Research Article

Automorphisms and Inner Automorphisms

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Let K be a field of characteristic not 2 and let $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ be central simple superalgebra over K , and let $*$ be superinvolution on $\mathcal{A}$. Our main purpose is to classify the group of automorphisms and inner automorphisms of $(\mathcal{A}, *)$ (i.e., commuting with $*$) by using the classical theorem of Skolem-Noether. Also we study two examples of groups of automorphisms and inner automorphisms on even central simple superalgebras with superinvolutions.

1. Introduction

An associative super ring $R = R_0 + R_1$ is nothing but a $(\mathbb{Z}/2\mathbb{Z})$ -graded associative ring. A $(\mathbb{Z}/2\mathbb{Z})$ -graded ideal $I = I_0 + I_1$ of an associative super ring R is called a *superideal* of R . An associative super ring R is *simple* if it has no nontrivial superideals. Let R be an associative super ring with $1 \in R_0$; then R is said to be a *division* super ring if all nonzero homogeneous elements are invertible; that is, every $0 \neq r_\alpha \in R_\alpha$ has an inverse r_α^{-1} necessarily in R_α .

Let K be a field of characteristic not 2; an associative $(\mathbb{Z}/2\mathbb{Z})$ -graded K -algebra $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is a *finite dimensional central simple* superalgebra over a field K , if $\widehat{Z}(\mathcal{A}) = K$ is the center of $\mathcal{A}$, where

$$\begin{aligned} (\widehat{Z}(\mathcal{A}))_\alpha &= \{a_\alpha \in \mathcal{A}_\alpha \mid a_\alpha b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha \quad \forall b_\beta \in \mathcal{A}_\beta\}, \end{aligned} \quad (1)$$

and the only superideals of $\mathcal{A}$ are (0) and $\mathcal{A}$ itself. We say that $\mathcal{A}_0 \cup \mathcal{A}_1$ is the set of all homogeneous elements of $\mathcal{A}$.

Finite dimensional central simple associative superalgebras over a field K are isomorphic to $\text{End } V \cong M_n(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1$ is a finite dimensional associative division superalgebra over K ; that is, all nonzero elements of $\mathcal{D}_\alpha$, $\alpha = 0, 1$, are invertible, and $V = V_0 + V_1$ is an n -dimensional $\mathcal{D}$ -superspace. If $\mathcal{D}_1 = \{0\}$, the grading of $M_n(\mathcal{D})$ is induced

by that of $V = V_0 + V_1$, $\mathcal{A} = M_{p+q}(\mathcal{D})$, $p = \dim_{\mathcal{D}} V_0$, and $q = \dim_{\mathcal{D}} V_1$, so $p + q$ is a nontrivial decomposition of n . Meanwhile, if $\mathcal{D}_1 \neq \{0\}$, then the grading of $M_n(\mathcal{D})$ is given by $(M_n(\mathcal{D}))_\alpha = M_n(\mathcal{D}_\alpha)$, $\alpha = 0, 1$.

For completeness, we recall the structure theorem for central simple associative division superalgebras.

Theorem 1 (Division Superalgebra Theorem [1, 2]). *If $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1$ is a finite dimensional associative division superalgebra over a field K , then exactly one of the following holds, where throughout $\mathcal{E}$ denotes a finite dimensional associative division algebra over K :*

- (i) $\mathcal{D} = \mathcal{D}_0 = \mathcal{E}$, and $\mathcal{D}_1 = \{0\}$.
- (ii) $\mathcal{D} = \mathcal{E} \otimes_K K[u]$, $u^2 = \lambda \in K^\times$, $\mathcal{D}_0 = \mathcal{E} \otimes K1$, and $\mathcal{D}_1 = \mathcal{E} \otimes Ku$.
- (iii) $\mathcal{D} = \mathcal{E}$ or $M_2(\mathcal{E})$; $u \in \mathcal{D}$ such that $u^2 = \lambda \in K/K^2$ ($\lambda \in K/\{\alpha + \alpha^2 \mid \alpha \in K\}$ if $\text{Char}(K) = 2$) $\mathcal{D}_0 = C_{\mathcal{D}}(u)$, $\mathcal{D}_1 = S_{\mathcal{D}}(u)$, where $C_{\mathcal{D}}(u) = \{d \in \mathcal{D} \mid du = ud\}$ and $S_{\mathcal{D}}(u) = \{d \in \mathcal{D} \mid du = u^\sigma d\}$, for some quadratic Galois extension $K[u] \subseteq \mathcal{D}$ with Galois automorphism σ . Moreover, in the second case, $u = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$ ($u = \begin{pmatrix} 0 & 1 \\ \lambda & 1 \end{pmatrix}$ if $\text{Char}(K) = 2$) and $K[u]$ does not embed in $\mathcal{E}$.

Following [3] we say that a division superalgebra $\mathcal{D}$ is *even* if $Z(\mathcal{D}) \cap \mathcal{D}_1 = \{0\}$; that is, $\mathcal{D}$ is even if its form is (i) or (iii),

and we say that $\mathcal{D}$ is *odd* otherwise; that is, $\mathcal{D}$ is odd if its form is (ii). Also, if $\mathcal{A} = M_n(\mathcal{D})$ is a finite dimensional central simple superalgebra over a field K , then we say that $\mathcal{A}$ is an even K -superalgebra if $\mathcal{D}$ is even division superalgebra and $\mathcal{A}$ is odd K -superalgebra if $\mathcal{D}$ is odd division superalgebra.

