_{1 \leq i, j \leq n} a_{ij}^{(n)} = 0$. Therefore, $a_{ij}^{(n)} = 0$, for $i, j = 1, 2, \dots, n$. That is, $A^n = \mathbf{0}$. Hence, by Theorem 3, A is nilpotent. $\square$

Proposition 18. Let $A \in M_n(L)$ be an idempotent matrix. Then the greatest characteristic root of A is $\bigvee_{1 \leq i, j \leq n} a_{ij}$.

Proof. We have $R^u(A) = e^T A^n e = e^T A e = \bigvee_{1 \leq i, j \leq n} a_{ij}$. Hence, the proof is complete. $\square$

Remark 19. Let $A \in M_n(L)$ be idempotent. Then 0 need not be a characteristic root of A .

Example 20. Consider the lattice $L = \{0, a, b, c, d, 1\}$, whose diagrammatical representation is shown in Figure 1.

It is easy to see that L is a distributive lattice.

Let $A = \begin{bmatrix} c & a \\ a & c \end{bmatrix} \in M_2(L)$. Then $A^2 = \begin{bmatrix} c & a \\ a & c \end{bmatrix} = A$. Hence, A is idempotent. Now the characteristic roots of A are those values of λ satisfying $\lambda \vee c = (\lambda \wedge c) \vee a \Rightarrow \lambda = b, c$.

Remark 21. Let $A, B \in M_n(L)$. Then the characteristic roots of AB and BA need not be the same.

Example 22. In Example 20, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} b & c \\ d & 1 \end{bmatrix} \in M_2(L)$. Then $AB = \begin{bmatrix} 0 & c \\ 1 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} c & c \\ c & d \end{bmatrix}$. Now the

characteristic roots of AB are those values of λ satisfying $\lambda \vee 0 = (\lambda \wedge 1) \vee c \Rightarrow \lambda = \lambda \vee c \Rightarrow \lambda = c, 1$ and the characteristic roots of BA are those values of λ satisfying $\lambda \vee a = (\lambda \wedge 1) \vee c \Rightarrow \lambda \vee a = \lambda \vee c \Rightarrow \lambda = b, c, 1$.

Lemma 23 (see [9]). *Let $A \in M_n(L)$. Then $p_n(A^T) = p_n(A)$ and $q_n(A^T) = q_n(A)$.*

Theorem 24. *Let $A \in M_n(L)$. Then the characteristic roots of A and A^T are the same.*

Proof. By Lemma 2(b), we have

$$\begin{aligned} R^u(A^T) &= e^T (A^T)^n e = e^T (A^n)^T e = (e^T (A^n)^T e)^T \\ &= e^T A^n e = R^u(A), \end{aligned} \quad (14)$$

and by Lemma 23,

$$\begin{aligned} r^l(A^T) &= (p_n(A^T) - q_n(A^T)) \\ &\quad \vee (q_n(A^T) - p_n(A^T)) \\ &= (p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A)) \\ &= r^l(A). \end{aligned} \quad (15)$$

Therefore, by Theorem 15, the characteristic roots of A and A^T are the same. $\square$

Theorem 25. *Let $A \in M_n(L)$ be a triangular (upper or lower) matrix. Then the characteristic roots of A are those values of $\lambda \in L$ satisfying*

$$\bigwedge_{1 \leq i \leq n} a_{ii} \leq \lambda \leq \bigvee_{1 \leq i \leq n} a_{ii}. \quad (16)$$

Proof. Assume that A is an upper triangular matrix. Then $a_{ij} = 0, i > j, i, j = 1, 2, \dots, n$.

Therefore, we have

$$\begin{aligned} R^u(A) &= e^T A^n e = \bigvee_{1 \leq i_1, i_2, \dots, i_{n+1} \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_{n+1}} \\ &= \bigvee_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_{n+1}}. \end{aligned} \quad (17)$$

For any $1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq n$, there must exist some $k \in \{1, 2, \dots, n\}$ such that $i_k = i_{k+1}$. Then $R^u(A) \leq \bigvee_{1 \leq i \leq n} a_{ii}$. On the other hand, since $a_{ii} = a_{ii} a_{ii} \cdots a_{ii}$ is a term in the expression $\bigvee_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_{n+1}}$, we have $R^u(A) = \bigvee_{1 \leq i \leq n} a_{ii}$.