In [1] Racine described all types of superinvolutions on $\mathcal{A} = M_{p+q}(\mathcal{D})$. It appears that if $*$ is a superinvolution on $\mathcal{A}$ such that $(\mathcal{A}_0, *)$ is simple algebra, then $p = q$ and $*$ is conjugate to the transpose involution. Otherwise, $*$ is conjugate to the orthosymplectic involution.

In [4] we proved that if $\mathcal{A}$ is a finite dimensional central simple associative superalgebra over a field K of characteristic not 2 such that $\mathcal{A}$ has a superinvolution of the first kind, then $\mathcal{A} = M_{p+q}(\mathcal{D})$, where $\mathcal{D}$ is a division algebra over K .

In [5] we proved that $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}_1 \neq \{0\}$ has a pseudosuperinvolution of the first kind if and only if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\text{BW}(K)$, where K is a field of characteristic not 2. But if K is a field of characteristic 2, and $\mathcal{A}$ is a central simple associative superalgebra over K , then a superinvolution (which is a pseudosuperinvolution) on $\mathcal{A}$ is just an involution on $\mathcal{A}$ respecting the grading. Moreover, if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\text{BW}(K)$, then the supercenter of $\mathcal{A}$ equals the center of $\mathcal{A}$ and $\otimes_K = \otimes_K$, which means that $\mathcal{A}$ is of order 2 in the Brauer group $\text{Br}(K)$. Thus, by theorem of Albert, $\mathcal{A}$ has an involution of the first kind, but since $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\text{BW}(K)$, $\mathcal{A}$ has superantiautomorphism of the first kind respecting the grading; therefore by [6, Chapter 8, Theorem 8.2] $\mathcal{A}$ has an involution of the first kind respecting the grading, which means that $\mathcal{A}$ has a superinvolution (which is a pseudosuperinvolution) of the first kind if and only if $\mathcal{A}$ is of order 2 in the Brauer-Wall group $\text{BW}(K)$.

Let K/k be a separable quadratic field extension over k with Galois group $\text{Gal}(K/k)$, where $\text{Gal}(K/k) = \{1, \sigma\}$ and $\theta^2 \in k - k^2$ and $\sigma(\theta) = -\theta$ ($\theta^2 \in k/\{\alpha + \alpha^2 \mid \alpha \in k\}$, $\sigma(\theta) = \theta + 1$ if $\text{Char}(k) = 2$). We recall a theorem of Albert-Reihm on the existence of K/k -involution (involution of the second kind) which states that finite dimensional central simple algebra $\mathcal{A}$ over K has a K/k -involution if and only if the corestriction of $\mathcal{A}$ splits over k . In [7] Elduque and Villa gave much better exposition and motivation for the whole theory of the existence of superinvolutions.

Throughout this work we say that if u is homogeneous element in $\mathcal{A}$, then $\psi_u : \mathcal{A} \rightarrow \mathcal{A}$ is an inner automorphism on $\mathcal{A}$, with conjugation by u . Also $\text{Aut}(\mathcal{A}, *)$, where $\mathcal{A}$ is a central simple superalgebra of any type, means the set of all automorphisms on $\mathcal{A}$ commuting with a given superinvolution $*$ defined on $\mathcal{A}$, and $\text{inAut}(\mathcal{A}, *)$ means the set of all inner automorphisms on $\mathcal{A}$ commuting with $*$.

In this paper we examine the characterization of automorphisms and inner automorphisms on $(\mathcal{A}, *)$, where $\mathcal{A}$ is a central simple superalgebra of any type, which commute with a given superinvolution $*$ defined on $\mathcal{A}$. Then we produce two examples of even superalgebras with superinvolution to investigate in detail groups of automorphisms and inner automorphisms in these examples.

2. Group of Automorphisms of Superalgebras

Definition 2. Let $\mathcal{A}$ be any K -superalgebra; we define the map $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ by

$$a_\alpha^\varphi = (-1)^\alpha a_\alpha \quad \forall a_\alpha \in \mathcal{A}_\alpha, \alpha = \bar{0}, \bar{1}. \quad (2)$$

This map, φ , is a superalgebra automorphism, called the sign automorphism, since

$$(a_\alpha b_\beta)^\varphi = (-1)^{\alpha+\beta} a_\alpha b_\beta = a_\alpha^\varphi b_\beta^\varphi \quad (3)$$

for all $a_\alpha \in \mathcal{A}_\alpha$ and $b_\beta \in \mathcal{A}_\beta$. The automorphism φ has order 2, if $\text{Char}(K) \neq 2$ (unless $\mathcal{A}_{\bar{1}} = 0$), and $\varphi = \text{id}_{\mathcal{A}}$ if $\text{Char}(K) = 2$.