For any $\sigma \in S_n$ with $i \leq \sigma(i)$, we have $\sigma(n) = n$ and $\sigma(i) \neq \sigma(j)$ for any $i \neq j$, and so $\sigma(i) = i$ for any i . In this case, $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = a_{11} a_{22} \cdots a_{nn}$. Then

$$\begin{aligned} p_n(A) &= \bigvee_{\sigma \in S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= \bigvee_{\sigma \in S_n^+, i \leq \sigma(i)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = a_{11} a_{22} \cdots a_{nn}, \end{aligned}$$

$$\begin{aligned} q_n(A) &= \bigvee_{\sigma \in S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= \bigvee_{\sigma \in S_n^-, i \leq \sigma(i)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = 0. \end{aligned} \quad (18)$$

Hence,

$$\begin{aligned} r^l(A) &= (p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A)) \\ &= \bigwedge_{1 \leq i \leq n} a_{ii}. \end{aligned} \quad (19)$$

By Theorem 15, the characteristic roots of A are those values of $\lambda \in L$ satisfying

$$\bigwedge_{1 \leq i \leq n} a_{ii} \leq \lambda \leq \bigvee_{1 \leq i \leq n} a_{ii}. \quad (20)$$

If A is a lower triangular matrix, then A^T is an upper triangular matrix. Proceedings as above, we get that the characteristic roots of A are those values of $\lambda \in L$ satisfying

$$\bigwedge_{1 \leq i \leq n} a_{ii} \leq \lambda \leq \bigvee_{1 \leq i \leq n} a_{ii}. \quad (21)$$

$\square$

Corollary 26. *Let $A \in M_n(L)$ be a diagonal matrix (i.e., $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$). Then the characteristic roots of A are those values of $\lambda \in L$ satisfying*

$$\bigwedge_{1 \leq i \leq n} a_{ii} \leq \lambda \leq \bigvee_{1 \leq i \leq n} a_{ii}. \quad (22)$$

Corollary 27. *Let $A \in M_n(L)$ be a scalar matrix (i.e., $A = \text{diag}(a, a, \dots, a)$, where $a \in L$). Then “ a ” is the only characteristic root of A .*

Proposition 28. *Let $A \in M_n(L)$. Then the set of all characteristic roots of A is L if and only if $b = c$ and $d = 1$.*

Proof. First assume that the set of all characteristic roots of A is L .

Then by Theorem 15, we have

$$\begin{aligned} r^l(A) &= (b - c) \vee (c - b) = 0 \\ &\Rightarrow b - c = 0, \quad c - b = 0 \\ &\Rightarrow b \leq c, \quad c \leq b \quad (\text{by Lemma 1(b)}) \\ &\Rightarrow b = c, \end{aligned} \quad (23)$$

$$R^u(A) = d \vee c = 1 \Rightarrow$$

$$d \vee b = 1 \Rightarrow$$

$$d = 1$$

(by Lemma 11).

Hence, $b = c$ and $d = 1$.

Conversely assume that $b = c$ and $d = 1$. Then the characteristic equation (10) may be written as

$$\begin{aligned}\lambda \vee b &= \lambda d \vee c \implies \\ \lambda \vee b &= \lambda \vee b,\end{aligned}\tag{24}$$

which is true for all $\lambda \in L$.

Hence, the set of all characteristic roots of A is L . $\square$

Proposition 29. Let $A \in M_n(L)$. Then 0 is a characteristic root of A if and only if $b = c$.

Proof. First assume that 0 is a characteristic root of A . Then (10) becomes

$$\begin{aligned}0 \vee b &= 0d \vee c \implies \\ b &= c.\end{aligned}\tag{25}$$

Conversely assume that $b = c$. Then (10) can be written as

$$\begin{aligned}\lambda \vee b &= \lambda d \vee b = (\lambda \vee b) (d \vee b) = (\lambda \vee b) d \\ &\text{(by Lemma 11)} \\ \implies (\lambda \vee b) &\leq d,\end{aligned}\tag{26}$$

which is true when $\lambda = 0$, by Lemma 11. Hence, 0 is a characteristic root of A . $\square$

Example 30. Consider $E \in M_n(L)$, $n \geq 2$. We have

$$\begin{aligned}p_k(E) &= \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is even}} E_{r_1 \sigma(r_1)} E_{r_2 \sigma(r_2)} \dots E_{r_k \sigma(r_k)} \right) = 1, \quad k = 1, 2, \dots, n, \\ q_1(E) &= 0 \\ q_k(E) &= \bigvee_{1 \leq r_1 < r_2 < \dots < r_k \leq n} \left(\bigvee_{\sigma \in S(r_1, r_2, \dots, r_k), \sigma \text{ is odd}} E_{r_1 \sigma(r_1)} E_{r_2 \sigma(r_2)} \dots E_{r_k \sigma(r_k)} \right) = 1, \quad k = 2, 3, \dots, n.\end{aligned}\tag{27}$$

Therefore, $b = c = 1$ and $d = 1$.

Hence, by Proposition 28, the set of all characteristic roots of E is L .

Lemma 31. If $A \in M_n(L)$ is invertible, then $|A| = 1$.