Lemma 3. Let $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{0}}v$ be an odd superalgebra over a field K with a superinvolution $*$, and let φ be the sign automorphism. Then

$$\text{Aut}(\mathcal{A}, *) = \text{inAut}(\mathcal{A}, *) \cup \varphi \cdot \text{inAut}(\mathcal{A}, *). \quad (4)$$

Proof. If $\xi \in \text{Aut}(\mathcal{A}, *)$, then by [4, Theorem 3.2] Skolem-Noether theorem forces that ξ or $\xi\varphi$ is inner but not both of them, so, $\xi \in \text{inAut}(\mathcal{A}, *)$ or $\xi\varphi \in \text{inAut}(\mathcal{A}, *)$. $\square$

Now, let $*$ be a superinvolution on an odd central simple superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{0}}v$ of any kind and let $\xi \in \text{inAut}(\mathcal{A}, *)$. Then $\xi = \psi_{c_\gamma}$ for some invertible $c_\gamma \in \mathcal{A}_\gamma$. But $\xi|_{\mathcal{A}_{\bar{0}}}$ is inner, so $\xi|_{\mathcal{A}_{\bar{0}}} = \psi_{b_{\bar{0}}}$. Therefore, $c_\gamma(b_{\bar{0}}^{-1}ab_{\bar{0}})c_\gamma^{-1} = a$, $\forall a \in \mathcal{A}_{\bar{0}}$, and $c_\gamma(b_{\bar{0}}^{-1}vb_{\bar{0}})c_\gamma^{-1} = v$ which implies that $\psi_{c_\gamma b_{\bar{0}}^{-1}} = \text{id}_{\mathcal{A}} \Rightarrow \psi_{c_\gamma} = \psi_{b_{\bar{0}}}$. So, we can choose γ to be $\bar{0}$, and hence $\psi_{c_{\bar{0}}} = \psi_{b_{\bar{0}}} \Leftrightarrow c_{\bar{0}} = \alpha b_{\bar{0}}$ for some $\alpha \in K$.

Now, $a_\alpha^{*\psi_{b_{\bar{0}}}} = a_\alpha^{\psi_{b_{\bar{0}}}} \Leftrightarrow b_{\bar{0}}^* b_{\bar{0}} = \lambda \in K \Leftrightarrow b_{\bar{0}}^* = \lambda b_{\bar{0}}^{-1}$. Let

$$G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} : b_{\bar{0}}^* = \alpha b_{\bar{0}}^{-1} \text{ for some } \alpha \in K^\times\}. \quad (5)$$

Let $\sim$ be a relation on G defined as follows: $a_{\bar{0}} \sim b_{\bar{0}} \Leftrightarrow b_{\bar{0}} a_{\bar{0}}^{-1} = \alpha \in K^\times$. Then one can easily show that $\sim$ is an equivalence relation on G , and $G/\sim = \{[b_{\bar{0}}] \mid b_{\bar{0}} \in G\}$, where $[b_{\bar{0}}] = \{\alpha b_{\bar{0}} \mid \alpha \in K^\times\}$, is a group with the well-defined operation $[b_{\bar{0}}][c_{\bar{0}}] = [c_{\bar{0}} b_{\bar{0}}]$.

Theorem 4. Let $\mathcal{A}$ be an odd central simple superalgebra over a field K with a superinvolution $*$; then $\text{inAut}(\mathcal{A}, *) \cong G/\sim$.

Proof. Let $\delta : \text{inAut}(\mathcal{A}, *) \rightarrow G/\sim$ such that $\delta(\psi_{b_{\bar{0}}}) = [b_{\bar{0}}]$; then

$$\begin{aligned} \delta(\psi_{b_{\bar{0}}}\psi_{c_{\bar{0}}}) &= \delta(\psi_{c_{\bar{0}}b_{\bar{0}}}) = [c_{\bar{0}}b_{\bar{0}}] = [b_{\bar{0}}][c_{\bar{0}}] \\ &= \delta(\psi_{b_{\bar{0}}})\delta(\psi_{c_{\bar{0}}}). \end{aligned} \quad (6)$$

So, δ is an onto homomorphism such that $\ker(\delta) = \psi_1$. Therefore, $\text{inAut}(\mathcal{A}, *) \cong G/\sim$. $\square$

Let $\mathcal{A} = M_n(\mathcal{D})$ be an even central simple superalgebra over K , where $\mathcal{D}$ is a nontrivial grading division superalgebra. Then by [1, Division Superalgebra Theorem]

$\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v$, where $uv = -vu$ and $Z(\mathcal{A}_{\bar{0}}) = K[u]$ and $C_{\mathcal{D}}(u)$ is the centralizer of u in $\mathcal{D}$.

Let $*$ be a superinvolution on $\mathcal{A}$, and let $\xi \in \text{Aut}(\mathcal{A}, *)$. Then by the Skolem-Noether theorem there exists an invertible element $c_\gamma \in \mathcal{A}_\gamma$ such that $a^\xi = c_\gamma a c_\gamma^{-1}$, $\forall a \in \mathcal{A}$. Since $a^{\xi*} = a^{*\xi}$, $\forall a \in \mathcal{A}$, we have $c_\gamma c_\gamma^* = \lambda$ for some $\lambda \in K$. If $\xi|_{\mathcal{A}_{\bar{0}}} = \psi_{c_\gamma}|_{\mathcal{A}_{\bar{0}}}$ is inner, then $\xi|_{\mathcal{A}_{\bar{0}}} = \psi_{b_0}$ for some invertible element $b_0 \in \mathcal{A}_{\bar{0}}$ which implies that $a^\xi = b_0 a b_0^{-1}$, $\forall a \in \mathcal{A}_{\bar{0}}$, and therefore $\psi_{b_0^{-1}c_\gamma} = id_{\mathcal{A}_{\bar{0}}}$. Let $x_\gamma = b_0^{-1}c_\gamma$; if $\gamma = \bar{1}$, then $x_\gamma a x_\gamma^{-1} = a \forall a \in \mathcal{A}_{\bar{0}}$. But $\mathcal{A}_{\bar{1}} = \mathcal{A}_{\bar{0}}x_\gamma$, so x_γ centralizes $\mathcal{A}_{\bar{1}}$; thus $x_\gamma \in Z(\mathcal{A}) = K$. This is a contradiction and therefore $\gamma = \bar{0}$. Therefore, for any $\xi \in \text{Aut}(\mathcal{A}, *)$ such that $\xi|_{\mathcal{A}_{\bar{0}}}$ is inner, $\xi = \psi_{b_0}$, for some invertible element $b_0 \in \mathcal{A}_{\bar{0}}$, and hence

$$\text{inAut}(\mathcal{A}, *) = \{\xi \in \text{Aut}(\mathcal{A}, *) : \xi = \psi_{b_0}, b_0 \in \mathcal{A}_{\bar{0}}\}. \quad (7)$$

Corollary 5. Let $\mathcal{A} = M_n(\mathcal{D})$ be an even central simple superalgebra over a field K , where

$$\mathcal{D} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v \quad (8)$$

and $uv = -vu$ is a nontrivial grading division superalgebra, and let $*$ be a superinvolution on $\mathcal{A}$. Then

$$\begin{aligned} \text{Aut}(\mathcal{A}, *) \\ = \begin{cases} \text{inAut}(\mathcal{A}, *) \cup \psi_v \cdot \text{inAut}(\mathcal{A}, *) : & \text{if } \psi_v * = * \psi_v \\ \text{inAut}(\mathcal{A}, *) : & \text{if } \psi_v * \neq * \psi_v. \end{cases} \quad (9) \end{aligned}$$

Proof. If $\psi_v * = * \psi_v$, then for any $\xi \in \text{Aut}(\mathcal{A}, *)$ such that $u^\xi = -u$ we get $u^{\xi\psi_v} = -u^{\psi_v} = u$ which implies that $\xi\psi_v|_{\mathcal{A}_{\bar{0}}}$ is inner and so $\xi\psi_v = \psi_{b_0}$ for some invertible b_0 in $\mathcal{A}_{\bar{0}}$. Thus $\xi = \psi_v \psi_{b_0}$. But if $\psi_v * \neq * \psi_v$, then it is easy to check that $\text{Aut}(\mathcal{A}, *) = \text{inAut}(\mathcal{A}, *)$. Therefore

$$\begin{aligned} \text{Aut}(\mathcal{A}, *) \\ = \begin{cases} \text{inAut}(\mathcal{A}, *) \cup \psi_v \cdot \text{inAut}(\mathcal{A}, *) : & \text{if } \psi_v * = * \psi_v \\ \text{inAut}(\mathcal{A}, *) : & \text{if } \psi_v * \neq * \psi_v. \end{cases} \quad (10) \end{aligned}$$

□

Theorem 6. Let $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v$, and let $uv = -vu$ be a superalgebra with a superinvolution $*$. Then $\text{inAut}(\mathcal{A}, *) \cong G/\sim$.

Proof. Let $\delta : \text{inAut}(\mathcal{A}, *) \rightarrow G/\sim$ such that $\delta(\psi_{b_0}) = [b_0]$. Then

$$\begin{aligned} \delta(\psi_{b_0}\psi_{c_0}) &= \delta(\psi_{c_0 b_0}) = [c_0 b_0] = [b_0][c_0] \\ &= \delta(\psi_{b_0})\delta(\psi_{c_0}). \end{aligned} \quad (11)$$

So, δ is an onto homomorphism, where $\ker \delta = id_{\mathcal{A}} = \psi_{\bar{1}}$. Therefore, $\text{inAut}(\mathcal{A}, *) \cong G/\sim$. □

Next we introduce this example to show that uu^* may equal λ with $\lambda \neq \alpha\alpha^*$ for any $\alpha \in K$.

Example 7. Let $k = \mathbb{Q}(e)$, let $K = k(\sqrt{2})$, and let $u = \sqrt{e}i$, $v = \sqrt[4]{2}j$, where $i^2 = -1$, $j^2 = -1$, and $ij = -ji$. Then $\mathcal{A} = (K + Ku) \oplus (Kv + Kuv)$ is a quaternion division superalgebra over field K . Let $*$ be a superinvolution on $\mathcal{A}$ defined by

$$(a + b\sqrt{2})^* = a - b\sqrt{2}, \forall a, b \in k; u^* = u; v^* = v; \text{ then } (uv)^* = vu = -uv.$$

Now, $\psi_u \in \text{Aut}(\mathcal{A}, *)$, since $uu^* = -e \in K$. But $uu^* \neq \alpha\alpha^*$, where $\alpha \in K$, because if

$$-e = (a + b\sqrt{2})(a + b\sqrt{2})^*, \quad (12)$$

where $a, b \in k$, then

$$\begin{aligned} -e &= (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2 \\ &= \left(\frac{p_1(e)}{q_1(e)}\right)^2 - 2\left(\frac{p_2(e)}{q_2(e)}\right)^2, \end{aligned} \quad (13)$$

where $p_i(e), q_i(e) \in \mathbb{Q}[e]$ and $q_i(e) \neq 0$. Hence,

$$\begin{aligned} -e(q_1(e))^2(q_2(e))^2 &= (p_1(e)q_2(e))^2 \\ &\quad - 2(p_2(e)q_1(e))^2. \end{aligned} \quad (14)$$

Since $\sqrt{2} \notin \mathbb{Q}$, the highest power of e in the right-hand side is even, but the highest power of e in the left-hand side is odd, a contradiction.