Proof. Assume A is invertible. Then by Theorem 4, $AA^T = I$. Using Theorems 7 and 8, we have $1 = |I| = |AA^T| = |A||A^T| = |A||A| = |A|$. Hence, $|A| = 1$. $\square$

Theorem 32. Let $A \in M_n(L)$ be an invertible matrix. Then the only characteristic root of A is 1.

Proof. We have

$$\begin{aligned}p_n(A) q_n(A) &= \left(\bigvee_{\sigma \in S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \right) \\ &\cdot \left(\bigvee_{\sigma \in S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \right).\end{aligned}\tag{28}$$

Each term u of $p_n(A)q_n(A)$ is of the form $a_{1\sigma(1)} \dots a_{n\sigma(n)} a_{1\mu(1)} \dots a_{n\mu(n)}$, where $\sigma \in S_n^+$, $\mu \in S_n^-$. Then there must exist some $j \in \{1, 2, \dots, n\}$ such that $\sigma(j) \neq \mu(j)$. Corresponding to that j , $a_{j\sigma(j)} a_{j\mu(j)} = 0$, by Theorem 5. Hence, $u = 0$. Therefore, we get

$$\begin{aligned}p_n(A) q_n(A) &= \left(\bigvee_{\sigma \in S_n^+} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \right) \\ &\cdot \left(\bigvee_{\sigma \in S_n^-} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \right) = 0.\end{aligned}\tag{29}$$

Similarly we can prove that $q_n(A)p_n(A) = 0$.

Therefore,

$$bc = \begin{cases} p_n(A) q_n(A), & \text{if } n \text{ is even} \\ q_n(A) p_n(A), & \text{if } n \text{ is odd} \end{cases} = 0.\tag{30}$$

Hence,

$$\begin{aligned}r^l(A) &= (b - c) \vee (c - b) = (b \vee c) - (bc) \\ &\text{(by Lemma 1(c))} \\ &= b \vee c - 0 = b \vee c = |A| = 1 \\ &\text{(by Lemma 31)}.\end{aligned}\tag{31}$$

Therefore, $r^l(A) = 1 = R^u(A)$.

Hence, the only characteristic root of A is 1. $\square$

Definition 33. Let $A, B \in M_n(L)$. If there exists an invertible matrix $P \in M_n(L)$ such that $B = P^{-1}AP$, then B is said to be similar to A .

Let $A, B \in M_n(L)$. If there exists an orthogonal matrix $Q \in M_n(L)$ such that $B = Q^{-1}AQ$, then B is said to be orthogonally similar to A .

Note that, If B is similar to A , then A is similar to B . From Theorem 4, B being similar to A is the same as B being orthogonally similar to A .

Theorem 34. Similar lattice matrices have the same characteristic roots.

Proof. Let $A, B \in M_n(L)$. Assume that B is similar to A . Then there exists an invertible matrix $P \in M_n(L)$ such that $B = P^{-1}AP = P^TAP$, by Theorem 4.

We have

$$\begin{aligned} R^u(B) &= e^T B^n e = e^T (P^T A P)^n e = e^T P^T A^n P e \\ &= (Pe)^T A^n (Pe) = e^T A^n e \quad (\text{by Theorem 5}) \quad (32) \\ &= R^u(A). \end{aligned}$$

We claim that $p_n(B) = p_n(A)$ and $q_n(B) = q_n(A)$.

To do this, we first prove $p_n(AP) = p_n(A)p_n(P) \vee q_n(A)q_n(P)$ for any $A, P \in M_n(L)$ and P is invertible.

We have

$$\begin{aligned} p_n(AP) &= \bigvee_{\sigma \in S_n^+} (AP)_{1\sigma(1)} (AP)_{2\sigma(2)} \cdots (AP)_{n\sigma(n)} \\ &= \bigvee_{\sigma \in S_n^+} \left(\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} P_{k_1\sigma(1)} P_{k_2\sigma(2)} \right. \\ &\quad \left. \cdots P_{k_n\sigma(n)} \right) = \bigvee_{\sigma \in S_n^+} \left(\bigvee_{\tau \in S_n} a_{1\tau(1)} a_{2\tau(2)} \right. \\ &\quad \left. \cdots a_{n\tau(n)} P_{\tau(1)\sigma(1)} P_{\tau(2)\sigma(2)} \cdots P_{\tau(n)\sigma(n)} \right) \\ &= \bigvee_{\tau \in S_n} a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} \left(\bigvee_{\sigma \in S_n^+} P_{\tau(1)\sigma(1)} P_{\tau(2)\sigma(2)} \right. \\ &\quad \left. \cdots P_{\tau(n)\sigma(n)} \right) = \left(\bigvee_{\tau \in S_n^+} a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} \right. \\ &\quad \left. \cdot \left(\bigvee_{\sigma \in S_n^+} P_{\tau(1)\sigma(1)} P_{\tau(2)\sigma(2)} \cdots P_{\tau(n)\sigma(n)} \right) \right) \\ &\quad \vee \left(\bigvee_{\tau \in S_n^-} a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} \left(\bigvee_{\sigma \in S_n^+} P_{\tau(1)\sigma(1)} P_{\tau(2)\sigma(2)} \right. \right. \\ &\quad \left. \left. \cdots P_{\tau(n)\sigma(n)} \right) \right) = \left(\bigvee_{\tau \in S_n^+} a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} p_n(P) \right) \\ &\quad \vee \left(\bigvee_{\tau \in S_n^-} a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} q_n(P) \right) = p_n(A) \\ &\quad \cdot p_n(P) \vee q_n(A) q_n(P). \end{aligned} \quad (33)$$