Now we try to classify the group of automorphisms of central simple superalgebras $\mathcal{A} = M_{2p}(\mathcal{D}_{\bar{0}})$, and $\mathcal{A} = M_{p+q}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K and $(p \neq q)$.

First let $\mathcal{A} = M_{2p}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K , and let $*$ be any superinvolution on $\mathcal{A}$. If $\xi \in \text{Aut}(\mathcal{A}, *)$, then, by [1, Isomorphism Theorem], $\xi = \psi_{M_\alpha}$ for some invertible element $M_\alpha \in \mathcal{A}_\alpha$ and

$$M_\alpha P_\gamma M_\alpha^{-1} = (M_\alpha P_\gamma M_\alpha^{-1})^*, \quad \forall P_\gamma \in \mathcal{A}_\gamma; \quad (15)$$

thus,

$$\begin{aligned} M_\alpha P_\gamma M_\alpha^{-1} &= (-1)^{\alpha(\alpha+\gamma)} (-1)^{\alpha\gamma} (M_\alpha^{-1})^* P_\gamma^* M_\alpha^* \\ &= (-1)^\alpha (M_\alpha^{-1})^* P_\gamma^* M_\alpha^* = (M_\alpha^*)^{-1} P_\gamma^* M_\alpha^*, \end{aligned} \quad (16)$$

which implies that $M_\alpha^* M_\alpha P_\gamma^* M_\alpha^{-1} (M_\alpha^*)^{-1} = P_\gamma^*$, $\forall P_\gamma$, and therefore $M_\alpha^* M_\alpha = \lambda I_{2p}$, where $\lambda \in K^\times$. If $*$ is a superinvolution of the first kind, then

$$\begin{aligned} (M_\alpha^* M_\alpha)^* &= (-1)^{\alpha^2} M_\alpha^* M_\alpha = (\lambda I_{2p})^* = \lambda I_{2p} \\ &= M_\alpha^* M_\alpha. \end{aligned} \quad (17)$$

Hence $\alpha^2 = \bar{0}$, which implies that $\alpha = \bar{0}$ and therefore

$$\begin{aligned} \text{Aut}(\mathcal{A}, *) &= \{\psi_M \mid \\ &M \text{ is an invertible element in } \mathcal{A}_{\bar{0}}, \text{ with } MM^* \\ &= \lambda I_{2p}\}. \end{aligned} \quad (18)$$

Moreover, if $*$ is a superinvolution of the second kind on $\mathcal{A}$, then

$$\begin{aligned} \text{Aut}(\mathcal{A}, *) &= \{\psi_M \mid \\ &M \text{ is an invertible element in } \mathcal{A}_{\bar{0}}, \text{ with } MM^* \\ &= \lambda I_{2p}\} \cup \{\psi_M \mid \\ &M \text{ is an invertible element in } \mathcal{A}_{\bar{1}}, \text{ with } MM^* \\ &= \gamma I_{2p}, \gamma^* = -\gamma\}. \end{aligned} \quad (19)$$

Theorem 8. Let $\mathcal{A} = M_{2p}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K , and let $*$ be any superinvolution on $\mathcal{A}$. If $*$ is a superinvolution of the first kind, then $\text{Aut}(\mathcal{A}, *) \cong G/\sim$, where $G/\sim$ is the group defined in Theorem 4. If $*$ is a superinvolution of the second kind, then $\text{Aut}(\mathcal{A}, *) \cong G/\sim$, where $G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} \mid b_{\bar{0}} b_{\bar{0}}^* = \alpha \in K^\times\} \cup \{b_{\bar{1}} \in \mathcal{A}_{\bar{1}} \mid b_{\bar{1}} b_{\bar{1}}^* = \lambda \in K^\times, \text{ where } \lambda^* = -\lambda\}$ and the equivalence relation $\sim$ on G is defined by $a_\alpha \sim b_\alpha \Leftrightarrow b_\alpha = \gamma a_\alpha$ for some $\gamma \in K^\times$.

Proof. If $*$ is a superinvolution of the first kind, then

$$\delta : \text{Aut}(\mathcal{A}, *) \longrightarrow \frac{G}{\sim}, \quad (\psi_{a_{\bar{0}}} \mapsto [a_{\bar{0}}]) \quad (20)$$

is a group isomorphism.

Similarly, if $*$ is a superinvolution of the second kind, then it is easy to check that $G/\sim$ is a group under the well-defined operation $[a_\alpha][b_\beta] = [b_\beta a_\alpha]$, and so

$$\delta : \text{Aut}(\mathcal{A}, *) \longrightarrow \frac{G}{\sim}, \quad (\psi_{a_\alpha} \mapsto [a_\alpha]) \quad (21)$$

is a group isomorphism. Thus in all cases we get $\text{Aut}(\mathcal{A}, *) \cong G/\sim$. $\square$

Second let $\mathcal{A} = M_{p+q}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K and $(p \neq q)$. Let $*$ be any superinvolution on $\mathcal{A}$ of any kind. If $\xi \in \text{Aut}(\mathcal{A}, *)$, then by [1, Isomorphism Theorem] $\xi = \psi_{M_\alpha}$, for some invertible element M_α in $\mathcal{A}_\alpha$, since $p \neq q$, the homogeneous invertible elements in $\mathcal{A}$ are the invertible elements in $\mathcal{A}_{\bar{0}}$. Thus $\alpha = \bar{0}$, and, therefore, if $\xi \in \text{Aut}(\mathcal{A}, *)$, then $\xi = \psi_M$, where M is an invertible element in $\mathcal{A}_{\bar{0}}$. Now, if $P_\gamma \in \mathcal{A}_\gamma$, then

$$\begin{aligned} MP_\gamma^* M^{-1} &= (MP_\gamma M^{-1})^* = (M^{-1})^* P_\gamma^* M^* \\ &= (M^*)^{-1} P_\gamma^* M^*. \end{aligned} \quad (22)$$

Therefore, $M^* MP_\gamma^* M^{-1} (M^*)^{-1} = P_\gamma^*$, $\forall P_\gamma \in \mathcal{A}_\gamma$, and hence $M^* M = \lambda I_{p+q}$, where $\lambda \in K^\times$. Thus it is easy to check that $\text{Aut}(\mathcal{A}, *) = \{\psi_M \mid M \text{ is an invertible element in } \mathcal{A}_{\bar{0}}, \text{ where } MM^* = \lambda I_{p+q}\} \cong G/\sim$, where $G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} \mid b_{\bar{0}} b_{\bar{0}}^* = \alpha \in K^\times\}$.

3. Examples of Group of Automorphisms of Even Superalgebras

In this section we classify in detail the groups of automorphisms of superalgebras $\mathcal{A} = M_{2p}(\mathcal{D})$, and $\mathcal{A} = M_{p+q}(\mathcal{D})$,

where $\mathcal{D} = \mathbb{H}$ or $\mathbb{R}$, where $\mathbb{H}$ is the Hamiltonian quaternion algebra and $\mathbb{R}$ is the field of real numbers.

Let $\mathcal{A} = M_{2p}(\mathcal{D})$, where $\mathcal{D} = \mathbb{H}$ or $\mathbb{R}$, and let $*$ be a superinvolution of the first kind on $\mathcal{A}$ such that $(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}})$ is simple; then by [1, Proposition 13] $*$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}, \quad (23)$$

where $\sim$ is the involution on $M_p(\mathcal{D})$ induced by $*$. Then $H = \text{Aut}(\mathcal{A}, *) = \{\psi_M \mid M \text{ is an invertible element in } \mathcal{A}_{\bar{0}}, \text{ where } MM^* = \alpha I_{2p}, \text{ and } \alpha \in \mathbb{R}\}$.

If $\alpha > 0$, then by replacing M by $(1/\sqrt{\alpha})M$ we get $MM^* = I_{2p}$.

If $\alpha < 0$, then by replacing M by $(1/\sqrt{-\alpha})M$ we get $MM^* = -I_{2p}$.

So $H = \{\psi_M \mid MM^* = I_{2p} \text{ or } -I_{2p}\}$.

Theorem 9. In the situation described above, let

$$G = \text{Aut}(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}}). \quad (24)$$

Then $H/\mathbb{R}^\times \cong G_0$, where G_0 is the connected component of the identity in G .

Proof. By [8, page 593], $G = G_0 \cup \psi_N G_0$, where $N = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and G_0 is the connected component of the identity. Let

$$\begin{aligned} \pi : H &\longrightarrow G_0 \\ (\psi_M &\mapsto \psi_M|_{\mathcal{A}_{\bar{0}}}). \end{aligned} \quad (25)$$

Then π is onto group homomorphism because if $g_0 \in G_0$, then $g_0 = \psi_M|_{\mathcal{A}_{\bar{0}}}$ for some invertible element $M \in \mathcal{A}_{\bar{0}}$ since G_0 is the connected component of the identity. Then it is easy to check that $MM^* = \alpha I_{2p}$ ($\alpha \in \mathbb{R}$) since $\psi_M^* = * \psi_M$.

If $\alpha > 0$, then by replacing M by $(1/\sqrt{\alpha})M$ we get $MM^* = I_{2p}$.

If $\alpha < 0$, then by replacing M by $(1/\sqrt{-\alpha})M$ we get $MM^* = -I_{2p}$. Therefore, $\psi_M \in H$ and $\pi(\psi_M) = g_0$. Also it is easy to check that $\ker \pi = \{\psi_M \mid M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_p \end{pmatrix}, \text{ where } \lambda, \gamma \in \mathbb{R}\}$. But

$$\begin{aligned} \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_p \end{pmatrix} \begin{pmatrix} \gamma I_p & 0 \\ 0 & \lambda I_p \end{pmatrix} &= \begin{pmatrix} \lambda \gamma I_p & 0 \\ 0 & \gamma \lambda I_p \end{pmatrix} = I_{2p} \\ \text{or } \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_p \end{pmatrix} \begin{pmatrix} \gamma I_p & 0 \\ 0 & \lambda I_p \end{pmatrix} &= \begin{pmatrix} \lambda \gamma I_p & 0 \\ 0 & \gamma \lambda I_p \end{pmatrix} \\ &= -I_{2p}. \end{aligned} \quad (26)$$