Similarly, we can prove $p_n(PA) = p_n(P)p_n(A) \vee q_n(P)q_n(A)$ and $q_n(AP) = q_n(PA) = p_n(P)q_n(A) \vee q_n(P)p_n(A)$.

Therefore,

$$\begin{aligned} p_n(B) &= p_n(P^T A P) \\ &= p_n(P^T A) p_n(P) \vee q_n(P^T A) q_n(P) \\ &= (p_n(P^T) p_n(A) \vee q_n(P^T) q_n(A)) p_n(P) \\ &\quad \vee (p_n(P^T) q_n(A) \vee q_n(P^T) p_n(A)) q_n(P) \end{aligned}$$

$$\begin{aligned} &= p_n(P) p_n(A) \vee q_n(P) p_n(A) \\ &\quad (\text{by Lemma 23 and Theorem 32}) \end{aligned}$$

$$\begin{aligned} &= (p_n(P) \vee q_n(P)) p_n(A) = |P| p_n(A) \\ &= p_n(A) \quad (\text{by Lemma 31}), \end{aligned}$$

$$\begin{aligned} q_n(B) &= q_n(P^T A P) \\ &= p_n(P^T A) q_n(P) \vee q_n(P^T A) p_n(P) \\ &= (p_n(P^T) p_n(A) \vee q_n(P^T) q_n(A)) q_n(P) \\ &\quad \vee (p_n(P^T) q_n(A) \vee q_n(P^T) p_n(A)) p_n(P) \\ &= q_n(P) q_n(A) \vee p_n(P) q_n(A) \\ &\quad (\text{by Lemma 23 and Theorem 32}) \\ &= (q_n(P) \vee p_n(P)) q_n(A) = |P| q_n(A) \\ &= q_n(A) \quad (\text{by Lemma 31}). \end{aligned} \quad (34)$$

Hence,

$$\begin{aligned} r^l(B) &= (p_n(B) - q_n(B)) \vee (q_n(B) - p_n(B)) \\ &= (p_n(A) - q_n(A)) \vee (q_n(A) - p_n(A)) \quad (35) \\ &= r^l(A). \end{aligned}$$

Therefore, by Theorem 15, the characteristic roots of A and B are the same. $\square$

4. Conclusion

In this paper, we have studied the characteristic roots of different types of lattice matrices which have further strengthened the theory of lattice matrices. There is a great scope for further development in this area. We hope the research in this direction would be able to throw light on a few new concepts in lattice matrices, thereby paving the way for applications in areas ranging from switching circuits to theoretical computer science.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The second author is thankful to the University Grants Commission, New Delhi, India, for the award of Teacher Fellowship under the XII Plan period.

References

- [1] Y. Givón, "Lattice matrices," *Information and Control*, vol. 7, no. 4, pp. 477–484, 1964.
- [2] K. L. Zhang, "Determinant theory for D01-lattice matrices," *Fuzzy Sets and Systems*, vol. 62, no. 3, pp. 347–353, 1994.
- [3] Y. Tan, "On the eigenproblem of matrices over distributive lattices," *Linear Algebra and Its Applications*, vol. 374, pp. 87–106, 2003.
- [4] Y.-J. Tan, "Eigenvalues and eigenvectors for matrices over distributive lattices," *Linear Algebra and Its Applications*, vol. 283, no. 1–3, pp. 257–272, 1998.
- [5] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, USA, 3rd edition, 1967.
- [6] G. Grätzer, *General Lattice Theory*, Birkhäuser, Basel, Switzerland, 2nd edition, 1998.
- [7] Y.-J. Tan, "On compositions of lattice matrices," *Fuzzy Sets and Systems*, vol. 129, no. 1, pp. 19–28, 2002.
- [8] Y. Tan, "On invertible matrices over antirings," *Linear Algebra and Its Applications*, vol. 423, no. 2–3, pp. 428–444, 2007.
- [9] E. E. Marenich, "Determinant theory for lattice matrices," *Journal of Mathematical Sciences*, vol. 193, no. 4, pp. 537–547, 2013.

Hindawi

Submit your manuscripts at
<http://www.hindawi.com>