So, $\gamma = 1/\lambda$ or $-1/\lambda$. Therefore, $\ker \pi = \{\psi_M \mid M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & (1/\lambda)I_p \end{pmatrix}, \text{ or } M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & (-1/\lambda)I_p \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}^\times\} \cong \mathbb{R}^\times$, since $\psi_M = \psi_{-M}$. Therefore $G_0 \cong H/\ker \pi \cong H/\mathbb{R}^\times$. $\square$

In the second case, let $\mathcal{A} = M_{p+q}(\mathcal{D})$, where $\mathcal{D} = \mathbb{H}$ or $\mathbb{R}$, and let $*$ be a superinvolution of the first kind on $\mathcal{A}$ such

that $(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}})$ is not simple; then by [1, Proposition 14] $*$ is defined by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} & -N\bar{z}^t M^{-1} \\ M\bar{y}^t N^{-1} & \tilde{w} \end{pmatrix}, \quad (27)$$

where $\tilde{x} = N\bar{x}^t N^{-1}$ ($\bar{N}^t = -N \in M_p(\mathcal{D})$) and $\tilde{w} = M\bar{w}^t M^{-1}$ ($\bar{M}^t = M \in M_q(\mathcal{D})$) and $\bar{\cdot}$ is the standard involution on $\mathcal{D}$.

Since $*$ is a superinvolution of the first kind we have $H = \text{Aut}(\mathcal{A}, *) = \{\psi_M \mid M \text{ is an invertible element in } \mathcal{A}_{\bar{0}}, \text{ where } MM^* = \alpha I_{p+q}, \text{ and } \alpha \in \mathbb{R}\}$. Similarly, $H = \{\psi_M \mid MM^* = I_{p+q} \text{ or } -I_{p+q}\}$.

We will start with the case $p \neq q$.

Theorem 10. *In the situation described above, let*

$$G = \text{Aut}(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}}). \quad (28)$$

Then $H/\ker \pi \cong \pi(H) \leq G$, where

$$\pi : H \longrightarrow G \quad (\psi_N \mapsto \psi_N|_{\mathcal{A}_{\bar{0}}}). \quad (29)$$

Proof. Since $\mathcal{A}_{\bar{0}}$ is semisimple and $p \neq q$, we get $G = \text{Aut}(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}}) = \{\psi_M \mid M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{ where } A, B \text{ are invertible matrices}\}$. If $\psi_M \in G$, then

$$\begin{aligned} \psi_M *|_{\mathcal{A}_{\bar{0}}} &= *|_{\mathcal{A}_{\bar{0}}} \psi_M \iff \\ MM^* &= \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix}, \quad (\lambda, \gamma \in \mathbb{R}); \text{ therefore} \\ \psi_M *|_{\mathcal{A}_{\bar{0}}} &= *|_{\mathcal{A}_{\bar{0}}} \psi_M \iff \end{aligned} \quad (30)$$

$$MM^* \in V = \left\{ I_{p+q}, -I_{p+q}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \right\}.$$

So $G = \{\psi_M \mid MM^* \in V\}$. Therefore,

$$\pi : H \longrightarrow G \quad (\psi_N \mapsto \psi_N|_{\mathcal{A}_{\bar{0}}}) \quad (31)$$

is a group homomorphism and $\ker \pi = \{\psi_N \mid \psi_N|_{\mathcal{A}_{\bar{0}}} = id_{\mathcal{A}_{\bar{0}}}\} = \{id_{\mathcal{A}}, \psi_M, M = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\}$. To see this let $N = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that $\psi_N \in \ker \pi$; then $NN^* = I_{p+q}$ or $-I_{p+q}$ (since $\psi_N \in H$), and

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \quad (32)$$

if and only if $A = \lambda I_p$ and $B = \gamma I_q$ ($\lambda, \gamma \in \mathbb{R}$); since

$$\begin{aligned} NN^* &= \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix} \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix} = \begin{pmatrix} \lambda^2 I_p & 0 \\ 0 & \gamma^2 I_q \end{pmatrix} \\ &\in \{I_{p+q}, -I_{p+q}\}, \end{aligned} \quad (33)$$

we get $NN^* = I_{p+q}$. Therefore, $\lambda^2 = \gamma^2 = 1 \Rightarrow \lambda, \gamma \in \{1, -1\}$, which implies that $\ker \pi = \{\psi_{I_{p+q}}, \psi_{-I_{p+q}}, \psi_N, \psi_M\}$, where $N = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$ and $M = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and hence $\ker \pi = \{id_{\mathcal{A}}, \psi_M\}$ since $\psi_{I_{p+q}} = \psi_{-I_{p+q}}$ and $\psi_N = \psi_M$. Therefore, $H/\ker \pi \cong \pi(H) \leq G$. $\square$

Theorem 11. *As in the theorem above, if $p \neq q$, then $G = \pi(H) \cup \psi_M|_{\mathcal{A}_{\bar{0}}} \pi(H)$ for a fixed invertible matrix $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that*

$$MM^* \in \left\{ \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \right\}. \quad (34)$$

Proof. Since $MM^* \neq I_{p+q}, -I_{p+q}$, then $\psi_M \notin H$, which implies that $\psi_M|_{\mathcal{A}_{\bar{0}}} \notin \pi(H)$. Let $N = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ such that

$$NN^* \in \left\{ \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \right\}; \quad (35)$$

then

$$\begin{aligned} (NM^{-1})(NM^{-1})^* &= NM^{-1}(M^{-1})^* N^* \\ &= \begin{cases} N \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} N^* : MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \\ N \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} N^* : MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \end{cases} \\ &= \begin{cases} \begin{pmatrix} X\tilde{X} & 0 \\ 0 & -Y\tilde{Y} \end{pmatrix} : MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \\ \begin{pmatrix} -X\tilde{X} & 0 \\ 0 & Y\tilde{Y} \end{pmatrix} : MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \end{cases} \end{aligned} \quad (36)$$

Therefore, $(NM^{-1})(NM^{-1})^* \in \{I_{p+q}, -I_{p+q}\}$; thus, by Theorem 10,

$$\psi_{NM^{-1}}|_{\mathcal{A}_{\bar{0}}} = \psi_{M^{-1}}|_{\mathcal{A}_{\bar{0}}} \psi_N|_{\mathcal{A}_{\bar{0}}} \in \pi(H) \cong \frac{H}{\ker \pi}. \quad (37)$$

Now, $\psi_N|_{\mathcal{A}_{\bar{0}}} = \psi_{NM^{-1}M}|_{\mathcal{A}_{\bar{0}}} = \psi_M|_{\mathcal{A}_{\bar{0}}} \psi_{M^{-1}}|_{\mathcal{A}_{\bar{0}}} \psi_N|_{\mathcal{A}_{\bar{0}}} = \psi_M|_{\mathcal{A}_{\bar{0}}} \underbrace{\psi_{NM^{-1}}|_{\mathcal{A}_{\bar{0}}}}_{\in \psi_M|_{\mathcal{A}_{\bar{0}}} \pi(H)}$. Therefore,

$$G = \pi(H) \cup \psi_M|_{\mathcal{A}_{\bar{0}}} \pi(H). \quad (38)$$

$\square$

For the case $p = q$, $G = \text{Aut}(\mathcal{A}_{\bar{0}}, *|_{\mathcal{A}_{\bar{0}}}) = G_0 \cup \psi_N|_{\mathcal{A}_{\bar{0}}} G_0$, where $N = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ (it is easy to check that $\psi_N|_{\mathcal{A}_{\bar{0}}} *|_{\mathcal{A}_{\bar{0}}} = *|_{\mathcal{A}_{\bar{0}}} \psi_N|_{\mathcal{A}_{\bar{0}}}$) and $G_0 = \{\psi_M|_{\mathcal{A}_{\bar{0}}} \mid MM^* \in V\}$ is the connected component of the identity in G . Now, let $\pi : H \rightarrow G_0$ ($\psi_L \mapsto \psi_L|_{\mathcal{A}_{\bar{0}}}$), where $H = \{\psi_M \mid MM^* = I_{p+q} \text{ or } -I_{p+q}\}$; then π is a group homomorphism and $\ker \pi = \{id_{\mathcal{A}}, \psi_{N'}\}$ where $N' = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$. Therefore, $H/\ker \pi \cong \pi(H) \leq G_0$.

Theorem 12. In the situation described above, if $p = q$, then

$$G_0 = \pi(H) \cup \psi_M|_{\mathcal{A}_0} \pi(H), \quad (39)$$

for a fixed invertible matrix $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, such that

$$MM^* \in \left\{ \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \right\}, \quad (40)$$

and G_0 is the connected component of the identity in G .

Proof. Consider $\psi_M|_{\mathcal{A}_0} \notin \pi(H)$ since $MM^* \in \left\{ \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} \right\}$. Let $N = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ such that $NN^* \in \left\{ \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} \right\}$; then

$$\begin{aligned} (NM^{-1})(NM^{-1})^* &= NM^{-1}(M^{-1})^* N^* \\ &= \begin{cases} N \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} N^*: & MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \\ N \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} N^*: & MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} \end{cases} \quad (41) \\ &= \begin{cases} \begin{pmatrix} X\bar{X} & 0 \\ 0 & -Y\bar{Y} \end{pmatrix}: & MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \\ \begin{pmatrix} -X\bar{X} & 0 \\ 0 & Y\bar{Y} \end{pmatrix}: & MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix}. \end{cases} \end{aligned}$$

Therefore, $(NM^{-1})(NM^{-1})^* \in \{I_{2p}, -I_{2p}\}$ and so

$$\psi_{NM^{-1}}|_{\mathcal{A}_0} = \psi_{M^{-1}}|_{\mathcal{A}_0} \psi_N|_{\mathcal{A}_0} \in \pi(H) \cong \frac{H}{\ker \pi}. \quad (42)$$

Now, $\psi_N|_{\mathcal{A}_0} = \psi_{NM^{-1}M}|_{\mathcal{A}_0} = \psi_M|_{\mathcal{A}_0} \psi_{M^{-1}}|_{\mathcal{A}_0} \psi_N|_{\mathcal{A}_0} = \psi_M|_{\mathcal{A}_0} \underbrace{\psi_{NM^{-1}}|_{\mathcal{A}_0}}_{\in \psi_M \pi(H)} \in \psi_M \pi(H)$. Therefore,

$$G_0 = \pi(H) \cup \psi_M|_{\mathcal{A}_0} \pi(H). \quad (43)$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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